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Research Article

Chebyshev Collocation Method for Parabolic Partial Integrodifferential Equations

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An efficient technique for solving parabolic partial integrodifferential equation is presented. This technique is based on Chebyshev polynomials and finite difference method. *A priori* error estimate for the proposed technique is deduced. Some examples are presented to illustrate the validity and efficiency of the presented method.

1. Introduction

Partial integrodifferential equations (PIDEs) are the equations that combine partial differentiation and integration of the unknown function. They are used in modeling several phenomena where the effect of memory must be considered.

In this work, we consider the parabolic PIDE of the following form

$$\frac{\partial u}{\partial t} + \alpha(x,t)\frac{\partial u}{\partial x} + \beta(x,t)\frac{\partial^2 u}{\partial x^2} = g(x,t) + \int_0^t k(t,s)\,u(x,s)\,ds, \qquad a < x < b,\ 0 < t \le T, \tag{1}$$

subject to the following initial and boundary conditions:

$$\begin{aligned} u(x,0) &= \phi(x), \quad x \in [a,b], \\ u(a,t) &= h(t), \\ u(b,t) &= l(t), \\ & t \in [0,T], \end{aligned} \tag{2}$$

where $u(x,t)$, $g(x,t)$, $\alpha(x,t)$, and $\beta(x,t)$ are continuous functions.

This class of equations appears in various fields of physics and engineering such as heat conduction [1], compression of poroviscoelastic media [2], reaction diffusion problems [3], and nuclear reactor dynamics [4].

PIDEs are solved by some numerical methods [5–12]. In this work, the Chebyshev polynomials are applied through finite difference method to obtain an approximate solution for problems (1) and (2).

Chebyshev polynomials were proposed by the Russian mathematician Chebyshev. Many authors depend on Chebyshev collocation method to solve different types of equations such as linear differential equations [13], systems of high-order linear differential equations with variable coefficients [14], systems of high-order linear Fredholm-Volterra integro-differential equations [15], fourth-order Sturm-Liouville problems [16], Troeschs problem [17], nonlinear differential equations [18], linear partial differential equations [19, 20], and nonlinear Fredholm-Volterra integrodifferential equations [21].

This paper is organized into six sections. Section 2 lists some notations and definitions of Chebyshev polynomials. Section 3 presents a description for the technique of solution to problems (1) and (2). In Section 4, we deduce *a priori* error estimate for the approximate solution. Section 5 is devoted to present some examples that illustrate the proposed technique of solution. Finally, Section 6 presents the conclusions of this study.

2. Fundamental Relations

On a general interval $x \in [a, b]$, the shifted Chebyshev polynomials take the following form:

$$T_n^*(x) = \cos\left(n \arccos\left(\frac{2}{b-a}\left(x - \frac{b+a}{2}\right)\right)\right), \quad (3)$$

and the Chebyshev collocation points are given by

$$x_k = \frac{b-a}{2}\left[\left(\frac{a+b}{b-a}\right) + \cos\left(\frac{k\pi}{N}\right)\right], \quad k = 0, 1, 2, \ldots, N. \quad (4)$$

The function $u(x)$ on $[a, b]$ is approximated using truncated shifted Chebyshev series in the form

$$u(x) = \sum_{r=0}^{N}{}' a_r^* T_r^*(x), \quad a \le x \le b. \quad (5)$$

The symbol sum with single prime indicates that the summation involves $(1/2)a_0^*$ rather than a_0^*. Similarly, the derivatives $u^{(k)}(x)$, $k = 0, 1, 2, 3, \ldots$, are written as

$$u^{(k)}(x) = \sum_{r=0}^{N}{}' a_r^{*(k)} T_r^*(x). \quad (6)$$

The function $u(x)$ and its derivatives can be written in matrix form as

$$u(x) = T^*(x) A^*, \qquad u^{(k)}(x) = T^*(x) A^{*(k)}, \quad (7)$$

where

$$T^*(x) = \left[T_0^*(x) \;\; T_1^*(x) \;\; T_2^*(x) \;\; \cdots \;\; T_N^*(x)\right], \qquad A^* = \left[\frac{1}{2}a_0^* \;\; a_1^* \;\; \cdots \;\; a_N^*\right]^\tau. \quad (8)$$

Matrix $A^{*(k)}$ can be obtained from A^* by the following relation [13]:

$$A^{*(k)} = \left(\frac{4}{b-a}\right)^k M^k A^*, \quad (9)$$

where

$$M = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{5}{2} & \cdots & m_1 \\ 0 & 0 & 2 & 0 & 4 & 0 & \cdots & m_2 \\ 0 & 0 & 0 & 3 & 0 & 5 & \cdots & m_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad (10)$$

where

$$\begin{aligned} m_1 &= \frac{N}{2}, \\ m_2 &= 0, \\ m_3 &= N, \\ &\text{if } N \text{ is odd}, \\ m_1 &= 0, \\ m_2 &= N, \\ m_3 &= 0, \\ &\text{if } N \text{ is even}. \end{aligned} \quad (11)$$

Then, $u^{(k)}(x)$ can be written in the form

$$u^{(k)}(x) = \left(\frac{4}{b-a}\right)^k M^k T^*(x) A^*. \quad (12)$$

3. Description of Chebyshev Collocation Method

The partial time-derivative in (1) is discretized using finite difference method. Denote the time step by δt and the value of the function at time $i\delta t$ by u_i. Then we have

$$\frac{\partial u}{\partial t} = \frac{u_{i+1} - u_i}{\delta t}. \quad (13)$$

The integral is handled numerically using the composite trapezoidal rule, and (1) is discretized into the following form:

$$\begin{aligned} &\frac{u_{i+1} - u_i}{\delta t} + \alpha_{i+1}(x) u_{i+1}' + \beta_{i+1}(x) u_{i+1}'' \\ &\quad = g_{i+1}(x) + \frac{1}{2}\delta t k_{i+1}(t_0) u_0 + \delta t \sum_{j=1}^{i} k_{i+1}(t_j) u_j \\ &\qquad + \frac{1}{2}\delta t k_{i+1}(t_{i+1}) u_{i+1}, \end{aligned} \quad (14)$$

which is simplified to

$$\begin{aligned} &\delta t \beta_{i+1}(x) u_{i+1}'' + \delta t \alpha_{i+1}(x) u_{i+1}' \\ &\quad + \left(1 - \frac{1}{2}(\delta t)^2 k_{i+1}(t_{i+1})\right) u_{i+1} = z_{i+1}(x), \end{aligned} \quad (15)$$

where

$$\begin{aligned} z_{i+1}(x) &= u_i + \frac{1}{2}(\delta t)^2 k_{i+1}(t_0) u_0 \\ &\quad + (\delta t)^2 \sum_{j=1}^{i} k_{i+1}(t_j) u_j + \delta t g_{i+1}(x). \end{aligned} \quad (16)$$

The approximate solution of (15) is computed using truncated Chebyshev series.

Theorem 1. *If the assumed approximate solution of problem (15) is described by (5), then the discrete Chebyshev system is given by*

$$\begin{aligned} &4^2\delta t\beta_{i+1}(x_k)T^*(x_k)M^2A^*_{i+1} \\ &\quad + 4\delta t\alpha_{i+1}(x_k)T^*(x_k)MA^*_{i+1} \\ &\quad + \left(1-\frac{1}{2}(\delta t)^2k_{i+1}(t_{i+1})\right)T^*(x_k)A^*_{i+1} \\ &= z_{i+1}(x_k), \end{aligned} \tag{17}$$

where x_k are the Chebyshev collocation points. The fundamental matrix form of the discrete Chebyshev system is given by

$$WA^*_{i+1} = Z_{i+1}, \tag{18}$$

where

$$\begin{aligned} W &= 4^2\delta t\beta_{i+1}T^*M^2 + 4\delta t\alpha_{i+1}T^*M \\ &\quad + \left(1-\frac{1}{2}(\delta t)^2k_{i+1}(t_{i+1})\right)T^*, \\ \alpha_{i+1} &= \begin{bmatrix} \alpha_{i+1}(x_0) & 0 & \cdots & 0 \\ 0 & \alpha_{i+1}(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{i+1}(x_N) \end{bmatrix}, \\ \beta_{i+1} &= \begin{bmatrix} \beta_{i+1}(x_0) & 0 & \cdots & 0 \\ 0 & \beta_{i+1}(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_{i+1}(x_N) \end{bmatrix}, \\ T^* &= \begin{bmatrix} T^*(x_0) & 0 & \cdots & 0 \\ 0 & T^*(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T^*(x_N) \end{bmatrix}, \\ Z_{i+1} &= \begin{bmatrix} z_{i+1}(x_0) \\ z_{i+1}(x_1) \\ \vdots \\ z_{i+1}(x_N) \end{bmatrix}, \\ A^*_{i+1} &= \begin{bmatrix} \frac{1}{2}(a^*_0)_{i+1} \\ (a^*_1)_{i+1} \\ \vdots \\ (a^*_N)_{i+1} \end{bmatrix}. \end{aligned} \tag{19}$$

The boundary conditions are integrated into system of (18) in the following form:

$$\begin{aligned} T^*(a)A^*_{i+1} &= h(t_{i+1}), \\ T^*(b)A^*_{i+1} &= l(t_{i+1}). \end{aligned} \tag{20}$$

The final form of the system is given by

$$\widetilde{W}A^*_{i+1} = \widetilde{Z}. \tag{21}$$

We construct the two matrices $\widetilde{W}$ and $\widetilde{Z}$ by replacing the first row and last row of the matrix $[W, Z]$ by the corresponding row of boundary conditions.

4. Error Analysis

In (1), the time is discretized using finite difference method. From Taylor series, we have

$$\left.\frac{\partial u}{\partial t}\right|_{t_{i+1}} = \frac{u_{i+1}-u_i}{\delta t} + O(\delta t). \tag{22}$$

Approximation of integral term using composite trapezoidal integration rule leads to an error of order $O((\delta t)^2)$. So (15) can be written in the form

$$\begin{aligned} &\delta t\beta_{i+1}(x)u''_{i+1} + \delta t\alpha_{i+1}(x)u'_{i+1} \\ &\quad + \left(1-\frac{1}{2}(\delta t)^2k_{i+1}(t_{i+1})\right)u_{i+1} \\ &= z_{i+1}(x) + O\left((\delta t)^2\right). \end{aligned} \tag{23}$$

Equation (23) shows that the truncation error due to time discretization of (1) is of order $O((\delta t)^2)$.

If $u(x,t)$ is the exact solution of (1), then $u(x,t_{i+1})$ is the exact solution of (23). Let $u_1(x,t_{i+1})$ and $u_2(x,t_{i+1})$ be the Chebyshev series solution of (15) and (23), respectively. If $u_1(x,t_{i+1}) = T(x)\overline{A}^*_{i+1}$, then the discrete Chebyshev system is given by

$$W\overline{A}^*_{i+1} = Z_{i+1} + O\left((\delta t)^2\right). \tag{24}$$

By subtracting (18) from (24), the following relation is obtained:

$$\overline{A}^*_{i+1} - A^*_{i+1} = W^{-1}O\left((\delta t)^2\right) = O\left((\delta t)^2\right). \tag{25}$$

Let $I^X_N(x,t_{i+1})$ be the Lagrange interpolating polynomial of order N of $u(x,t_{i+1})$ on the grid X of Chebyshev collocation points. Then $I^X_N(x,t_{i+1})$ is given by

$$I^X_N(x,t_{i+1}) = \sum_{k=0}^{N} u(x_k,t_{i+1})\,l_i(x), \tag{26}$$

where

$$l_i(x_j) = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{27}$$

Theorem 2 (see [22]). *If $f \in C^{N+1}[a,b]$, then for any grid X of $N+1$ nodes and for any $x \in [a,b]$ the interpolation error at x is given by*

$$f(x,t_{i+1}) - I_N^X(x,t_{i+1}) = \frac{f^{(N+1)}(\xi,t_{i+1})}{(N+1)!}\omega_{N+1}^X(x), \quad (28)$$

where $\xi \in [a,b]$ and $\omega_{N+1}^X(x)$ are the nodal polynomial given by

$$\omega_{N+1}^X(x) = \prod_{j=0}^{N}(x - x_j). \quad (29)$$

Let $u(x,t_{i+1}) = I_N(x,t_{i+1}) + H_N(x,t_{i+1})$. Then $I_N(x,t_{i+1})$ is the solution of the following equation:

$$\begin{aligned}\delta t\beta_{i+1}(x) I_N''(x,t_{i+1}) &+ \delta t\alpha_{i+1}(x) I_N'(x,t_{i+1}) \\ &+ \left(1 - \frac{1}{2}(\delta t)^2 k_{i+1}(t_{i+1})\right) I_N(x,t_{i+1}) \\ &= z_{i+1}(x) + \Delta z_{i+1}(x) + O\left((\delta t)^2\right),\end{aligned} \quad (30)$$

where

$$\begin{aligned}\Delta z_{i+1}(x) = \delta t\beta_{i+1}(x) H_N''(x,t_{i+1}) &\\ + \delta t\alpha_{i+1}(x) H_N'(x,t_{i+1}) &\\ + \left(1 - \frac{1}{2}(\delta t)^2 k_{i+1}(t_{i+1})\right) H_N(x,t_{i+1}).&\end{aligned} \quad (31)$$

$I_N(x,t_{i+1})$ can be written as $I_N(x,t_{i+1}) = T(x)A_{i+1}^{*\prime}$. So the discrete Chebyshev system of (30) is given by

$$WA_{i+1}^{*\prime} = Z + \Delta Z + O\left((\delta t)^2\right); \quad (32)$$

by subtracting (18) from (32), we obtain

$$A_{i+1}^{*\prime} - A_{i+1}^{*} = W^{-1}\Delta Z + O\left((\delta t)^2\right). \quad (33)$$

Theorem 3. *Let $u_1(x,t_{i+1})$ be the Chebyshev series solution of (15) and $u(x,t_{i+1})$ be the exact solution of (23). If $u(x,t_{i+1})$ is sufficiently smooth, then*

$$\begin{aligned}&\left|u(x,t_{i+1}) - u_1(x,t_{i+1})\right| \\ &\quad \le \left|H_N(x,t_{i+1})\right| + \|T(x)\| \left\|W^{-1}\right\| \|\Delta Z\| \\ &\qquad + O\left((\delta t)^2\right).\end{aligned} \quad (34)$$

Proof. Adding and subtracting the polynomial $u_2(x,t_{i+1})$ yield

$$\begin{aligned}&\left|u(x,t_{i+1}) - u_1(x,t_{i+1})\right| \\ &\quad \le \left|u(x,t_{i+1}) - u_2(x,t_{i+1})\right| \\ &\qquad + \left|u_2(x,t_{i+1}) - u_1(x,t_{i+1})\right|.\end{aligned} \quad (35)$$

Table 1: l_∞ error of Example 1 at $\delta t = 0.0025$, $t = 0.5$, and $\kappa = 1$, $\lambda = -1$.

x	Chebyshev collocation method $N = 10$	Backward-Euler scheme [12] $N = 100$
0.31	$0.5545E-04$	$0.632306E-04$
0.63	$0.1070E-03$	$0.120272E-03$
0.94	$0.1467E-03$	$0.165540E-03$
1.26	$0.1730E-03$	$0.194604E-03$
1.57	$0.1817E-03$	$0.204619E-03$

The upper bound for the second term can be found using some properties of norm

$$\begin{aligned}&\left|u_2(x,t_{i+1}) - u_1(x,t_{i+1})\right| = \left|T(x)\overline{A}_{i+1}^{*} - T(x)A_{i+1}^{*}\right| \\ &\quad \le \|T(x)\| \left|\overline{A}_{i+1}^{*} - A_{i+1}^{*}\right| \le \|T(x)\| O\left((\delta t)^2\right) \\ &\quad \le O\left((\delta t)^2\right).\end{aligned} \quad (36)$$

The upper bound for the second term can be found by adding and subtracting the polynomial $I_N(x,t_{i+1})$ following the work in [20].

$$\begin{aligned}&\left|u(x,t_{i+1}) - u_2(x,t_{i+1})\right| \\ &\quad \le \left|u(x,t_{i+1}) - I_N(x,t_{i+1})\right| \\ &\qquad + \left|I_N(x,t_{i+1}) - u_2(x,t_{i+1})\right| \\ &\quad \le \left|H_N(x,t_{i+1})\right| + \left|T(x)A_{i+1}^{*\prime} - T(x)\overline{A}_{i+1}^{*}\right| \\ &\quad \le \left|H_N(x,t_{i+1})\right| + \|T(x)\| \left|A_{i+1}^{*\prime} - \overline{A}_{i+1}^{*}\right| \\ &\quad \le \left|H_N(x,t_{i+1})\right| + \|T(x)\| \left\|W^{-1}\right\| \|\Delta Z\| \\ &\qquad + O\left((\delta t)^2\right).\end{aligned} \quad (37)$$

Summing up these two upper bounds yields (34). □

5. Numerical Examples

Example 1 (see [12]). Consider the parabolic PIDE (1) with the initial condition

$$u(x,0) = \sin(x) \quad 0 \le x \le \pi \quad (38)$$

and with boundary conditions

$$u(0,t) = u(\pi,t) = 0, \quad 0 \le t \le 1 \quad (39)$$

and with $\alpha = 0$, $\beta = -\kappa$, $k(x,t) = \lambda\exp(x-t)$, and $g(x,t) = 0$. The exact solution of this example is $u(x,t) = e^{(-\theta t)}(\cos(\nu t) + ((1-\theta)/\nu)\sin(\nu t))\sin(x)$, where $\theta = (\kappa+1)/2$ and $\nu = (\sqrt{-\kappa^2 + 2\kappa - 1 - 4\lambda})/2$.

The maximum absolute error is tabulated in Table 1 for Chebyshev polynomial together with the results of [12] at

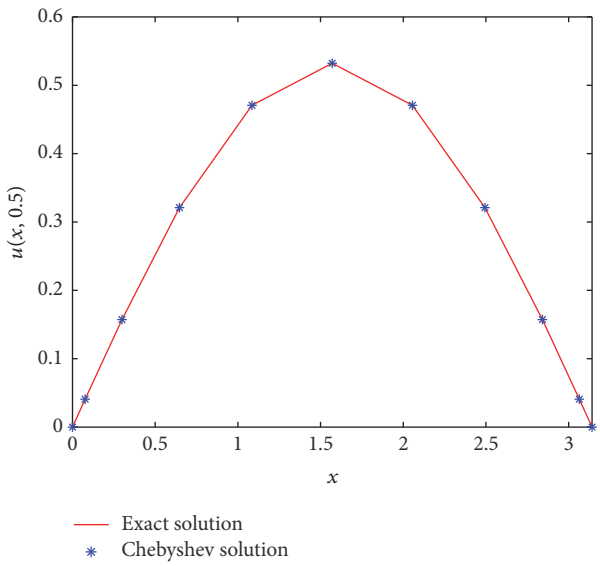

FIGURE 1: Exact and Chebyshev approximated solutions of Example 1 at $t = 0.5$, $\kappa = 1$, $\lambda = -1$.

TABLE 2: l_∞ error of Example 1 at $\delta t = 10^{-3}$, $N = 10$, $\kappa = 2$, and $\lambda = -2$.

t	l_∞ error
0.2	$1.72E-04$
0.4	$2.75E-04$
0.6	$3.09E-04$
0.8	$2.88E-04$
1	$2.33E-04$

$\kappa = 1$ and $\lambda = -1$. The graphs of exact and approximate solutions for $t = 0.5$ are illustrated in Figure 1.

The maximum absolute error is tabulated in Table 2 for Chebyshev polynomial at $\kappa = 2$ and $\lambda = -2$.

Example 2. Consider the parabolic PIDE (1) with the initial condition

$$u(x,0) = e^{-k(x-0.5)^2} \quad 0 \le x \le 1 \tag{40}$$

and with boundary conditions

$$u(0,t) = u(1,t) = \cos(t)\, e^{-k/4}, \quad 0 \le t \le 1 \tag{41}$$

and with $\alpha = 0$, $\beta = 1$, $k(x,t) = (t^2 - x^2)$, and $g(x,t) = \cos(t)e^{-k(x-0.5)^2}(2t + 4k^2(x-0.5)^2 - 2k)$. The exact solution of this example is given by $u(x,t) = \cos(t)e^{-k(x-0.5)^2}$.

The maximum absolute error is tabulated in Table 3 for Chebyshev polynomial at different times. The graphs of exact and approximate solutions for $t = 1$ and $k = 100$ are illustrated in Figure 2.

Example 3 (see [10]). Consider the parabolic PIDE (1) with the initial condition

$$u(x,0) = x \quad 0 \le x \le 1 \tag{42}$$

TABLE 3: l_∞ error of Example 2 at $\delta t = 10^{-4}$, $N = 75$, and $k = 100$.

t	l_∞ error
0.2	$2.59E-06$
0.4	$3.29E-06$
0.6	$3.33E-06$
0.8	$3.03E-06$
1	$2.514E-06$

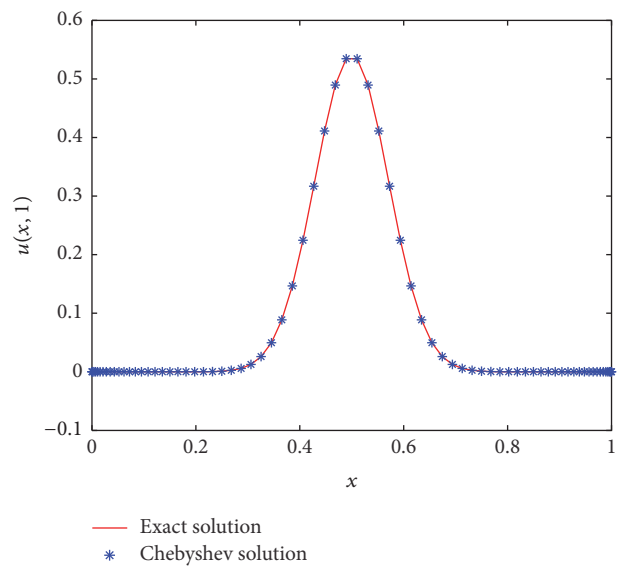

FIGURE 2: Exact and Chebyshev approximated solutions of Example 2 at $t = 1$, $k = 100$.

and with boundary conditions

$$u(0,t) = 0,$$
$$u(1,t) = e^{-t}, \tag{43}$$
$$0 \le t \le 1$$

and with $\alpha = 0$, $\beta = -x^2/2$, $k(x,t) = e^{(x-t)}$, and $g(x,t)$ which can be chosen such that the exact solution in this example is $u(x,t) = xe^{-tx}$.

The maximum absolute error is tabulated in Table 4 for Chebyshev polynomial together with the results of [10]. The graphs of exact and approximate solutions for $t = 1$ are illustrated in Figure 3.

Example 4. Consider the parabolic PIDE (1) with the initial condition

$$u(x,0) = e^{-kx}\sin(\omega\pi x) \quad 0 \le x \le 1 \tag{44}$$

and with boundary conditions

$$u(0,t) = 0,$$
$$u(1,t) = 0, \tag{45}$$
$$0 \le t \le 1$$

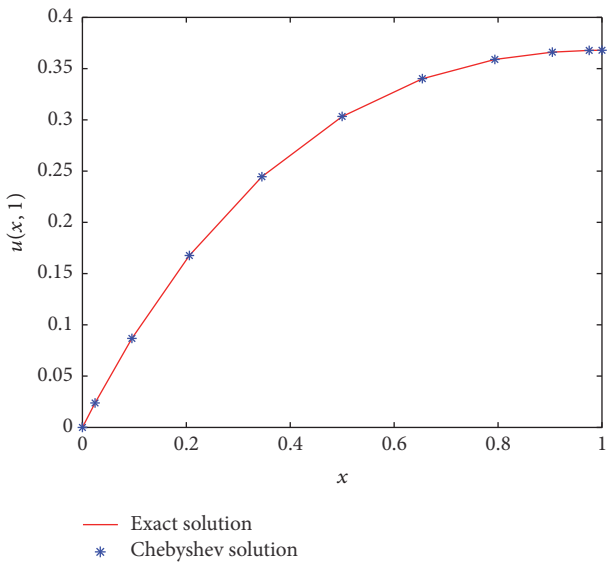

Figure 3: Exact and Chebyshev approximated solutions of Example 3 at $t = 1$.

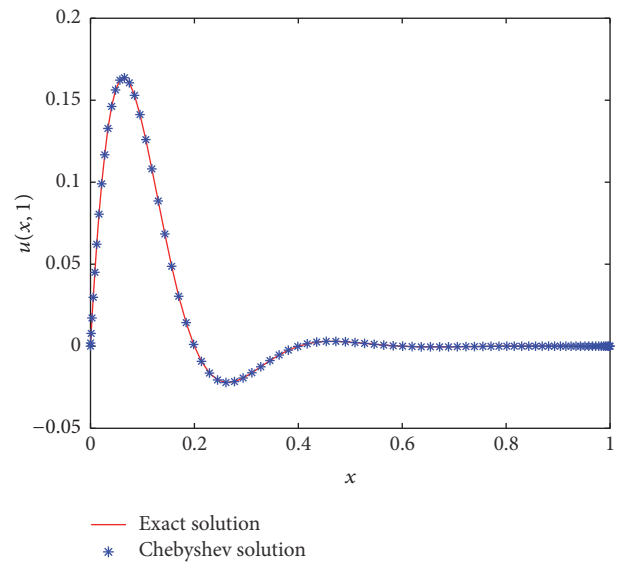

Figure 4: Exact and Chebyshev approximated solutions of Example 4 at $t = 1$, $k = 10$, $\omega = 5$.

Table 4: l_∞ error of Example 3 at $\delta t = 10^{-3}$.

t	Chebyshev collocation method $N = 10$	Radial basis functions [10] $N = 40$
0.1	$1.5265E-5$	$1.8818E-5$
0.2	$2.0744E-5$	$2.6480E-5$
0.3	$2.4945E-5$	$3.0188E-5$
0.4	$2.7769E-5$	$3.1915E-5$
0.5	$2.9610E-5$	$3.2470E-5$
0.6	$3.08267E-5$	$3.2421E-5$
0.7	$3.1628E-5$	$3.2001E-5$
0.8	$3.2145E-5$	$3.1393E-5$
0.9	$3.2466E-5$	$3.0699E-5$
1	$3.2649E-5$	$2.9994E-5$

Table 5: l_∞ error of Example 4 at $\delta t = 10^{-3}$.

t	l_∞ error $k = 10, \omega = 5, N = 15$	l_∞ error $k = 10, \omega = 50, N = 85$
0.2	$3.20E-06$	$1.66E-05$
0.4	$2.62E-06$	$1.38E-05$
0.6	$2.15E-06$	$1.13E-05$
0.8	$1.76E-06$	$9.27E-06$
1	$1.43E-06$	$7.60E-06$

and with $\alpha = 0$, $\beta = -1$, $k(x,t) = e^{-xt}$, and $g(x,t)$ which can be chosen such that the exact solution in this example is $u(x,t) = e^{-t}e^{-kx}\sin(\omega\pi x)$.

The maximum absolute error is tabulated in Table 5 for Chebyshev polynomial at different values of k and ω. The graphs of exact and approximate solutions for $t = 1$, at $k = 10$, $\omega = 5$, and $k = 10$, $\omega = 50$ are illustrated in Figures 4 and 5, respectively.

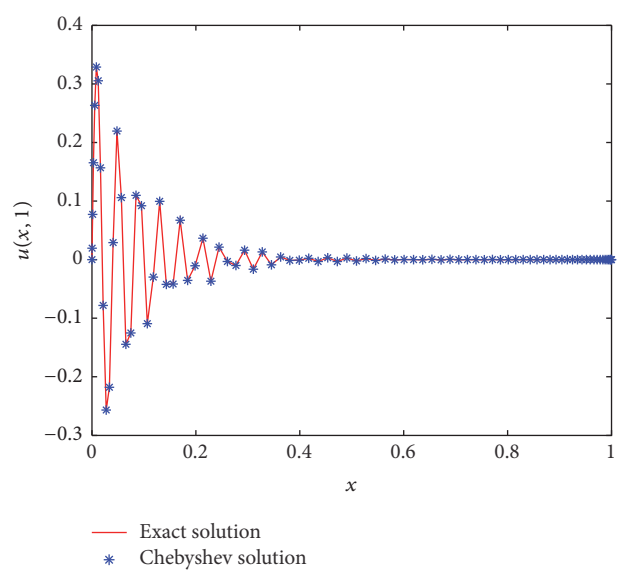

Figure 5: Exact and Chebyshev approximated solutions of Example 4 at $t = 1$, $k = 10$, $\omega = 50$.

6. Conclusions

Chebyshev collocation method is successfully used for solving parabolic partial integrodifferential equation. This method reduced the considered problem into linear system of algebraic equations that can be solved successively to obtain a numerical solution at varied time levels. Numerical examples show that the results of above scheme are in good agreement with the exact ones. Comparisons with the results obtained by using radial basis functions and backward-Euler scheme show that the Chebyshev method yields good results with fewer number of iterations. The method also renders good results for problems where solution exhibits fast changes but

with larger number of iterations as anticipated by the error estimate. Moreover, the above scheme can be developed to solve nonlinear parabolic partial integrodifferential equation.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

References

[1] M. Dehghan and F. Shakeri, "Solution of parabolic integro-differential equations arising in heat conduction in materials with memory via He's variational iteration technique," *International Journal for Numerical Methods in Biomedical Engineering*, vol. 26, no. 6, pp. 705–715, 2010.

[2] G. J. Habetler and R. L. Schiffman, "A finite difference method for analyzing the compression of poro-viscoelastic media," *Computing*, vol. 6, pp. 342–348, 1970.

[3] H. Engler, "On some parabolic integro-differential equations: existence and asymptotics of solutions," in *Equadiff 82*, H. W. Knobloch and K. Schmitt, Eds., vol. 1017 of *Lecture Notes in Mathematics*, pp. 161–167, 1983.

[4] C. V. Pao, "Solution of a nonlinear integrodifferential system arising in nuclear reactor dynamics," *Journal of Mathematical Analysis and Applications*, vol. 48, pp. 470–492, 1974.

[5] F. Fakhar-Izadi and M. Dehghan, "The spectral methods for parabolic Volterra integro-differential equations," *Journal of Computational and Applied Mathematics*, vol. 235, no. 14, pp. 4032–4046, 2011.

[6] C. E. Greenwell-Yanik and G. Fairweather, "Analyses of spline collocation methods for parabolic and hyperbolic problems in two space variables," *SIAM Journal on Numerical Analysis*, vol. 23, no. 2, pp. 282–296, 1986.

[7] J.-P. Kauthen, "The method of lines for parabolic partial integro-differential equations," *Journal of Integral Equations and Applications*, vol. 4, no. 1, pp. 69–81, 1992.

[8] S. Larsson, V. Thomée, and L. B. Wahlbin, "Numerical solution of parabolic integro-differential equations by the discontinuous Galerkin method," *Mathematics of Computation*, vol. 67, no. 221, pp. 45–71, 1998.

[9] J. Ma, "Finite element methods for partial Volterra integro-differential equations on two-dimensional unbounded spatial domains," *Applied Mathematics and Computation*, vol. 186, no. 1, pp. 598–609, 2007.

[10] Z. Avazzadeh, Z. Beygi Rizi, F. M. Maalek Ghaini, and G. B. Loghmani, "A numerical solution of nonlinear parabolic-type Volterra partial integro-differential equations using radial basis functions," *Engineering Analysis with Boundary Elements*, vol. 36, no. 5, pp. 881–893, 2012.

[11] A. F. Soliman, A. M. El-asyed, and M. S. El-Azab, "On the numerical solution of partial integro-differential equations," *Mathematical Sciences Letters*, vol. 1, no. 1, pp. 71–80, 2012.

[12] A. Filiz, "Numerical Solution of Parabolic Volterra integro-differential equations via backward Euler scheme," *Journal of Computational and Applied Mathematics*, vol. 3, no. 6, pp. 277–282, 2013.

[13] M. Sezer and M. Kaynak, "Chebyshev polynomial solutions of linear differential equations," *International Journal of Mathematical Education in Science and Technology*, vol. 27, no. 4, pp. 607–611, 1996.

[14] A. Akyüz and M. Sezer, "Chebyshev polynomial solutions of systems of high-order linear differential equations with variable coefficients," *Applied Mathematics and Computation*, vol. 144, no. 2-3, pp. 237–247, 2003.

[15] A. Akyüz and M. Sezer, "A Chebyshev collocation method for the solution of linear integro-differential equations," *International Journal of Computer Mathematics*, vol. 72, no. 4, pp. 491–507, 1999.

[16] M. El-Gamel and M. Sameeh, "An efficient technique for finding the eigenvalues of fourth-order Sturm-Liouville problems," *Applied Mathematics*, vol. 3, no. 8, pp. 920–925, 2012.

[17] M. El-Gamel and M. Sameeh, "A Chebyshev collocation method for solving Troesch's problem," *International Journal of Mathematics and Computer Applications Research*, vol. 3, pp. 23–32, 2013.

[18] A. Akyüz-Daşcıoğlu and H. Çerdik-Yaslan, "The solution of high-order nonlinear ordinary differential equations by Chebyshev series," *Applied Mathematics and Computation*, vol. 217, no. 12, pp. 5658–5666, 2011.

[19] C. Keşan, "Chebyshev polynomial solutions of second-order linear partial differential equations," *Applied Mathematics and Computation*, vol. 134, no. 1, pp. 109–124, 2003.

[20] G. Yuksel, O. R. Isik, and M. Sezer, "Error analysis of the Chebyshev collocation method for linear second-order partial differential equations," *International Journal of Computer Mathematics*, vol. 92, no. 10, pp. 2121–2138, 2015.

[21] H. Çerdk-Yaslan and A. Akyüz-Yaslan, "Chebyshev polynomial solution of nonlinear Fredholm Volterra integro-differential equations," *Journal of Science and Arts*, vol. 6, pp. 89–101, 2006.

[22] A. Quarteroni, R. Sacco, and F. Saleri, *Numerical Mathematics*, vol. 37 of *Texts in Applied Mathematics*, Springer, Berlin, Germany, 2nd edition, 2007.