Research Article

A Comparative Approach to the Solution of the Zabolotskaya-Khokhlov Equation by Iteration Methods

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We employed different iteration methods like Homotopy Analysis Method (HAM), Adomian Decomposition Method (ADM), and Variational Iteration Method (VIM) to find the approximate solution to the Zabolotskaya-Khokhlov (ZK) equation. Iteration methods are used to solve linear and nonlinear PDEs whose classical methods are either very complex or too limited to apply. A comparison study has been made to see which of these methods converges to the approximate solution rapidly. The result revealed that, amongst these methods, ADM is more effective and simpler tool in its nature which does not require any transformation or linearization.

1. Introduction

In science, physics, and engineering, many problems are modeled by the means of nonlinear evolutions equations especially in plasma physics, fluid dynamics, biology, and nonlinear acoustics. In this paper, we consider one such nonlinear partial differential equation which is $(1 + 1)$ dimensional Zabolotskaya-Khokhlov (ZK) equation (ZK for short)

$$u_{xt} = uu_{xx} + \alpha (u_x)^2. \quad (\text{A})$$

This equation is one of the basic equations in nonlinear acoustic and nonlinear wave theory. It is named after two Russian mathematicians R. V. Khokhlov and E. A. Zabolotskaya who derived it for the first time almost four decades ago and presented an approximate solution which described some interesting features of nonlinear wave beams [1]. In recent years, an extensive research is being made in nonlinear acoustic due to the development of new medical devices for nonlinear diagnostic ultrasound imaging, for noninvasive destruction of tumors in acoustic surgery and for kidney stone comminution. All of the above applications rely on the focusing acoustic waves in nonlinear medium which can be studied by the ZK equation [2]. The ZK equation describes the propagation of a confined wave beam or sound beam in nonlinear medium without dispersion or absorption

and it investigates the beam deformation associated with the properties of the nonlinear medium [3, 4]. This equation has applications in many fields of life such as it is used to simulate the estimation of fish stock abundance, discrimination between fish species, the effect of excess attenuation which occurs due to the nonlinear sound propagation in water, absorption, and diffraction in the focused sound beams [5–8].

The solution of the ZK equation will shed light on some new features in the behavior of nonlinear beam. Many authors [9, 10] have found the infinitesimal symmetries and exact solution of the ZK equation by means of Lie Algebras, but the analytical or approximate solution of the ZK equation in the form of an infinite series has not been obtained before. Keeping this limitation in our mind, in this work, we will find the analytical solution of the ZK equation in the form of an infinite series with the help of iteration methods. This analytical solution would be extremely useful in the areas of theoretical physics, applied mathematics, and engineering as it will express the acoustic pressure of the beam in terms of propagation coordinate and time explicitly.

As we know that many physical problems are modeled by means of partial differential equations (PDEs) like Helmholtz equation, heat equation, wave equation, gas equation, and so forth [11–15]. Of these PDEs, some are linear, but many

are quasi-linear or nonlinear [16] whose analytical solution techniques are limited to apply. There is no general method of solving such PDEs; however an extensive research is being made to find exact or approximate solutions of nonlinear partial differential equations with the aid of iteration methods like variational principal [17], decomposition solution [18, 19], and Adomian Decomposition Method [20–23]. These iteration methods are analytical tool that approximate the solution of such PDEs in terms of an infinite series which converges rapidly to the exact solution of the PDE in just few iterations. The accuracy and the speed of convergence of the resulting series are surprisingly high and one needs only a few iterations to approximate the exact solution as shown by the example given in Section 4.

Consider the initial and boundary conditions for (A)

$$\begin{aligned} u_x(x, 0) &= f(x), \\ u(0, t) &= g(t), \end{aligned} \quad (1)$$

where $f(x)$ and $g(t)$ are two known functions.

Integrating (A) twice and applying the initial conditions (1), we can write

$$\begin{aligned} u(x, t) &= F(x, t) + \int_0^x \int_0^t F_1(u(\zeta, \tau)) d\tau d\zeta \\ &+ \alpha \int_0^x \int_0^t F_2(u(\zeta, \tau)) d\tau d\zeta, \end{aligned} \quad (2)$$

where

$$\begin{aligned} F(x, t) &= g(t) + \int_0^x f(\zeta) d\zeta, \\ F_1(u(x, t)) &= u(x, t) \frac{\partial^2 u(x, t)}{\partial x^2}, \\ F_2(u(x, t)) &= \left(\frac{\partial u(x, t)}{\partial x} \right)^2. \end{aligned} \quad (3)$$

In (2), we assume that $F(x, t)$ is bounded function for all t in $[0, T]$ and x in $[a, b]$, where $T, a, b \in \mathbb{R}$. The functions $F_1(u(x, t))$ and $F_2(u(x, t))$ are Lipschitz continuous such that

$$\begin{aligned} |F_i(u) - F_i(u^*)| &\leq K_i |u - u^*|, \quad i = 1, 2, \\ k_1 &= T(b-a)(\alpha K_1 + K_2), \\ k_2 &= 1 - T(1 - k_1). \end{aligned} \quad (4)$$

2. Existence and Uniqueness

In this section, we will analyze uniqueness and existence of the solution to the ZK equation.

Theorem 1. Equation (A) has a unique solution for $0 < k_1 < 1$.

Proof. We suppose that u and u^* are two solutions of (2), and then

$$\begin{aligned} |u - u^*| &= \left| \left(F(x, t) + \int_0^x \int_0^t F_1(u(\zeta, \tau)) d\tau d\zeta \right. \right. \\ &+ \alpha \int_0^x \int_0^t F_2(u(\zeta, \tau)) d\tau d\zeta \Big) - \left(F(x, t) \right. \\ &+ \int_0^x \int_0^t F_1(u^*(\zeta, \tau)) d\tau d\zeta \\ &+ \alpha \int_0^x \int_0^t F_2(u^*(\zeta, \tau)) d\tau d\zeta \Big) \Big|, \\ |u - u^*| &\leq \int_0^x \int_0^t |F_1(u(\zeta, \tau)) - F_1(u^*(\zeta, \tau))| d\tau d\zeta \\ &+ \alpha \int_0^x \int_0^t |F_2(u(\zeta, \tau)) - F_2(u^*(\zeta, \tau))| d\tau d\zeta \\ &- \int_0^x \int_0^t |F_1(u^*(\zeta, \tau))| d\tau d\zeta \\ &- \alpha \int_0^x \int_0^t |F_2(u^*(\zeta, \tau))| d\tau d\zeta, \\ |u - u^*| &\leq \int_0^x \int_0^t |F_1(u(\zeta, \tau)) - F_1(u^*(\zeta, \tau))| d\tau d\zeta \\ &+ \alpha \int_0^x \int_0^t |F_2(u(\zeta, \tau)) - F_2(u^*(\zeta, \tau))| d\tau d\zeta, \\ |u - u^*| &\leq (K_1 + K_2\alpha) T(b-a) |u(\zeta, \tau) - u^*(\zeta, \tau)|, \\ |u - u^*| &\leq k_1 |u - u^*|, \\ (1 - k_1) |u - u^*| &\leq 0. \end{aligned} \quad (5)$$

Since $0 < k_1 < 1$, $|u - u^*| = 0 \Rightarrow u = u^*$ which proves that (A) has a unique solution. $\square$

3. Analysis of the Methods

In this section, we will introduce the iteration methods which we will apply on the ZK equation. We will apply Homotopy Analysis Method (HAM), Adomian Decomposition Method (ADM), and Variational Iteration Method (VIM) to solve the said equation.

3.1. Homotopy Analysis Method (HAM). To illustrate the basic idea of the Homotopy Analysis Method, we consider the following nonlinear differential equation:

$$A(u(x, t)) = 0, \quad (6)$$

where A is any differential or integral operator and $u(x, t)$ is the unknown function. Using Homotopy Analysis Method, we can construct the so-called zeroth-order deformation equation

$$\begin{aligned} (1 - p) [L(\phi(x, t, p)) - L(u_0(x, t))] \\ = phH(x, t) N(\phi(x, t, p)), \end{aligned} \quad (7)$$

where $u_0(x, t)$ is an initial guess to the exact solution $u(x, t)$ of the given differential equation and $p \in [0, 1]$ is the embedding parameter, whereas $h \neq 0$ is an auxiliary parameter, L is an auxiliary linear operator, $H(x, t)$ is an auxiliary function, and $\phi(x, t, p)$ is the unknown function. Observe that when $p = 0$, relation (7) turns out to be $\phi(x, t, 0) = u_0(x, t)$, and when $p = 1$ it becomes $\phi(x, t, 1) = u(x, t)$. So, as the embedding parameter p changes from 0 to 1, we see that $\phi(x, t, p)$ continuously changes from the initial condition $u_0(x, t)$ to the exact solution $u(x, t)$ of the differential equations.

Using Taylor theorem, we can express $\phi(x, t, p)$ as a power series of p

$$\phi(x, t, p) = u_0(x, t) + \sum_{k=1}^{\infty} p^k u_k, \quad (8)$$

where

$$u_k(x, t) = \frac{1}{k!} \left. \frac{\partial^k \phi(x, t, p)}{\partial p^k} \right|_{p=0}. \quad (9)$$

It is obvious that the convergence of the above relation depends mainly on the auxiliary parameter h . With the proper choice of the initial guess $u_0(x, t)$, the power series (8) converges to the exact solution at $p = 1$

$$u(x, t) = \phi(x, t, 1) = u_0(x, t) + \sum_{k=1}^{\infty} u_k. \quad (10)$$

Next, we substitute (8) into (7) to get

$$(1-p)L \left[\sum_{k=1}^{\infty} p^k u_k \right] = pH(x, t) N(\phi(x, t, p)). \quad (11)$$

Differentiating (11) k times with respect to p and dividing both sides by $k!$ and finally applying $p = 0$, we can write

$$\begin{aligned} & \frac{\partial^k}{\partial p^k} \left((1-p)L \left[\sum_{k=1}^{\infty} p^k u_k \right] \right) \\ &= \frac{\partial^k}{\partial p^k} [pH(x, t) N(\phi(x, t, p))], \\ & \frac{\partial^k}{\partial p^k} \left(L \left[\sum_{k=1}^{\infty} p^k u_k \right] \right) - \frac{\partial^k}{\partial p^k} \left(pL \left[\sum_{k=1}^{\infty} p^k u_k \right] \right) \\ &= \frac{\partial^k}{\partial p^k} [pH(x, t) N(\phi(x, t, p))], \end{aligned} \quad (12)$$

$$\begin{aligned} & L[u_k(x, t) - \chi_k u_{k-1}(x, t)] \\ &= hH(x, t) R_k[u_{k-1}(x, t)], \end{aligned}$$

where

$$\chi_k = \begin{cases} 0, & k \leq 1 \\ 1, & k > 1, \end{cases} \quad (13)$$

$$R_k[u_{k-1}(x, t)] = \frac{1}{(k-1)!} \left. \frac{\partial^{k-1} N[\phi(x, t, p)]}{\partial p^{k-1}} \right|_{p=0}.$$

The higher order deformation equation (12) governs L and so $R_k[u_{k-1}(x, t)]$ is valid for any nonlinear operator N . So, we can let

$$\begin{aligned} N[\phi(x, t)] &= u(x, t) - F(x, t) \\ &- \int_0^x \int_0^t F_1(u_k(\zeta, \tau)) d\tau d\zeta \\ &- \alpha \int_0^x \int_0^t F_2(u_k(\zeta, \tau)) d\tau d\zeta \end{aligned} \quad (14)$$

which then becomes

$$\begin{aligned} R_k[u_{k-1}(x, t)] &= u_{k-1}(x, t) - F(x, t) \\ &- \int_0^x \int_0^t F_1(u_k(\zeta, \tau)) d\tau d\zeta \\ &- \alpha \int_0^x \int_0^t F_2(u_k(\zeta, \tau)) d\tau d\zeta. \end{aligned} \quad (15)$$

Substituting (15) into (12), we have

$$\begin{aligned} L[u_k(x, t) - \chi_k u_{k-1}(x, t)] &= hH(x, t) \left(u_{k-1}(x, t) \right. \\ &- F(x, t) - \int_0^x \int_0^t F_1(u_k(\zeta, \tau)) d\tau d\zeta \\ &- \alpha \int_0^x \int_0^t F_2(u_k(\zeta, \tau)) d\tau d\zeta \left. \right). \end{aligned} \quad (16)$$

Now, taking $F(x, t)$ as initial guess $u_0(x, t)$, $h = -1$, $\alpha = 1$, and $H(x, t) = 1$, together with using the value of χ_k , we can write the recurrence relation from the last equation as follows:

$$\begin{aligned} u_0(x, t) &= F(x, t), \\ u_{k+1}(x, t) &= \int_0^x \int_0^t F_1(u_k(\zeta, \tau)) d\tau d\zeta \\ &+ \int_0^x \int_0^t F_2(u_k(\zeta, \tau)) d\tau d\zeta \quad k \geq 1. \end{aligned} \quad (17)$$

The set of equations in (17) can be solved by simple integration. The resulting components u_i , $i = 1, 2, \dots$ will then be substituted in (8) with p as unity, which will lead towards the solution of (A) in the form of an infinite series whose convergence is sure in most cases in its own domain.

As we see, the initial and boundary conditions play vital role in determining the series solution and different set of conditions will result in different series solutions. Though we do not have general criteria of imposing restrictions at these conditions at the start, one should be careful in choosing initial conditions.

3.2. Adomian Decomposition Method (ADM). Adomian Decomposition Method was first introduced by Adomian in 1984 [18, 19]. Like HAM, in this method, we split the nonlinear partial differential equation into two parts, namely, linear and nonlinear. Applying the inverse linear operator on both sides

of the equation and decomposing the unknown function in the form of a series, we get a solution. In ADM, we also decompose the nonlinear terms in the form of Adomian polynomials and use a special formula to find these unknown terms due to which, no doubt, we need to do extra calculation than we do in HAM.

Consider the following equation:

$$A(u(x, t)) = 0. \quad (18)$$

Equation (18) can be split into the following form:

$$L(u(x, t)) + N(u(x, t)) = 0, \quad (19)$$

where L and N are linear and nonlinear differential operators, respectively. Consider that the inverse operator L^{-1} exists, and applying it on (19), we get

$$u(x, t) = u_0(x, t) - L^{-1}(N[u(x, t)]), \quad (20)$$

where $u_0(x, t)$ is the initial guess of the equation.

By the Adomian Decomposition Method [16], we can define the unknown function

$$u = \sum_{k=0}^{\infty} u_k. \quad (21)$$

The nonlinear term of the equation is given by the formula [18]

$$N = \sum_{n=0}^{\infty} A_n, \quad n \geq 0, \quad (22)$$

where $A_n, n \geq 0$ are the Adomian polynomials which can be retrieved by the formula

$$A_n(u_0, u_1, \dots, u_n) = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N \left(\sum_{i=0}^n \lambda^i u_i \right) \right]_{\lambda=0}. \quad (23)$$

Substituting (21) and (22) into (20), we can write

$$\sum_{k=0}^{\infty} u_k(x, t) = u_0(x, t) - L^{-1} \left(\sum_{n=0}^{\infty} A_n \right). \quad (24)$$

Relation (24) can be used to determine the solution of (18).

3.3. Variational Iteration Method (VIM). Under Variational Iteration Method, we break the given (6) into the following form:

$$N(u(x, t)) + L(u(x, t)) = 0, \quad (25)$$

where N and L are nonlinear and linear operators, respectively. According to VIM, a correctional functional can be constructed as follows:

$$\begin{aligned} u_{n+1}(x, t) &= u_n(x, t) \\ &+ \int_0^t \lambda(x, \tau) [N(u_n(x, \tau)) + L(u_n(x, \tau))] d\tau, \end{aligned} \quad (26)$$

$n \geq 0,$

where λ is the Lagrange multiplier which can be obtained optimally by using variation theory. The function $u_n(x, t)$ is the restricted variation; that is $\delta u_n(x, t) = 0$, which helps us to determine the Lagrange multiplier together with integration by parts. The exact solution may be obtained by

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t). \quad (27)$$

Applying (26) on (2), we can write

$$\begin{aligned} u_{n+1}(x, t) &= u_n(x, t) + L^{-1} \left[\lambda \left\{ u_n(x, t) - F(x, t) \right. \right. \\ &\quad \left. \left. - \int_0^x \int_0^t F_1(u_n(\zeta, \tau)) d\tau d\zeta \right. \right. \\ &\quad \left. \left. - \alpha \int_0^x \int_0^t F_2(u_n(\zeta, \tau)) d\tau d\zeta \right\} \right], \quad n \geq 0, \end{aligned} \quad (28)$$

where $L^{-1}[\cdot] = \int_0^t [\cdot] d\tau$.

Next we find the value of Lagrange multiplier by taking variation on (28)

$$\begin{aligned} \delta u_{n+1}(x, t) &= \delta u_n(x, t) + \delta L^{-1} \left[\lambda \left\{ u_n(x, t) - F(x, t) \right. \right. \\ &\quad \left. \left. - \int_0^x \int_0^t F_1(u_n(\zeta, \tau)) d\tau d\zeta \right. \right. \\ &\quad \left. \left. - \alpha \int_0^x \int_0^t F_2(u_n(\zeta, \tau)) d\tau d\zeta \right\} \right], \quad n \geq 0. \end{aligned} \quad (29)$$

From (29), due to stationary conditions, we get, after a bit of calculations,

$$\lambda' = 0 \quad \text{or} \quad (30)$$

$$1 + \lambda = 0$$

which means that the Lagrange multiplier is -1 , so from (28) we have

$$\begin{aligned} u_{n+1}(x, t) &= u_n(x, t) - L^{-1} \left[u_n(x, t) - F(x, t) \right. \\ &\quad \left. - \int_0^x \int_0^t F_1(u_n(\zeta, \tau)) d\tau d\zeta \right. \\ &\quad \left. - \alpha \int_0^x \int_0^t F_2(u_n(\zeta, \tau)) d\tau d\zeta \right], \quad n \geq 0. \end{aligned} \quad (31)$$

We can find the successive values of $u_n(x, t)$ from relation (31).

3.4. Theorem. The series solution $u(x, t) = \sum_{k=0}^{\infty} u_k(x, t)$ using VIM converges with $0 < k_1 < 1$, $0 < k_2 < 1$.

Proof. Consider

$$\begin{aligned} u_{n+1}(x, t) &= u_n(x, t) - L^{-1} \left[u_n(x, t) - F(x, t) \right. \\ &\quad \left. - \int_0^x \int_0^t F_1(u_n(\zeta, \tau)) d\tau d\zeta \right. \\ &\quad \left. - \alpha \int_0^x \int_0^t F_2(u_n(\zeta, \tau)) d\tau d\zeta \right]. \end{aligned} \quad (32)$$

Removing the subscripts, we can again write

$$\begin{aligned} u(x, t) = & u(x, t) - L^{-1} \left[u(x, t) - F(x, t) \right. \\ & - \int_0^x \int_0^t F_1(u(\zeta, \tau)) d\tau d\zeta \\ & \left. - \alpha \int_0^x \int_0^t F_2(u(\zeta, \tau)) d\tau d\zeta \right]. \end{aligned} \quad (33)$$

Subtracting (33) from (32), we get

$$\begin{aligned} u_{n+1}(x, t) - u(x, t) = & u_n(x, t) - u(x, t) \\ & - L^{-1} \left[u_n(x, t) - u(x, t) \right. \\ & - \int_0^x \int_0^t [F_1(u_n(\zeta, \tau)) - F_1(u(\zeta, \tau))] d\tau d\zeta \\ & \left. - \alpha \int_0^x \int_0^t [F_2(u_n(\zeta, \tau)) - F_2(u(\zeta, \tau))] d\tau d\zeta \right]. \end{aligned} \quad (34)$$

Now, let

$$\begin{aligned} e_{n+1} &= u_{n+1}(x, t) - u(x, t), \\ e_n &= u_n(x, t) - u(x, t), \end{aligned} \quad (35)$$

$$|e_n(x, t^*)| = \max_t e_n(x, t).$$

Now, observe that e_n is decreasing with respect to t , so applying mean value theorem, we can write from (34) the following:

$$\begin{aligned} e_{n+1}(x, t) = & e_n(x, t) - L^{-1} \left[e_n(x, t) \right. \\ & - \int_0^x \int_0^t [F_1(u_n(\zeta, \tau)) - F_1(u(\zeta, \tau))] d\tau d\zeta \\ & \left. - \alpha \int_0^x \int_0^t [F_2(u_n(\zeta, \tau)) - F_2(u(\zeta, \tau))] d\tau d\zeta \right], \\ e_{n+1}(x, t) \leq & e_n(x, t) - L^{-1} [e_n(x, t) \\ & - L^{-1} |e_{n+1}(x, t)| T(b-a)(\alpha K_1 - K_2)], \end{aligned} \quad (36)$$

$$\begin{aligned} e_{n+1}(x, t) \leq & e_n(x, t) - [Te_n(x, \eta) \\ & - T(b-a)(\alpha K_1 - K_2)L^{-1}L^{-1}|e_{n+1}(x, t)|], \\ & 0 < \eta < 1, \end{aligned}$$

$$\begin{aligned} e_{n+1}(x, t) &\leq e_n(x, t) - T(1 - k_1)|e_{n+1}(x, t^*)|, \\ e_{n+1}(x, t) &\leq (1 - T(1 - k_1))|e_{n+1}(x, t^*)|, \\ e_{n+1}(x, t) &\leq k_2|e_{n+1}(x, t^*)|. \end{aligned}$$

Therefore,

$$\begin{aligned} \|e_{n+1}(x, t)\| &= \max |e_{n+1}(x, t)| \leq k_2 \max \|e_n(x, t^*)\| \\ &\leq k_2 \|e_n(x, t^*)\|. \end{aligned} \quad (37)$$

Since $0 < k_2 < 1$, $\|e_n(x, t^*)\| \rightarrow 0$, so the series converges and hence the proof. $\square$

4. Application and Results

In this section, we solve the ZK equation by the above-mentioned methods. We also compare the results obtained to see which method is more sufficient and converges rapidly to the exact solution.

4.1. Example. Consider ZK equation (A) with $\alpha = 1$ together with initial and boundary conditions given as follows:

$$\begin{aligned} u_{xt} &= uu_{xx} + (u_x)^2, \\ f(x) &= 2x, \quad g(t) = 0. \end{aligned} \quad (38)$$

Here $u = u(x, t)$ is the unknown function to be determined which represents the acoustic pressure and x is the propagation coordinate. The term u_{xt} denotes the time evolution of the wave, while the nonlinear terms uu_{xx} and u_x^2 describe the steepening of the wave.

Using HAM. We apply HAM on (38) to get the following iteration scheme due to (17):

$$\begin{aligned} u_0(x, t) &= F(x, t) = g(t) + \int_0^x f(\zeta) d\zeta \\ &= 0 + \int_0^x 2\zeta d\zeta = x^2, \end{aligned} \quad (39)$$

$$\begin{aligned} u_{k+1}(x, t) = & + \int_0^x \int_0^t F_1(u_k(\zeta, \tau)) d\tau d\zeta \\ & + \int_0^x \int_0^t F_2(u_k(\zeta, \tau)) d\tau d\zeta \quad k \geq 1. \end{aligned}$$

Using the above relation, we can find the other components as follows:

$$\begin{aligned} u_1(x, t) &= 2x^3t, \\ u_2(x, t) &= 4x^4t^2, \\ u_3(x, t) &= \frac{80}{7}x^9t^7, \\ u_4(x, t) &= \frac{3840}{49}x^{17}t^{15}. \end{aligned} \quad (40)$$

Substituting these values in (8) and making p unity, we get the solution of (38) in the form of the following infinite series:

$$\begin{aligned} u(x, t) &= \sum_{n=0}^{\infty} u_n \\ &= x^2 + 2x^3t + 4x^4t^2 + \frac{80}{7}x^9t^7 \\ &\quad + \frac{3840}{49}x^{17}t^{15} \dots \end{aligned} \quad (41)$$

This represents the acoustic pressure of the beam in terms of propagation coordinate and time.

Using ADM. Let $L = \partial^2 / \partial x \partial t$ and $N = uu_{xx} + (u_x)^2$ so that (38) can be written as

$$L[u] = N[u]. \quad (42)$$

Consider that the inverse operator L^{-1} exists and is defined as

$$L^{-1}[\cdot] = \int_0^x \int_0^t [\cdot] dt dx. \quad (43)$$

Applying L^{-1} on (42) and using the initial conditions, we get

$$u(x, t) = x^2 + L^{-1}(N[u]). \quad (44)$$

By the Adomian Decomposition Method, we can define the unknown function

$$u = \sum_{k=0}^{\infty} u_k. \quad (45)$$

The nonlinear term of the equation is given by the formula

$$N = \sum_{n=0}^{\infty} A_n, \quad n \geq 0, \quad (46)$$

where $A_n, n \geq 0$ are the Adomian polynomials which can be retrieved by the formula

$$A_n(u_0, u_1, \dots, u_n) = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N \left(\sum_{i=0}^n \lambda^i u_i \right) \right]_{\lambda=0}. \quad (47)$$

Substituting (45) and (46) into (44), we can write

$$\sum_{k=0}^{\infty} u_k(x, t) = x^2 + L^{-1} \left(\sum_{n=0}^{\infty} A_n \right). \quad (48)$$

From (48), we get the recurrence relation as

$$\begin{aligned} u_0(x, t) &= x^2, \\ u_{n+1}(x, t) &= L^{-1}(A_n). \end{aligned} \quad (49)$$

Now, using formula (47), we can find the first few Adomian polynomials as

$$\begin{aligned} A_0 &= 6x^2, \\ A_1 &= 40x^3t, \\ A_2 &= 210x^4t^2, \\ A_3 &= 1008x^5t^3. \end{aligned} \quad (50)$$

Applying the recurrence relation (49) together with the inverse operator (43) on the set of equations given in (50) to

get the components of $u(x, t)$ and inserting these components in (45), we get the solution to (38) as

$$\begin{aligned} u_0 &= x^2, \\ u_1(x, t) &= 2x^3t, \\ u_2(x, t) &= 5x^4t^2, \\ u_3(x, t) &= 14x^5t^3, \\ u_4(x, t) &= 42x^6t^4. \end{aligned} \quad (51)$$

Substituting these values in (8) and making p unity, we get the series

$$\begin{aligned} u(x, t) &= \sum_{n=0}^{\infty} u_n \\ &= x^2 + 2x^3t + 5x^4t^2 + 14x^5t^3 + 42x^6t^4 + \dots \end{aligned} \quad (52)$$

The exact solution to (38) can be written in a closed form as follows:

$$u(x, t) = \sum_{n=1}^{\infty} \frac{(2n)!}{n!(n+1)!} x^{n+1} t^{n-1} = \frac{4x^2}{(1 + \sqrt{1 - 4xt})^2}. \quad (53)$$

Using VIM. Using VIM, we can find the recurrence relation given as follows:

$$\begin{aligned} u_{n+1}(x, t) &= u_n(x, t) - L^{-1} \left[u_n(x, t) - F(x, t) \right. \\ &\quad \left. - \int_0^x \int_0^t F_1(u_n(\zeta, \tau)) d\tau d\zeta \right. \\ &\quad \left. - \alpha \int_0^x \int_0^t F_2(u_n(\zeta, \tau)) d\tau d\zeta \right], \quad n \geq 0. \end{aligned} \quad (54)$$

Since

$$\begin{aligned} u_0(x, t) &= F(x, t) = g(t) + \int_0^x f(\zeta) d\zeta \\ &= 0 + \int_0^x 2\zeta d\zeta = x^2, \end{aligned} \quad (55)$$

we can do few more iterations and have the successive components of $u_{n+1}(x, t)$ as follows:

$$\begin{aligned} u_1(x, t) &= x^2 + x^3t^2, \\ u_2(x, t) &= x^2 + \left(2t^2 - \frac{1}{3}t^3 \right) x^3 + \frac{5}{12}x^4t^4 + \frac{1}{10}x^5t^6, \\ u_3(x, t) &= x^2 + \left(3t^2 - t^3 + \frac{1}{12}t^4 \right) x^3 \\ &\quad + \left(\frac{5}{4}t^4 - \frac{1}{6}t^5 \right) x^4 \end{aligned}$$

$$\begin{aligned}
& + \left(\frac{7}{12}t^6 - \frac{23}{210}t^7 + \frac{1}{168}t^8 \right) x^5 \\
& - \frac{35}{2592}x^6t^9 \\
& + \left(\frac{413}{16200}t^{10} - \frac{2}{825}t^{11} \right) x^7 \\
& + \left(\frac{7}{60}t^6 + \frac{2}{352}t^{12} \right) x^8 + \frac{1}{3640}x^9t^{14}.
\end{aligned} \tag{56}$$

Consequently, the third-order approximate solution of (38) using VIM is given by

$$\begin{aligned}
u(x, t) & \simeq u_3(x, t) \\
& = x^2 + \left(3t^2 - t^3 + \frac{1}{12}t^4 \right) x^3 \\
& + \left(\frac{5}{4}t^4 - \frac{1}{6}t^5 \right) x^4 \\
& + \left(\frac{7}{12}t^6 - \frac{23}{210}t^7 + \frac{1}{168}t^8 \right) x^5 - \frac{35}{2592}x^6t^9 \\
& + \left(\frac{413}{16200}t^{10} - \frac{2}{825}t^{11} \right) x^7 \\
& + \left(\frac{7}{60}t^6 + \frac{2}{352}t^{12} \right) x^8 + \frac{1}{3640}x^9t^{14}.
\end{aligned} \tag{57}$$

Table 1 shows the comparison between the approximate solutions (41), (51), and (57) and the exact solution (53) of (38). From the table, we see that the approximate solution obtained by ADM is closer to the exact solution than others. After ADM, the approximate solution by HAM is better than that of VIM. Since we take the third-order solution from VIM, it is suggested that higher order solution obtained by VIM may rapidly lead to convergence towards the solution. If we look at the table carefully, we will see that these approximate solutions are close to the exact one for the values $x = 0.1$ to 0.3 and after this the approximate solutions seem to be moving away to the exact solution. Since the exact solution of the equation given in (53) has the domain $1 - 4xt > 0 \Rightarrow xt < 0.25$ and by fixing $t = 0.2$, we get that the restriction on x is $x < 1.25$. So, as the values of x are getting close to 1.25 , the solution is diverging away.

Table 2 shows the absolute error of these solutions at different values of x . Again, we see that the approximate solution of ADM has less percentage error than others. The percentage error is increasing as x is getting close to 1.25 due to the domain of x .

In Figure 1, the graph shows the approximate solutions by the iterations methods together with the exact solution (purple line). We see that the approximate solution by ADM (red line) is closer to the exact solution followed by HAM

TABLE 1: The values of $u(x, t)$ at $t = 0.2$, $x = 0.1-0.9$.

	u_{HAM}	u_{ADM}	u_{VIM}	Exact
$x = 0.1$	0.01040032	0.01042119	0.01011233	0.01042119
$x = 0.2$	0.04321024	0.04356014	0.04090019	0.04356076
$x = 0.3$	0.10087776	0.10274115	0.09304346	0.10275264
$x = 0.4$	0.18592772	0.19214213	0.16722674	0.19223594
$x = 0.5$	0.30100029	0.31705000	0.26413946	0.31754163
$x = 0.6$	0.44888979	0.4841644	0.3844759	0.48612181
$x = 0.7$	0.63258414	0.70194985	0.5289352	0.70843802
$x = 0.8$	0.85530539	0.98103624	0.69822148	1.00000000
$x = 0.9$	1.12055235	1.33466772	0.89304379	1.38562172

TABLE 2: The absolute error of $u(x, t)$ at $t = 0.2$, $x = 0.1-0.9$.

	u_{HAM}	u_{ADM}	u_{VIM}
$x = 0.1$	0.00002087171	0.00000000451	0.00030886334
$x = 0.2$	0.00035052253	0.00000062180	0.00266056974
$x = 0.3$	0.00187487820	0.00001149230	0.00029081445
$x = 0.4$	0.00187480000	0.00001150000	0.00970910000
$x = 0.5$	0.00630820000	0.00009380000	0.02500920000
$x = 0.6$	0.01654130000	0.00049160000	0.05340210000
$x = 0.7$	0.03723200000	0.00195740000	0.10164590000
$x = 0.8$	0.07585390000	0.00648810000	0.17950280000
$x = 0.9$	0.14469460000	0.01896380000	0.30177850000

FIGURE 1: The exact solution is in purple line, HAM is in blue, ADM is in red, and VIM is in green.

(blue line) and VIM (green line). In Figure 2, the error plot is given and we again see that the absolute error by ADM (red line) is the smallest, while the absolute error of VIM (green line) is the highest.

5. Conclusion

A comparative study has been made to seek the semianalytical solution of the ZK equation with initial and boundary conditions. The solution of the ZK equation is obtained in the form of an infinite series that converges rapidly in its domain. This form of solution gives the acoustic pressure of the beam in terms of propagation coordinate and time more explicitly. Since such form of the solution has not been obtained before, we believe that this will open new ways to understand the propagation of the confined beam in the nonlinear medium. Different iteration methods like HAM, ADM, and VIM have

FIGURE 2: The error graph of HAM is in blue, ADM is in red, and VIM is in green.

been applied and compared. The result revealed that ADM generated surprisingly effective results.

Conflict of Interests

The authors declare that they there is no conflict of interests regarding the publication of this paper.

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