Research Article

Two Kinds of Darboux-Bäcklund Transformations for the q -Deformed KdV Hierarchy with Self-Consistent Sources

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Two kinds of Darboux-Bäcklund transformations (DBTs) are constructed for the q -deformed N th KdV hierarchy with self-consistent sources (q -NKdVHSCS) by using the q -deformed pseudodifferential operators. Note that one of the DBTs provides a nonauto Bäcklund transformation for two q -deformed N th KdV equations with self-consistent sources (q -NKdVHSCS) with different degree. In addition, the soliton solution to the first nontrivial equation of q -KdVHSCS is also obtained.

1. Introduction

The q -deformed integrable systems are regarded as the q -deformation of the related classical ones. The q -deformation is performed by using the q -derivative ∂_q to take the place of usual derivative ∂_x and it reduces to a classical integrable system as $q \rightarrow 1$. In recent years, some q -deformed integrable systems, especially the q -deformed N th KdV hierarchy (q -NKdVH) and the q -deformed KP hierarchy (q -KPH), have attracted much interest both in mathematics and in physics [1–17]. It was shown that q -NKdVH inherited some integrable structures from the classical N th KdV hierarchy, such as infinite conservation law [2], bi-Hamiltonian structure [3, 4], tau function [5, 6], Darboux-Bäcklund transformation [7], and q -Miura transformation [8]. In 1999, some elementary DBTs of the q -NKdVH (also called q -deformed Gelfand-Dickey hierarchy) were constructed by using the q -deformed pseudodifferential operators. The formula for the n -times repeated DBTs was also presented, which produces the new soliton solutions to the q -NKdVH [7]. For q -KPH, its bi-Hamiltonian structure, tau function, additional symmetries, q -effect in q -soliton, Virasoro constraints of tau function and integrable extension, and q -Bäcklund transformation were also explored in [9–17]. In 2008, based on the symmetry constraint for q -KPH, the new extension of this hierarchy

was considered [16]. Two kinds of reductions of this new extended q -KP hierarchy were also studied, which give many $1 + 1$ dimensional q -deformed soliton equations with self-consistent sources [16]. For example, the n -reduction $L_{<0}^n = 0$ gives q -NKdVHSCS. However, to our knowledge, the DBTs and the soliton solution for q -NKdVHSCS still remain unexplored. It is known that the DBT is an important property to characterize the integrability of the hierarchy. Thus, it is necessary for us to explore the DBT for q -NKdVHSCS. We think our research results will deepen our understanding on soliton solutions of this hierarchy.

The outline of this paper is as follows. In Section 2, some notations in the q -calculus and the definition of the q -NKdVHSCS are briefly reviewed. In Section 3, we aim at the construction of auto DBTs for q -NKdVHSCS. In Section 4, the nonauto DBTs for q -NKdVHSCS are constructed. In Section 5, one soliton solution to the first nontrivial equation of q -NKdVHSCS is obtained by using nonauto DBTs. Section 6 is devoted to a brief summary.

2. The q -Deformed N th KdV Hierarchy with Self-Consistent Sources (q -NKdVHSCS)

In this section, we briefly review some notations in the q -calculus and the definition of the q -NKdVHSCS.

The q -derivative operator ∂_q and q -shift operator θ are defined by

$$[\partial_q f(x)] = \frac{f(qx) - f(x)}{x(q-1)}, \quad (1)$$

$$\theta(f(x)) = f(qx). \quad (2)$$

In this paper, we introduce two notations: $[Pf]$ and $P \circ f = Pf$, in which P is a q -pseudo-differential operator (q -PDO) given by

$$P = \sum_{i=-\infty}^n p_i \partial_q^i. \quad (3)$$

$[Pf]$ denotes P acting on the function f , while Pf indicates the multiplication of P and f ; that is, $\partial_q f = \theta(f)\partial_q + [\partial_q f]$.

It can be easily shown from (1) that when $q \rightarrow 1$, ∂_q reduces to the ordinary differential operator ∂_x and that θ and ∂_q do not commute but satisfy

$$[\partial_q \theta^k(f)] = q^k \theta^k [\partial_q f], \quad k \in \mathbb{Z}. \quad (4)$$

Let ∂_q^{-1} be the formal inverse of ∂_q such as $\partial_q \partial_q^{-1} f = \partial_q^{-1} \partial_q f = f$. In general, the q -deformed Leibnitz rule holds

$$\partial_q^n f = \sum_{k \geq 0} \binom{n}{k}_q \theta^{n-k} (\partial_q^k f) \partial_q^{n-k}, \quad n \in \mathbb{Z}, \quad (5)$$

where q -number and q -binomial are defined by

$$\begin{aligned} (n)_q &= \frac{q^n - 1}{q - 1}, \\ \binom{n}{k}_q &= \frac{(n)_q (n-1)_q \cdots (n-k+1)_q}{(1)_q (2)_q \cdots (k)_q}, \\ \binom{n}{0}_q &= 1. \end{aligned} \quad (6)$$

For a q -PDO $P = \sum_{i=-\infty}^n p_i \partial_q^i$, we separate P into the differential part $P_+ = \sum_{i=0}^n p_i \partial_q^i$ and the integral part $P_- = \sum_{i=-1} p_i \partial_q^i$. The conjugate operation P^* is given by

$$P^* = \sum_{i=-\infty}^n (\partial_q^*)^i p_i, \quad (7)$$

where $\partial_q^* = -\partial_q \theta^{-1} = -(1/q) \partial_{1/q}$, $(\partial_q^{-1})^* = (\partial_q^*)^{-1} = -\theta \partial_q^{-1}$.

The q -exponential function $E_q(x)$ is defined as

$$E_q(x) = \exp \left(\sum_{k=1}^{\infty} \frac{(1-q)^k}{k(1-q^k)} x^k \right), \quad (8)$$

satisfying $[\partial_q^k E_q(xz)] = z^k E_q(xz)$, $k \in \mathbb{Z}$.

The extended q -KPH was given by [16]

$$L_{t_n} = [B_n, L], \quad B_n = (L^n)_{\geq 0}, \quad (9a)$$

$$L_{\tau_k} = \left[B_k + \sum_{j=1}^m \phi_j \partial_q^{-1} \psi_j, L \right], \quad B_k = (L^k)_{\geq 0}, \quad k \neq n, \quad (9b)$$

$$\begin{aligned} \phi_{j,t_n} &= [B_n \phi_j], \\ \psi_{j,t_n} &= -[B_n^* \psi_j], \end{aligned} \quad (9c)$$

$$j = 1, \dots, m,$$

where $L = \partial_q + u_0 + u_1 \partial_q^{-1} + u_2 \partial_q^{-2} + \cdots$ and the coefficients u_i ($i = 0, 1, \dots$) are the functions of $t = (x, t_1, \dots)$.

The commutativity of (9a), (9b), and (9c) leads to the zero-curvature representation of q -KPH ((9a), (9b), and (9c)). As the n -reduction of the extended q -KPH, the q -NKdVHSCS is defined as follows [16]:

$$B_{n,\tau_k} = \left[B_k + \sum_{j=1}^m \phi_j \partial_q^{-1} \psi_j, B_n \right], \quad B_k = (B_n)_{\geq 0}^{k/n} \quad (10a)$$

$$[B_n \phi_j] = \lambda_j^n \phi_j, \quad (10b)$$

$$[B_n^* \psi_j] = \mu_j^n \psi_j, \quad j = 1, \dots, m. \quad (10c)$$

Under (10b) and (10c), the Lax representation for (10a) is

$$[B_n \varphi] = \lambda \varphi, \quad (11a)$$

$$\varphi_{\tau_k} = \left[\left(B_k + \sum_{j=1}^m \phi_j \partial_q^{-1} \psi_j \right) \varphi \right]. \quad (11b)$$

We find that when $m = 0$, q -NKdVHSCS ((10a), (10b), and (10c) and (11a) and (11b)) can be reduced to the q -NKdVH and its related Lax representation, respectively. In addition, when $n = 2$, $k = 1$ ((10a), (10b), and (10c)) becomes the first nontrivial soliton equation of q -KdVHSCS given by

$$v_{1,\tau_1} + e_1 + \sum_{j=1}^m g_{j1} = 0, \quad (12a)$$

$$v_{0,\tau_3} + e_0 + \sum_{j=1}^m g_{j0} = 0, \quad (12b)$$

$$\theta(v_1) - v_1 + u_0 - \theta^2(u_0) = 0, \quad (12c)$$

$$[(\partial_q^2 + v_1 \partial_q + v_0) \varphi_j] - \lambda_j^2 \varphi_j = 0, \quad (12d)$$

$$[(\partial_q^2 + v_1 \partial_q + v_0)^* \psi_j] - \mu_j^2 \psi_j = 0, \quad j = 1, \dots, m. \quad (12e)$$

3. The Auto Darboux-Bäcklund Transformation (DBT) for q -NKdVHSCS

In this section, we will focus on the construction of auto DBT for q -NKdVHSCS.

Theorem 1. Assume B_n, ϕ_j, ψ_j ($j = 1, \dots, m$) be the solution of q -NKdVHSCS ((10a), (10b), and (10c)), and h_1 satisfies (11a) and (11b) with $\lambda = \lambda_1^n$; the DBT is defined by

$$\bar{B}_n = T_1 B_n T_1^{-1} = \partial_q^n + \bar{v}_{n-1} \partial_q^{n-1} + \dots + \bar{v}_1 \partial_q + \bar{v}_0, \quad (13a)$$

$$\bar{\phi} = [T_1 \phi] = \frac{W_q[h_1, \phi]}{h_1}, \quad (13b)$$

$$\bar{\phi}_j = [T_1 \phi_j] = \frac{W_q[h_1, \phi_j]}{h_1}, \quad (13c)$$

$$\bar{\psi}_j = [(T_1^{-1})^* \psi_j] = -\frac{\theta(\Omega(h_1, \psi_j))}{\theta(h_1)}, \quad (13d)$$

$j = 1, \dots, m,$

Then $\bar{B}_n, \bar{\phi}, \bar{\phi}_j, \bar{\psi}_j$ ($j = 1, \dots, m$) satisfy (10b) and (10c) and (11a) and (11b) and hence are the solution of q -NKdVHSCS ((10a), (10b), and (10c)), where $T_1 = \theta(h_1) \partial_q h_1^{-1} = \partial_q - \alpha_1$, $\alpha_1 = [\partial_q h_1]/h_1$, and $W_q[h_1, \phi_j]$ and $\Omega_q[h_1, \psi_j]$ are defined as follows:

$$W_q[\phi_1, \phi] = \begin{vmatrix} h_1 & \phi \\ [\partial_q h_1] & [\partial_q \phi] \end{vmatrix}, \quad (14)$$

$$\Omega(h_1, \psi_j) = [\partial_q^{-1} h_1 \psi_j].$$

Remark 2. Here it should be pointed out that the formula holds

$$\begin{aligned} \bar{B}_k &= T_1 B_k T_1^{-1} + T_{1, \tau_k} T_1^{-1}, \\ B_k &= (B_n)_{\geq 0}^{k/n}, \\ \bar{B}_k &= (\bar{B}_n)_{\geq 0}^{k/n}, \end{aligned} \quad (15)$$

where the gauge operator T_1 is defined above. The proof has been given in [7].

Proof. (1) We firstly show that $\bar{B}_n, \bar{\phi}_j, \bar{\psi}_j$ ($j = 1, \dots, m$) satisfy (10b) and (10c).

Noting that B_n, ϕ_j, ψ_j ($j = 1, \dots, m$) are the solution of (10a), (10b), and (10c), we have

$$\begin{aligned} [B_n \phi_j] &= \lambda_j^n \phi_j, \\ [B_n^* \psi_j] &= \mu_j^n \psi_j. \end{aligned} \quad (16)$$

Hence,

$$\begin{aligned} [\bar{B}_n \bar{\phi}_j] - \lambda_j^n \bar{\phi}_j &= [T_1 B_n T_1^{-1} [T_1 \phi_j]] - \lambda_j^n \bar{\phi}_j \\ &= [T_1 [B_n \phi_j]] - \lambda_j^n \bar{\phi}_j \\ &= \lambda_j^n ([T_1 \phi_j] - \bar{\phi}_j) = 0, \\ [\bar{B}_n^* \bar{\psi}_j] - \mu_j^n \bar{\psi}_j &= [(T_1^{-1})^* B_n^* T_1^* [(T_1^{-1})^* \psi_j]] \\ &\quad - \mu_j^n \bar{\psi}_j = [(T_1^{-1})^* B_n^* \psi_j] - \mu_j^n \bar{\psi}_j \\ &= 0. \end{aligned} \quad (17)$$

(2) We finally show that $\bar{B}_n, \bar{\phi}, \bar{\phi}_j, \bar{\psi}_j$ satisfy (11a) and (11b).

Since the proof of (11a) is the same as the case (1), we only need to verify that $\bar{B}_n, \bar{\phi}, \bar{\phi}_j, \bar{\psi}_j$ satisfy (11b); that is,

$$\bar{\phi}_{\tau_k} - \left[\left(\bar{B}_k + \sum_{j=1}^m \bar{\phi}_j \partial_q^{-1} \bar{\psi}_j \right) \bar{\phi} \right] = 0, \quad \bar{B}_k = (\bar{B}_n)_{\geq 0}^{k/n}. \quad (18)$$

Noting that $\bar{\phi}_{\tau_k} = [T_1 \phi]_{\tau_k} = [T_{1, \tau_k} \phi] + [T_1 \phi_{\tau_k}]$ and $\phi_{\tau_k} = [(B_k + \sum_{j=1}^m \phi_j \partial_q^{-1} \psi_j) \phi]$, we get

$$\begin{aligned} \bar{\phi}_{\tau_k} - \left[\left(\bar{B}_k + \sum_{j=1}^m \bar{\phi}_j \partial_q^{-1} \bar{\psi}_j \right) \bar{\phi} \right] &= [T_{1, \tau_k} \phi] + [T_1 \phi_{\tau_k}] \\ &\quad - \left[\left(\bar{B}_k + \sum_{j=1}^m \bar{\phi}_j \partial_q^{-1} \bar{\psi}_j \right) [T_1 \phi] \right] = \left[\left(T_{1, \tau_k} + T_1 B_k \right. \right. \\ &\quad \left. \left. - \bar{B}_k T_1 + \sum_{j=1}^m T_1 \phi_j \partial_q^{-1} \psi_j - \sum_{j=1}^m \bar{\phi}_j \partial_q^{-1} \bar{\psi}_j T_1 \right) \phi \right]. \end{aligned} \quad (19)$$

According to Remark 2, we have

$$T_{1, \tau_k} + T_1 B_k - \bar{B}_k T_1 = 0. \quad (20)$$

Next, we prove

$$\sum_{j=1}^m (T_1 \phi_j \partial_q^{-1} \psi_j - \bar{\phi}_j \partial_q^{-1} \bar{\psi}_j T_1) = 0. \quad (21)$$

Since $T_1 = \partial_q - \alpha_1$ and $\partial_q \phi_j = \theta(\phi_j) \partial_q + [\partial_q \phi_j]$, then $\forall j$, we obtain by the tedious computation

$$\begin{aligned} T_1 \phi_j \partial_q^{-1} \psi_j - \bar{\phi}_j \partial_q^{-1} \bar{\psi}_j T_1 &= (\partial_q - \alpha_1) \phi_j \partial_q^{-1} \psi_j - \bar{\phi}_j \partial_q^{-1} \bar{\psi}_j \partial_q + \bar{\phi}_j \partial_q^{-1} \bar{\psi}_j \alpha_1 \\ &= \theta(\phi_j) \psi_j + \bar{\phi}_j \frac{\Omega(h_1, \psi_j)}{h_1}. \end{aligned} \quad (22a)$$

In addition, we also have

$$\begin{aligned}
 \theta(\phi_j) \psi_j + \bar{\phi}_j \frac{\Omega(h_1, \psi_j)}{h_1} &= \frac{[\theta(\phi_j) \partial_q \Omega(\phi_1, \psi_j)] + [T_1 h_1] \Omega(h_1, \psi_j)}{h_1} \\
 &= \frac{[\partial_q \phi_j \Omega(h_1, \psi_j)] - [\partial_q \phi_j] \Omega(h_1, \psi_j) + ([\partial_q \phi_j] - \alpha_1 \phi_j) \Omega(h_1, \psi_j)}{h_1} \\
 &= \frac{[(\partial_q - \alpha_1) \phi_j \Omega(h_1, \psi_j)]}{h_1} = \frac{[T_1 [(\phi_j \partial_q^{-1} \psi_j) h_1]]}{h_1}.
 \end{aligned} \tag{22b}$$

Substituting (22b) into (22a) leads to

$$\begin{aligned}
 \sum_{j=1}^m (T_1 \phi_j \partial_q^{-1} \psi_j - \bar{\phi}_j \partial_q^{-1} \bar{\psi}_j T_1) \\
 = \frac{1}{h_1} \left[T_1 \left[\sum_{j=1}^m (\phi_j \partial_q^{-1} \psi_j) h_1 \right] \right].
 \end{aligned} \tag{23}$$

Since h_1 is the solution of (11a) and (11b) with $\lambda = \lambda_1^n$, we have

$$\begin{aligned}
 h_{1, \tau_k} &= \left[\left(B_k + \sum_{j=1}^m \phi_j \partial_q^{-1} \psi_j \right) h_1 \right] \Rightarrow \\
 \left[\left(\sum_{j=1}^m \phi_j \partial_q^{-1} \psi_j \right) h_1 \right] &= h_{1, \tau_k} - [B_k h_1].
 \end{aligned} \tag{24}$$

Moreover, by the property of determinant, we have

$$[T_1 h_1] = \frac{W_q[h_1, h_1]}{h_1} = 0. \tag{25}$$

Differentiating both sides of (25) with respect to τ_k yields

$$\begin{aligned}
 [T_{1, \tau_k} \phi_1] + [T_1 \phi_{1, \tau_k}] &= 0 \Rightarrow \\
 [T_1 \phi_{1, \tau_k}] &= -[T_{1, \tau_k} \phi_1].
 \end{aligned} \tag{26}$$

From (23), (24), and (26), we have

$$\begin{aligned}
 \sum_{j=1}^m (T_1 \phi_j \partial_q^{-1} \psi_j - \bar{\phi}_j \partial_q^{-1} \bar{\psi}_j T_1) \\
 = -\frac{1}{\phi_1} [(T_{1, \tau_k} + T_1 B_k) \phi_1] = -\frac{1}{\phi_1} [\bar{B}_k [T_1 \phi_1]] \\
 = 0.
 \end{aligned} \tag{27}$$

This completes the proof. $\square$

Obviously, Theorem 1 provides an auto DBT for q -NKdVHSCS ((10a), (10b), and (10c)). However, this DBT does not enable us to obtain the new solution of q -NKdVHSCS ((10a), (10b), and (10c)). So we have to seek for nonauto DBTs between the two q -NKdVHSCS ((10a), (10b), and (10c)) with different degrees of sources.

4. The Nonauto DBTs of q -NKdVHSCS

In this section, we will construct the nonauto DBTs of q -NKdVHSCS ((10a), (10b), and (10c)), which enables us to obtain the new solution of q -NKdVHSCS from the known solution of q -NKdVH.

Theorem 3. Given B_n, ϕ_j, ψ_j ($j = 1, \dots, m$) the solution for q -NKdVHSCS ((10a), (10b), and (10c)), let $f_1, g_1 \equiv \phi_{m+1}$ be two independent eigenfunctions of (11a) and (11b) with $\lambda = \lambda_{m+1}^n$. Let $b_1(\tau_k)$ be a function of τ_k such that $b_1(\tau_k)_{\tau_k} = (-1)^{m+1} \beta_1(\tau_k) \eta_1(\tau_k)$. Denote $h_1 = f_1 + b_1(\tau_k) g_1$.

The DBT is defined by

$$\bar{B}_n = T_1 B_n T_1^{-1} = \partial_q^n + \bar{v}_{n-1} \partial_q^{n-1} + \dots + \bar{v}_1 \partial_q + \bar{v}_0, \tag{28a}$$

$$\bar{\phi} = [T_1 \phi] = \frac{W_q[h_1, \phi]}{h_1}, \tag{28b}$$

$$\bar{\phi}_j = [T_1 \phi_j] = \frac{W_q[h_1, \phi_j]}{h_1}, \tag{28c}$$

$$\bar{\psi}_j = [(T_1^{-1})^* \psi_j] = -\frac{\theta(\Omega(h_1, \psi_j))}{\theta(h_1)}, \tag{28d}$$

$$j = 1, \dots, m,$$

$$\bar{\phi}_{m+1} = -\beta_1(\tau_k) [T_1 g_1], \tag{28e}$$

$$\bar{\psi}_{m+1} = (-1)^{m+1} \eta_1(\tau_k) \frac{1}{\theta(h_1)}, \tag{28f}$$

where $T_1 = \partial_q - \alpha_1$, $\alpha_1 = [\partial_q h_1]/h_1$, and then $\bar{B}_n, \bar{\phi}, \bar{\phi}_j, \bar{\psi}_j$ ($j = 1, \dots, m$), $\bar{\phi}_{m+1}, \bar{\psi}_{m+1}$ satisfy (10b) and (11a) and (11b) with m replaced by $m+1$; hence $\bar{B}_n, \bar{\phi}_j, \bar{\psi}_j$ ($j = 1, \dots, m$), $\bar{\phi}_{m+1}, \bar{\psi}_{m+1}$ are the solution of q -NKdVHSCS ((10a), (10b), and (10c)) with m replaced by $m+1$.

Proof. (1) We firstly show that $\bar{B}_n, \bar{\phi}, \bar{\phi}_j, \bar{\psi}_j$ ($j = 1, \dots, m$), $\bar{\phi}_{m+1}, \bar{\psi}_{m+1}$ are the solution of (10b) and (10c).

With the same proof as Theorem 1, $\bar{B}_n, \bar{\phi}, \bar{\phi}_j, \bar{\psi}_j$ ($j = 1, \dots, m$) can be shown to be the solution of (10b) and (10c).

Here we only need to show that $\bar{\phi}_{m+1}$, $\bar{\psi}_{m+1}$ are also the solution of (10b) and (10c). Consider

$$\begin{aligned} [\bar{B}_n \bar{\phi}_{m+1}] &= -\beta_1(\tau_k) [T_1 B_n T_1^{-1} [T_1 g_1]] \\ &= -\beta_1(\tau_k) [T_1 [B_n g_1]] \\ &= -\lambda_{m+1}^n \beta_1(\tau_k) [T_1 g_1] = \lambda_{m+1}^n \bar{\phi}_{m+1}. \end{aligned} \quad (29)$$

Taking a proper solution ψ_{m+1} of (10c) with $\mu = \mu_{m+1}^n$ such that $\Omega(h_1, \psi_{m+1}) = -1$, then we get

$$\begin{aligned} [\bar{B}_n^* \bar{\psi}_{m+1}] &= (-1)^{m+1} \eta_1(\tau_k) \\ &\cdot \left[(T_1^{-1})^* B_n^* T_1^* \left(\frac{1}{\theta(h_1)} \right) \right] = (-1)^{m+1} \eta_1(\tau_k) \\ &\cdot \left[(T_1^{-1})^* B_n^* T_1^* \left(-\frac{\Omega(h_1, \psi_{m+1})}{\theta(h_1)} \right) \right]. \end{aligned} \quad (30)$$

Noting that $[(T_1^{-1})^* \psi_{m+1}] = -(\Omega(h_1, \psi_{m+1}))/\theta(h_1)$, we derive from (30)

$$\begin{aligned} [\bar{B}_n^* \bar{\phi}_{m+1}] &= (-1)^{m+1} \eta_1(\tau_k) \left[(T_1^{-1})^* B_n^* T_1^* \left[(T_1^{-1})^* \psi_{m+1} \right] \right] \\ &= (-1)^{m+1} \eta_1(\tau_k) \left[(T_1^{-1})^* [B_n^* \psi_{m+1}] \right] \\ &= \mu_{m+1}^n \bar{\psi}_{m+1}. \end{aligned} \quad (31)$$

(2) We finally show that $\bar{B}_n, \bar{\phi}_j, \bar{\psi}_j$ ($j = 1, \dots, m$), $\bar{\phi}_{m+1}, \bar{\psi}_{m+1}$ are the solution of (11a) and (11b) with m replaced by $m+1$. Evidently we only need to prove $\bar{B}_n, \bar{\phi}_j, \bar{\psi}_j$ ($j = 1, \dots, m$), $\bar{\phi}_{m+1}, \bar{\psi}_{m+1}$ satisfy (11b); that is, $\bar{\phi}_{\tau_k} - [\bar{B}_k + \sum_{j=1}^{m+1} \bar{\phi}_j \partial_q^{-1} \bar{\psi}_j] \bar{\phi} = 0$.

Noting that $\bar{\phi} = [T_1 \phi] \Rightarrow \bar{\phi}_{\tau_k} = [T_1 \phi]_{\tau_k} = [T_{1, \tau_k} \phi] + [T_1 \phi_{\tau_k}]$, we have

$$\begin{aligned} \bar{\phi}_{\tau_k} - \left[\left(\bar{B}_k + \sum_{j=1}^{m+1} \bar{\phi}_j \partial_q^{-1} \bar{\psi}_j \right) \bar{\phi} \right] &= \left[\left(T_{1, \tau_k} + T_1 B_k \right. \right. \\ &\left. \left. - \bar{B}_k T_1 + \sum_{j=1}^m T_1 \phi_j \partial_q^{-1} \psi_j - \sum_{j=1}^{m+1} \bar{\phi}_j \partial_q^{-1} \bar{\psi}_j T_1 \right) \phi \right]. \end{aligned} \quad (32a)$$

From (15), a direct computation leads to

$$T_{1, \tau_k} + T_1 B_k - \bar{B}_k T_1 + \frac{b_1(\tau_k)_{\tau_k} [T_1 g_1]}{h_1} = 0. \quad (32b)$$

Noticing that

$$\begin{aligned} \sum_{j=1}^m T_1 \phi_j \partial_q^{-1} \psi_j - \sum_{j=1}^{m+1} \bar{\phi}_j \partial_q^{-1} \bar{\psi}_j T_1 \\ = \sum_{j=1}^m (T_1 \phi_j \partial_q^{-1} \psi_j - \bar{\phi}_j \partial_q^{-1} \bar{\psi}_j T_1) \\ - \bar{\phi}_{m+1} \partial_q^{-1} \bar{\psi}_{m+1} T_1, \end{aligned} \quad (33a)$$

then $\forall j = 1, \dots, m$, we obtain by the tedious computation

$$\begin{aligned} T_1 \phi_j \partial_q^{-1} \psi_j - \bar{\phi}_j \partial_q^{-1} \bar{\psi}_j T_1 \\ = \theta(\phi_j) \psi_j + \bar{\phi}_j \frac{\theta(\Omega(h_1, \psi_j))}{\theta(h_1)}. \end{aligned} \quad (33b)$$

Substituting (33b) into (33a), we get

$$\begin{aligned} \sum_{j=1}^m T_1 \phi_j \partial_q^{-1} \psi_j - \sum_{j=1}^{m+1} \bar{\phi}_j \partial_q^{-1} \bar{\psi}_j T_1 \\ = \sum_{j=1}^m \left(\theta(\phi_j) \psi_j + \bar{\phi}_j \frac{\theta(\Omega(h_1, \psi_j))}{\theta(h_1)} \right) \\ - \bar{\phi}_{m+1} \partial_q^{-1} \bar{\psi}_{m+1} \partial_q + \bar{\phi}_{m+1} \partial_q^{-1} \bar{\psi}_{m+1} \alpha_1 \\ = \sum_{j=1}^m \left(\theta(\phi_j) \psi_j + \bar{\phi}_j \frac{\theta(\Omega(h_1, \psi_j))}{\theta(h_1)} \right) \\ - (-1)^{m+1} \eta_1(\tau_k) \bar{\phi}_{m+1} \partial_q^{-1} \left(\partial_q \frac{1}{h_1} - \left[\partial_q \frac{1}{h_1} \right] \right) \\ + \bar{\phi}_{m+1} \partial_q^{-1} \bar{\psi}_{m+1} \alpha_1 \\ = \sum_{j=1}^m \left(\theta(\phi_j) \psi_j + \bar{\phi}_j \frac{\theta(\Omega(h_1, \psi_j))}{\theta(h_1)} \right) \\ + (-1)^{m+1} \frac{\beta_1(\tau_k) \eta_1(\tau_k) [T_1 g_1]}{h_1}. \end{aligned} \quad (33c)$$

In addition, since f_1, g_1 are the solutions of (11a) and (11b), we have

$$\begin{aligned} f_{1, \tau_k} &= \left[\left(B_k + \sum_{j=1}^m \phi_j \partial_q^{-1} \psi_j \right) f_1 \right], \\ g_{1, \tau_k} &= \left[\left(B_k + \sum_{j=1}^m \phi_j \partial_q^{-1} \psi_j \right) g_1 \right]; \end{aligned} \quad (34)$$

hence

$$\begin{aligned} h_{1, \tau_k} &= f_{1, \tau_k} + b_1(\tau_k)_{\tau_k} g_{1, \tau_k} \\ &\quad + b_1(\tau_k)_{\tau_k} g_1 \\ &= \left[\left(B_k + \sum_{j=1}^m \phi_j \partial_q^{-1} \psi_j \right) h_1 \right] \\ &\quad + b_1(\tau_k)_{\tau_k} g_1 \Rightarrow \end{aligned} \quad (35)$$

$$\left[\sum_{j=1}^m (\phi_j \partial_q^{-1} \psi_j) h_1 \right] = h_{1, \tau_k} - [B_k h_1] - b_1(\tau_k)_{\tau_k} g_1.$$

Noting $[T_1 h_1] = W_q[h_1, h_1]/h_1 = 0$ and differentiating both sides of this equation with respect to τ_k lead to

$$\begin{aligned} [T_{1,\tau_k} h_1] + [T_1 h_{1,\tau_k}] &= 0 \implies \\ [T_1 h_{1,\tau_k}] &= -[T_{1,\tau_k} h_1]. \end{aligned} \quad (36)$$

Rewriting (33c) leads to

$$\begin{aligned} \sum_{j=1}^m T_1 \phi_j \partial_q^{-1} \psi_j - \sum_{j=1}^{m+1} \bar{\phi}_j \partial_q^{-1} \bar{\psi}_j T_1 \\ = \frac{\sum_{j=1}^m [T_1 (\phi_j \partial_q^{-1} \psi_j) h_1]}{h_1} \\ + (-1)^{m+1} \frac{\beta_1(\tau_k) \eta_1(\tau_k) [T_1 g_1]}{h_1}. \end{aligned} \quad (37a)$$

Combining (32a) and (32b) and (35) and (36), we get

$$\begin{aligned} \frac{\sum_{j=1}^m [T_1 (\phi_j \partial_q^{-1} \psi_j) h_1]}{h_1} &= \frac{T_1 [\sum_{j=1}^m (\phi_j \partial_q^{-1} \psi_j) h_1]}{h_1} \\ &= \frac{[T_1 h_{1,\tau_k}] - [T_1 B_k h_1] - b_1(\tau_k)_{\tau_k} [T_1 g_1]}{h_1} \\ &= -\frac{[T_{1,\tau_k} h_1] + [T_1 B_k h_1]}{h_1} - \frac{b_1(\tau_k)_{\tau_k} [T_1 g_1]}{h_1} \\ &= -\frac{[B_k [T_1 h_1]]}{h_1} + \frac{b_1(\tau_k)_{\tau_k} [T_1 g_1]}{h_1} \\ &\quad - \frac{b_1(\tau_k)_{\tau_k} [T_1 g_1]}{h_1} = -\frac{[B_k [T_1 h_1]]}{h_1} = 0. \end{aligned} \quad (37b)$$

Substituting (32b), (37a), and (37b) into (32a), we have

$$\begin{aligned} \bar{\phi}_{\tau_k} - \left[\left(\bar{B}_k + \sum_{j=1}^{m+1} \bar{\phi}_j \partial_q^{-1} \bar{\psi}_j \right) \bar{\phi} \right] \\ = -\frac{b_1(\tau_k)_{\tau_k} [T_1 g_1]}{h_1} \\ + (-1)^{(m+1)} \frac{\beta_1(\tau_k) \eta_1(\tau_k) [T_1 g_1]}{h_1}. \end{aligned} \quad (38)$$

Noting $b_1(\tau_k)_{\tau_k} = (-1)^{m+1} \beta_1(\tau_k) \eta_1(\tau_k)$, we immediately get from (38)

$$\bar{\phi}_{\tau_k} - \left[\left(\bar{B}_k + \sum_{j=1}^{m+1} \bar{\phi}_j \partial_q^{-1} \bar{\psi}_j \right) \bar{\phi} \right] = 0. \quad (39)$$

This completes the proof. $\square$

Theorem 4 (the N -times repeated nonauto DBT). *Given $B_n, \phi_1, \dots, \phi_m, \psi_1, \dots, \psi_m$ are the solution for q -NKdVHSCS*

((10a), (10b), and (10c)), $f_1, \dots, f_N, g_1, \dots, g_N$ are independent eigenfunctions of (11a) and (11b) with $\lambda = \lambda_{m+1}^n, \dots, \lambda_{m+N}^n$. $b_i(\tau_k)$, $i = 1, \dots, N$, are functions of τ_k such that $b_i(\tau_k)_{\tau_k} = (-1)^{m+N} \beta_i(\tau_k) \eta_i(\tau_k)$.

Denote $h_i = f_i + b_i(\tau_k) g_i$. The N -times repeated DBT is defined by

$$\begin{aligned} B_n^{(N)} &= T_N B_n T_N^{-1} = \partial_q^n + v_{n-1}^{(N)} \partial_q^{n-1} + \dots + v_1^{(N)} \partial_q \\ &\quad + v_0^{(N)}, \end{aligned} \quad (40a)$$

$$\phi^{(N)} = [T_N \phi] = \frac{W_q[h_1, h_2, \dots, h_N, \phi]}{W_q[h_1, h_2, \dots, h_N]}, \quad (40b)$$

$$\phi_j^{(N)} = [T_N \phi_j] = \frac{W_q[h_1, h_2, \dots, h_N, \phi_j]}{W_q[h_1, h_2, \dots, h_N]}, \quad (40c)$$

$$\begin{aligned} \psi_j^{(N)} &= \left[(T_N^{-1})^* \psi_j \right] = -\frac{\theta(G_q[h_1, h_2, \dots, h_N, \psi_j])}{\theta(W_q[h_1, h_2, \dots, h_N])}, \\ j &= 1, \dots, m, \end{aligned} \quad (40d)$$

$$\phi_{m+i}^{(N)} = -\beta_i(\tau_s) [T_N g_i], \quad (40e)$$

$$\begin{aligned} \psi_{m+i}^{(N)} &= (-1)^{m+i} \eta_i(\tau_k) \\ &\quad \cdot \frac{\theta(W_q[h_1, \dots, h_{i-1}, h_{i+1}, \dots, h_N])}{\theta(W_q[h_1, h_2, \dots, h_N])}, \\ i &= 1, \dots, N, \end{aligned} \quad (40f)$$

where

$$\begin{aligned} T_N &= \frac{1}{W_q[h_1, h_2, \dots, h_N]} \\ &\quad \cdot \begin{bmatrix} h_1 & h_2 & \dots & h_N & 1 \\ [\partial_q h_1] & [\partial_q h_2] & \dots & [\partial_q h_N] & \partial_q \\ \vdots & \vdots & \vdots & \vdots & \\ [\partial_q^N h_1] & [\partial_q^N h_2] & \dots & [\partial_q^N h_N] & \partial_q^N \end{bmatrix}, \\ G_q &[h_1, h_2, \dots, h_N] \end{aligned}$$

$$= \begin{vmatrix} h_1 & h_2 & \dots & h_N \\ \vdots & \vdots & \vdots & \vdots \\ [\partial_q^{N-2} h_1] & [\partial_q^{N-2} h_2] & \dots & [\partial_q^{N-2} h_N] \\ [\partial_q^{-1} h_1 \psi_j] & [\partial_q^{-1} h_2 \psi_j] & \dots & [\partial_q^{-1} h_N \psi_j] \end{vmatrix},$$

$$T_N = D_N D_{N-1} \dots D_1,$$

$$D_k = (\partial_q - \alpha_k^{(k-1)}),$$

$$\alpha_i^{(k)} = \frac{[\partial_q h_i^{(k)}]}{h_i^{(k)}},$$

$$h_i^{(k)} = [T_k h_i], \quad k = 0, 1, \dots, N-1; \quad (41)$$

then $B_n^{(N)}, \phi_j^{(N)}, \psi_j^{(N)}$, ($j = 1, \dots, m$) satisfy (10b) and (10c) and (11a) and (11b) with m replaced by $m + N$; hence $B_n^{(N)}, \phi_j^{(N)}, \psi_j^{(N)}$, ($j = 1, \dots, m$), $\phi_{m+i}^{(N)}, \psi_{m+i}^{(N)}$, ($i = 1, \dots, N$) are the solution of q -NKdVHSCS ((10a), (10b), and (10c)) with m replaced by $m + N$.

Proof. With the same method as Theorem 3, we can show that $\phi_j^{(N)}, \psi_j^{(N)}$, ($j = 1, \dots, m$), $\phi_{m+i}^{(N)}, \psi_{m+i}^{(N)}$, ($i = 1, \dots, N$) satisfy (10b), (10c), and (11a). Here we only need to show $B_n^{(N)}, \phi_j^{(N)}, \psi_j^{(N)}$, ($j = 1, \dots, m$), $\phi_{m+i}^{(N)}, \psi_{m+i}^{(N)}$, ($i = 1, \dots, N$) satisfy (11b). Next we will show it by the mathematical induction method. Theorem 3 indicates $B_n^{(N)}, \phi_j^{(N)}, \psi_j^{(N)}$, ($j = 1, \dots, m$), $\phi_{m+i}^{(N)}, \psi_{m+i}^{(N)}$, ($i = 1, \dots, N$) satisfy (11b) in the case of $N = 1$.

Provided that $B_l^{(N)}, \phi_j^{(l)}, \psi_j^{(l)}, \phi_{m+i}^{(l)}, \psi_{m+i}^{(l)}$ satisfy (11b) for $l \leq N-1$,

$$\phi_{\tau_k}^{(l)} = \left[\left(B_k^{(l)} + \sum_{j=1}^{m+l} \phi_j^{(l)} \partial_q^{-1} \psi_j^{(l)} \right) \phi^{(l)} \right], \quad (42a)$$

$$B_k^{(l)} = (B_n^{(l)})_{\geq 0}^{k/n},$$

$$b_j(\tau_k)_{\tau_k} = (-1)^{(l+i)} \beta_i(\tau_k) \eta_i(\tau_k), \quad (42b)$$

$$l = 1, \dots, N-1.$$

Noticing that $\phi^{(N)} = [D_N \phi^{(N-1)}]$, then when $l = N$, we have

$$\begin{aligned} \phi_{\tau_k}^{(N)} &= \left[\left(B_k^{(N)} + \sum_{j=1}^{m+N} \phi_j^{(N)} \partial_q^{-1} \psi_j^{(N)} \right) \phi^{(N)} \right] \\ &= [D_N \phi^{(N-1)}]_{\tau_k} - \left[\left(B_k^{(N)} D_N \right. \right. \\ &\quad \left. \left. + \sum_{j=1}^{m+N} \phi_j^{(N)} \partial_q^{-1} \psi_j^{(N)} D_N \right) \phi^{(N-1)} \right] = \left[\left(D_{N, \tau_k} \right. \right. \\ &\quad \left. \left. + D_N B_k^{(N-1)} - B_k^{(N)} D_N \right. \right. \\ &\quad \left. \left. + \sum_{j=1}^{m+N-1} D_N \phi_j^{(N-1)} \partial_q^{-1} \psi_j^{(N-1)} \right. \right. \\ &\quad \left. \left. - \sum_{j=1}^{m+N} \phi_j^{(N)} \partial_q^{-1} \psi_j^{(N)} D_N \right) \phi^{(N-1)} \right]; \end{aligned} \quad (43)$$

simplifying $\sum_{j=1}^{m+N-1} D_N \phi_j^{(N-1)} \partial_q^{-1} \psi_j^{(N-1)}$ —
 $\sum_{j=1}^{m+N} \phi_j^{(N)} \partial_q^{-1} \psi_j^{(N)} D_N$ leads to

$$\begin{aligned} &\sum_{j=1}^{m+N-1} D_N \phi_j^{(N-1)} \partial_q^{-1} \psi_j^{(N-1)} - \sum_{j=1}^{m+N} \phi_j^{(N)} \partial_q^{-1} \psi_j^{(N)} D_N \\ &= \sum_{j=1}^{m+N-1} (D_N \phi_j^{(N-1)} \partial_q^{-1} \psi_j^{(N-1)} - \phi_j^{(N)} \partial_q^{-1} \psi_j^{(N)} D_N) \\ &\quad - \phi_{m+N}^{(N)} \partial_q^{-1} \psi_{m+N}^{(N)} D_N. \end{aligned} \quad (44a)$$

From (40f), we obtain

$$\begin{aligned} \psi_{m+N}^{(N)} &= (-1)^{m+N} \eta_N(\tau_k) \frac{\theta(W_q[h_1, h_2, \dots, h_{i-1}])}{\theta(W_q[h_1, h_2, \dots, h_N])} \\ &= (-1)^{m+N} \frac{\eta_N(\tau_k)}{\theta([T_{N-1} h_N])} \\ &= (-1)^{m+N} \frac{\eta_N(\tau_k)}{\theta(h_N^{(N-1)})}. \end{aligned} \quad (44b)$$

Substituting (44b) into (44a) yields

$$\begin{aligned} &\sum_{j=1}^{m+N-1} D_N \phi_j^{(N-1)} \partial_q^{-1} \psi_j^{(N-1)} - \sum_{j=1}^{m+N} \phi_j^{(N)} \partial_q^{-1} \psi_j^{(N)} D_N \\ &= \sum_{j=1}^{m+N-1} (D_N \phi_j^{(N-1)} \partial_q^{-1} \psi_j^{(N-1)} - \phi_j^{(N)} \partial_q^{-1} \psi_j^{(N)} D_N) \\ &\quad - (-1)^{m+N} \frac{\eta_N(\tau_k) \phi_{m+N}^{(N)}}{h_N^{(N-1)}} \\ &\quad + (-1)^{m+N} \eta_N(\tau_k) \phi_{m+N}^{(N)} \partial_q^{-1} \left[\partial_q \frac{1}{h_N^{(N-1)}} \right] \\ &\quad + \phi_{m+N}^{(N)} \partial_q^{-1} \psi_{m+N}^{(N)} \alpha_N^{N-1} \\ &= \sum_{j=1}^{m+N-1} (D_N \phi_j^{(N-1)} \partial_q^{-1} \psi_j^{(N-1)} - \phi_j^{(N)} \partial_q^{-1} \psi_j^{(N)} D_N) \\ &\quad - (-1)^{m+N} \eta_N(\tau_k) \phi_{m+N}^{(N)} \partial_q^{-1} \frac{[\partial_q h_N^{(N-1)}]}{\theta(h_N^{(N-1)}) h_N^{(N-1)}} \\ &\quad + \phi_{m+N}^{(N)} \partial_q^{-1} \psi_{m+N}^{(N)} \alpha_N^{N-1} \\ &= \sum_{j=1}^{m+N-1} (D_N \phi_j^{(N-1)} \partial_q^{-1} \psi_j^{(N-1)} - \phi_j^{(N)} \partial_q^{-1} \psi_j^{(N)} D_N) \\ &\quad + (-1)^{m+N} \frac{\beta_N(\tau_k) \eta_N(\tau_k) [T_N g_N]}{h_N^{(N-1)}}. \end{aligned} \quad (45)$$

From (37a) for one DBT D_N , we have

$$\begin{aligned} & \sum_{j=1}^{m+N-1} \left(D_N \phi_j^{(N-1)} \partial_q^{-1} \psi_j^{(N-1)} - \sum_{j=1}^{m+N} \phi_j^{(N)} \partial_q^{-1} \psi_j^{(N)} D_N \right) \\ &= \frac{1}{h_N^{(N-1)}} \left[D_N \left[\sum_{j=1}^{m+N-1} (\phi_j^{(N-1)} \partial_q^{-1} \psi_j^{(N-1)}) \right. \right. \\ & \quad \left. \left. \cdot h_N^{(N-1)} \right] \right]. \end{aligned} \quad (46)$$

Note that $h_N^{(N-1)}$ satisfies

$$\begin{aligned} & h_{N,\tau_k}^{(N-1)} \\ &= \left[\left(B_k^{(N-1)} + \sum_{j=1}^{m+N-1} \phi_j^{(N-1)} \partial_q^{-1} \psi_j^{(N-1)} \right) h_N^{(N-1)} \right] \\ & \quad + b_N(\tau_k) g_N^{(N-1)} \Rightarrow \\ & \left[\left(\sum_{j=1}^{m+N-1} \phi_j^{(N-1)} \partial_q^{-1} \psi_j^{(N-1)} \right) h_N^{(N-1)} \right] \\ &= h_{N,\tau_k}^{(N-1)} - [B_k^{(N-1)} h_N^{(N-1)}] - b_N(\tau_k) g_N^{(N-1)}, \end{aligned} \quad (47a)$$

and that

$$[T_N h_N] = [D_N [T_{N-1} h_N]] = [D_N h_N^{(N-1)}] = 0. \quad (47b)$$

Differentiating both sides of (47b) with respect to τ_k yields

$$\begin{aligned} & [D_{N,\tau_k} h_N^{(N-1)}] + [D_N h_{N,\tau_k}^{(N-1)}] = 0 \Rightarrow \\ & [D_N h_{N,\tau_k}^{(N-1)}] = -[D_{N,\tau_k} h_N^{(N-1)}]; \end{aligned} \quad (48)$$

we obtain

$$\begin{aligned} & \frac{1}{h_N^{(N-1)}} \left[D_N \left[\sum_{j=1}^{m+N-1} (\phi_j^{(N-1)} \partial_q^{-1} \psi_j^{(N-1)}) h_N^{(N-1)} \right] \right] \\ &= -\frac{[B_k^{(N-1)} [D_N h_N^{(N-1)}]]}{h_N^{(N-1)}} = 0. \end{aligned} \quad (49)$$

Combining (43), (45), (46), and (49), we get

$$\begin{aligned} & \phi_{\tau_k}^{(N)} - \left[\left(B_k^{(N)} + \sum_{j=1}^{m+N} \phi_j^{(N)} \partial_q^{-1} \psi_j^{(N)} \right) \phi^{(N)} \right] \\ &= (-1)^{m+N} \frac{\beta_N(\tau_k) \eta_N(\tau_k) [T_N g_N]}{h_N^{(N-1)}} \\ & \quad - \frac{b_N(\tau_k) [T_N g_N]}{h_N^{(N-1)}} = 0. \end{aligned} \quad (50)$$

This completes the proof. $\square$

5. Soliton Solution of q -KdVHSCS

It is known that KdV equation is the first nontrivial equation of the KdV hierarchy. However, the first nontrivial equation of q -KdVHSCS is not the q -KdVHSCS but (12a), (12b), (12c), (12d), and (12e). In this section, we aim to construct the soliton solution to (12a), (12b), (12c), (12d), and (12e). In order to get the soliton solution of (12a), (12b), (12c), (12d), and (12e), the following proposition is firstly presented.

Proposition 5. Let f_1, g_1 be two independent wave functions of (12e), $h_1 \equiv f_1 + b_1(\tau_1) g_1$, under the nonauto DBT, and the transformed coefficients are given by

$$\bar{v}_1 - v_1 = x(q-1)(\bar{v}_0 - v_0), \quad (51)$$

where

$$\begin{aligned} \bar{v}_0 = v_0 + \left[\partial_q \left(\theta \left(\frac{f_{1,\tau_1} + b_1(\tau_1) g_{1,\tau_1}}{h_1} \right) \right. \right. \\ \left. \left. + \frac{f_{1,\tau_1} + b_1(\tau_1) g_{1,\tau_1}}{h_1} \right) \right]. \end{aligned} \quad (52)$$

Proof. It was shown in [7] that formula (51) holds for (12a), (12b), (12c), (12d), and (12e) and that

$$\bar{v}_0 - v_0 = [\partial_q (v_1 + \alpha_1 + \theta(\alpha_1))]. \quad (53)$$

Noting that $h_1 = f_1 + b_1(\tau_1) g_1$, $(B_2)_{\geq 0}^{1/2} = B_1 = \partial_q + u_0$, then we have

$$\begin{aligned} h_{1,\tau_1} &= \left[\left((B_2)_{\geq 0}^{1/2} + \sum_{j=1}^m \phi_j \partial_q^{-1} \psi_j \right) h_1 \right] + b_1(\tau_1)_{\tau_1} g_1 \\ &= [\partial_q h_1] + u_0 h_1 + \sum_{j=1}^m \phi_j \Omega(\psi_j, h_1) + b_1(\tau_1)_{\tau_1} g_1. \end{aligned} \quad (54)$$

From (54), we get

$$\begin{aligned} u_0 &= \frac{h_{1,\tau_1} - [\partial_q h_1]}{h_1} - \frac{1}{h_1} \sum_{j=1}^m \phi_j \Omega(\psi_j, h_1) \\ & \quad - b_1(\tau_1)_{\tau_1} \frac{g_1}{h_1}. \end{aligned} \quad (55)$$

Noticing that (12c) implies

$$v_1 = \theta(u_0) + u_0, \quad (56)$$

we have

$$\begin{aligned}
 \bar{v}_0 - v_0 &= \left[\partial_q \left(\theta \left(\frac{h_{1,\tau_1} - [\partial_q h_1]}{h_1} \right) + \frac{h_{1,\tau_1} - [\partial_q h_1]}{h_1} \right. \right. \\
 &\quad - \theta \left(\frac{1}{h_1} \sum_{j=1}^m \phi_j \Omega(\psi_j, h_1) + b_1(\tau_1)_{\tau_1} \frac{g_1}{h_1} \right) \\
 &\quad - \left(\frac{1}{h_1} \sum_{j=1}^m \phi_j \Omega(\psi_j, h_1) + b_1(\tau_1)_{\tau_1} \frac{g_1}{h_1} \right) + \frac{[\partial_q h_1]}{h_1} \\
 &\quad \left. \left. + \theta \left(\frac{[\partial_q h_1]}{h_1} \right) \right) \right] = \left[\partial_q \left(\theta \left(\frac{h_{1,\tau_1}}{h_1} \right) \right) + \frac{h_{1,\tau_1}}{h_1} \right. \\
 &\quad - \theta \left(\frac{1}{h_1} \sum_{j=1}^m \phi_j \Omega(\psi_j, h_1) + b_1(\tau_1)_{\tau_1} \frac{g_1}{h_1} \right) \\
 &\quad \left. - \left(\frac{1}{h_1} \sum_{j=1}^m \phi_j \Omega(\psi_j, h_1) + b_1(\tau_1)_{\tau_1} \frac{g_1}{h_1} \right) \right] \\
 &= \left[\partial_q \left(\theta \left(\frac{f_{1,\tau_1} + b_1(\tau_1) g_{1,\tau_1}}{h_1} \right) \right. \right. \\
 &\quad \left. \left. + \frac{f_{1,\tau_1} + b_1(\tau_1) g_{1,\tau_1}}{h_1} \right) \right] \\
 &\quad - \left[\partial_q \left(\theta \left(\frac{1}{h_1} \sum_{j=1}^m \phi_j \Omega(\psi_j, h_1) \right) \right. \right. \\
 &\quad \left. \left. + \left(\frac{1}{h_1} \sum_{j=1}^m \phi_j \Omega(\psi_j, h_1) \right) \right) \right] \\
 &= \left[\partial_q \left(\theta \left(\frac{f_{1,\tau_1} + b_1(\tau_1) g_{1,\tau_1}}{h_1} \right) \right. \right. \\
 &\quad \left. \left. + \frac{f_{1,\tau_1} + b_1(\tau_1) g_{1,\tau_1}}{h_1} \right) \right] + (q\theta + 1) \\
 &\quad \cdot \left[\partial_q \left(\frac{1}{h_1} \sum_{j=1}^m \phi_j \Omega(\psi_j, h_1) \right) \right].
 \end{aligned} \tag{57}$$

Next we consider

$$\begin{aligned}
 &\left[\partial_q \left(\frac{1}{h_1} \sum_{j=1}^m \phi_j \Omega(\psi_j, h_1) \right) \right] \\
 &= \sum_{j=1}^m \left(\theta \left(\frac{\phi_j}{h_1} \right) [\partial_q \Omega(\psi_j, h_1)] \right. \\
 &\quad \left. + \left[\partial_q \frac{\phi_j}{h_1} \right] \Omega(\psi_j, h_1) \right) = \sum_{j=1}^m \left(\frac{\theta(\phi_j) \psi_j h_1}{\theta(h_1)} \right.
 \end{aligned}$$

$$\begin{aligned}
 &\left. + \frac{[\partial_q \phi_j] h_1 - \phi_j [\partial_q h_1]}{\theta(h_1) h_1} \Omega(\psi_j, h_1) \right) = \frac{1}{\theta(h_1)} \\
 &\cdot \sum_{j=1}^m \left(\theta(\phi_j) \psi_j h_1 + ([\partial_q \phi_j] - \alpha_1 \phi_j) \Omega(\psi_j, h_1) \right) \\
 &= \frac{h_1}{\theta(h_1)} \left(\sum_{j=1}^m \theta(\phi_j) \psi_j + \bar{\phi}_j \frac{\Omega(\psi_j, h_1)}{h_1} \right).
 \end{aligned} \tag{58}$$

Noting (37b), we can immediately derive

$$\left[\partial_q \left(\frac{1}{h_1} \sum_{j=1}^m \phi_j \Omega(\psi_j, h_1) \right) \right] = 0. \tag{59}$$

Hence we obtain from (57)

$$\begin{aligned}
 \bar{v}_0 - v_0 &= \left[\partial_q \left(\theta \left(\frac{f_{1,\tau_1} + b_1(\tau_1) g_{1,\tau_1}}{h_1} \right) \right. \right. \\
 &\quad \left. \left. + \frac{f_{1,\tau_1} + b_1(\tau_1) g_{1,\tau_1}}{h_1} \right) \right].
 \end{aligned} \tag{60}$$

This completes the proof. $\square$

Next we will start from the trivial solution to (12a), (12b), (12c), (12d), and (12e) without sources, that is, $v_0 = v_1 = 0$, and use Theorem 3 and Proposition 5 to construct one soliton solution to (12a), (12b), (12c), (12d), and (12e) with $m = 1$. When $v_0 = v_1 = 0$, then $B_2 = \partial_q^2$; hence, the wave functions f_1, g_1 of Lax operator $B_2 = \partial_q^2$ satisfy

$$\begin{aligned}
 [\partial_q^2 \varphi] &= \lambda_1^2 \varphi, \\
 \phi_{\tau_1} &= [\partial_q \varphi].
 \end{aligned} \tag{61}$$

We take the solution f_1, g_1 of system (61) as follows:

$$\begin{aligned}
 f_1 &= E_q(p_1 x) \exp(p_1 \tau_1), \\
 g_1 &= E_q(-p_1 x) \exp(-p_1 \tau_1),
 \end{aligned} \tag{62}$$

where $E_q(x)$ denotes the q -exponential function satisfying

$$[\partial_q E_q(p_1 x)] = p_1 E_q(p_1 x), \tag{63}$$

with an equivalent form

$$E_q(x) = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} x^k. \tag{64}$$

Noting $h_1 \equiv f_1 + b_1(\tau_1)g_1$, where f_1, g_1 are defined by (62), we get from (51) and (52)

$$\begin{aligned} \bar{v}_0 &= \frac{p_1(h_1\theta^2(f_1 - b_1(\tau_1)g_1) - (f_1 - b_1(\tau_1)g_1)\theta^2(h_1))}{x(q-1)h_1\theta^2(h_1)}, \end{aligned} \quad (65a)$$

$$\begin{aligned} \bar{v}_1 &= \frac{p_1(h_1\theta^2(f_1 - b_1(\tau_1)g_1) - (f_1 - b_1(\tau_1)g_1)\theta^2(h_1))}{h_1\theta^2(h_1)}. \end{aligned} \quad (65b)$$

In addition, by Theorem 3, we obtain

$$\bar{\phi}_1 = -\beta_1(\tau_1) \frac{h_1[\partial_q g_1] - g_1[\partial_q h_1]}{h_1^2}, \quad (65c)$$

$$\bar{\psi}_1 = -\frac{\eta_1(\tau_1)}{\theta_1(h_1)}, \quad (65d)$$

where $b_1(\tau_1)$, $\beta_1(\tau_1)$, and $\eta_1(\tau_1)$ satisfy

$$b_1(\tau_1)_{\tau_1} = -\beta_1(\tau_1)\eta_1(\tau_1). \quad (65e)$$

Then (65a), (65b), (65c), (65d), and (65e) present one soliton solution of (12a), (12b), (12c), (12d), and (12e) with $m = 1$. In particular, when $b_1(\tau_1) = c$, where c is an arbitrary constant, (65a), (65b), (65c), (65d), and (65e) can be reduced to one soliton solution to the first nontrivial equation of the q -KdV hierarchy [7]. Certainly, we also use Theorem 4 and Proposition 5 to construct the multisoliton solution to (12a), (12b), (12c), (12d), and (12e). But owing to the complexity of the computation, we omit it here.

6. Summary

As n -reduction of the extended q -deformed KP hierarchy, q -NKdVHSCS is explored in this paper. Two kinds of DBTs are constructed and the soliton solution to the first nontrivial equation of q -KdVHSCS is also obtained. We find that one of the DBTs provides a nonauto Bäcklund transformation for the two q -NKdVHSCS with different degree, which enables us to obtain the new solution of q -NKdVHSCS from the known solution of q -NKdVH. Noting that we only investigate DBT and solution of q -NKdVHSCS, other integrable structures will be studied in our forthcoming paper such as infinite conservation law, tau function, and Hamiltonian structure.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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