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Research Article

Lax Triples for Integrable Surfaces in Three-Dimensional Space

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We study Lax triples (i.e., Lax representations consisting of three linear equations) associated with families of surfaces immersed in three-dimensional Euclidean space $\mathbb{E}^3$. We begin with a natural integrable deformation of the principal chiral model. Then, we show that all deformations linear in the spectral parameter λ are trivial unless we admit Lax representations in a larger space. We present an explicit example of triply orthogonal systems with Lax representation in the group Spin(6). Finally, the obtained results are interpreted in the context of the soliton surfaces approach.

1. Introduction

We consider geometric problems associated with Lax triples, that is, Lax representations consisting of three linear equations with a spectral parameter. They can be interpreted as integrable deformations of problems corresponding to one of the involved Lax pairs. The problem of finding integrable deformations of surfaces in $\mathbb{E}^3$ is an interesting and nontrivial task. There are just few papers on that subject; see, for example, [1–3]. However, we point out that the central result of [1], namely, Theorem 3.2, is wrong. This theorem claims that for any augmented system of Gauss-Mainardi-Codazzi equations there exists an explicit Lax representation with the spectral parameter: system (3.6) in [1]. Unfortunately, one can easily check that the spectral parameter can be easily eliminated from this system by performing simple algebraic calculations.

In this paper we suggest another methodology. We start from Lax representations of prescribed form in order to obtain special cases of integrable systems. We checked two cases: Lax representations with three different simple poles (an integrable deformation of the principal chiral model; see Section 2) and Lax representations linear in the spectral parameter. All $SU(2)$-valued Lax representations linear in λ turned out to be trivial; see Section 3. However, there exists a nontrivial Lax representation in a larger space for a special class of orthogonal nets in $\mathbb{E}^3$ [4]; see also Section 4. Finally, in Section 5, we shortly present more general context for studying integrable differential geometry: soliton surfaces approach [5] and Lie point symmetries for introducing the spectral parameter [6, 7].

Throughout this paper we usually use the Lie group $SU(2)$ instead of $SO(3)$, taking into account the isomorphism of corresponding Lie algebras: $so(3) \simeq su(2)$. We can assume, for instance,

$$\begin{pmatrix} 0 & -a_3 & -a_2 \\ a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix} \longleftrightarrow -\frac{i}{2}\begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}. \quad (1)$$

$SU(2)$ is double covering of $SO(3)$, so all our results can be projected on $SO(3)$ when necessary. In Section 4 we make use of another isomorphism, $su(2) \simeq \mathrm{spin}(3)$. Actually, also $SU(2) \simeq \mathrm{Spin}(3)$.

2. An Integrable Deformation of the Principal Chiral Model

The principal chiral model is defined by the following system of two equations [8–10]:

$$\begin{aligned} A_t &= \frac{1}{2}[B, A], \\ B_x &= \frac{1}{2}[A, B], \end{aligned} \quad (2)$$

where A, B are elements of a Lie algebra g. Equations (2) are equivalent to

$$A_t - B_x + [A, B] = 0, \qquad A_t + B_x = 0. \tag{3}$$

The chiral model has the Lax representation

$$\Psi_x = \frac{A}{1-\lambda}\Psi, \qquad \Psi_t = \frac{B}{1+\lambda}\Psi, \tag{4}$$

where $\Psi = \Psi(x, t; \lambda) \in G$ (G is the corresponding Lie group). The chiral model is integrable in the sense of the theory of solitons. In particular, Darboux transformation and multisoliton solutions are known [10]. Denoting $\Phi = \Psi(x, t; 0)$ we can express A and B in terms of Φ:

$$A = \Phi_x \Phi^{-1}, \qquad B = \Phi_t \Phi^{-1}. \tag{5}$$

Then the first equation of (3) is identically satisfied and system (3) reduces to a single equation for Φ:

$$\left(\Phi_x \Phi^{-1}\right)_t + \left(\Phi_t \Phi^{-1}\right)_x = 0. \tag{6}$$

By changing variables $\xi = (1/2)(x + t)$, $\eta = (i/2)(t - x)$, we transform (6) into

$$\left(\Phi_\xi \Phi^{-1}\right)_\xi + \left(\Phi_\eta \Phi^{-1}\right)_\eta = 0. \tag{7}$$

Solutions to this equation are harmonic maps from $\mathbb{R}^2$ into G provided that ξ, η are real [9, 11].

We propose an extension of the principal chiral model which is derived from the following Lax representation:

$$\Psi_x = \frac{A}{1-\lambda}\Psi, \qquad \Psi_t = \frac{B}{1+\lambda}\Psi, \qquad \Psi_y = \frac{\kappa C}{\kappa-\lambda}\Psi, \tag{8}$$

where $\kappa \in \mathbb{R}$ (we assume $|\kappa| \neq 1$), $\Psi = \Psi(x, t, y; \lambda)$ takes values in a Lie group G, and A, B, C belong to the corresponding Lie algebra g. In particular, we can take $G = SU(2)$. This case yields an integrable deformation of surfaces in $\mathbb{E}^3$. Compatibility conditions for (8) read

$$\begin{aligned} A_t &= \frac{1}{2}[B, A], \\ B_y &= \frac{\kappa}{1+\kappa}[C, B], \\ C_x &= \frac{1}{1-\kappa}[A, C], \\ B_x &= \frac{1}{2}[A, B], \\ C_t &= \frac{1}{1+\kappa}[B, C], \\ A_y &= \frac{\kappa}{1-\kappa}[A, C]. \end{aligned} \tag{9}$$

Denoting $\Phi = \Psi(x, t, y; 0)$ we have, similarly to the chiral model case,

$$A = \Phi_x \Phi^{-1}, \qquad B = \Phi_t \Phi^{-1}, \qquad C = \Phi_y \Phi^{-1}. \tag{10}$$

Then, compatibility conditions (9) reduce to three equations for one function Φ:

$$\begin{aligned} \left(\Phi_x \Phi^{-1}\right)_t + \left(\Phi_t \Phi^{-1}\right)_x &= 0, \\ \left(\Phi_x \Phi^{-1}\right)_y - \kappa\left(\Phi_y \Phi^{-1}\right)_x &= 0, \\ \left(\Phi_t \Phi^{-1}\right)_y + \kappa\left(\Phi_y \Phi^{-1}\right)_t &= 0. \end{aligned} \tag{11}$$

Proposition 1. *Any solution $\Phi_0(x, t)$ of the chiral model (6) admits an extension (unique up to translations in y) to a solution Φ of the deformed chiral model (11) such that $\Phi(x, t, y_0) = \widehat{\Phi}_0(x, t)$ for some y_0.*

Proof. System (11) is equivalent to

$$\begin{aligned} \left(\Phi_x \Phi^{-1}\right)_t + \left(\Phi_t \Phi^{-1}\right)_x &= 0, \\ (1-\kappa)\Phi_{xy} - \Phi_x \Phi^{-1}\Phi_y + \kappa \Phi_y \Phi^{-1}\Phi_x &= 0, \\ (1+\kappa)\Phi_{ty} - \Phi_t \Phi^{-1}\Phi_y - \kappa \Phi_y \Phi^{-1}\Phi_t &= 0. \end{aligned} \tag{12}$$

The last two equations of (12) can be rewritten as a linear system for one unknown Φ_y:

$$\frac{\partial}{\partial x}\Phi_y = \frac{A\Phi_y - \kappa\Phi_y\widehat{A}}{1-\kappa}, \qquad \frac{\partial}{\partial t}\Phi_y = \frac{B\Phi_y + \kappa\Phi_y\widehat{B}}{1+\kappa}, \tag{13}$$

where $\widehat{A} = \Phi^{-1}A\Phi$ and $\widehat{B} = \Phi^{-1}B\Phi$. Therefore, let us consider linear system

$$\frac{\partial \Gamma}{\partial x} = \frac{A\Gamma - \kappa\Gamma\widehat{A}}{1-\kappa}, \qquad \frac{\partial \Gamma}{\partial t} = \frac{B\Gamma + \kappa\Gamma\widehat{B}}{1+\kappa}. \tag{14}$$

We will show that the compatibility conditions for (14) are equivalent to the first equation of (12) which, in turn, is

equivalent to (3). Compatibility conditions $\Phi_{yx} = \Phi_{xy}$ can be written as

$$P\Phi_y - \Phi_y Q = 0, \tag{15}$$

where

$$\begin{aligned} P &= (1+\kappa) A_t + AB - (1-\kappa) B_x - BA, \\ Q &= \kappa(1+\kappa)\widehat{A}_t + \kappa^2 \widehat{B}\widehat{A} - \kappa^2 \widehat{A}\widehat{B} + \kappa(1-\kappa)\widehat{B}_x. \end{aligned} \tag{16}$$

We immediately see that (3) imply $P = 0$. In order to show $Q = 0$ we compute

$$\begin{aligned} \widehat{A}_t &= \Phi^{-1}\left(A_t + [A,B]\right)\Phi, \\ \widehat{B}_x &= \Phi^{-1}\left(B_x + [B,A]\right)\Phi, \\ \left[\widehat{A},\widehat{B}\right] &= \Phi^{-1}[A,B]\Phi. \end{aligned} \tag{17}$$

Therefore,

$$\begin{aligned} \widehat{A}_t - \widehat{B}_x - \left[\widehat{A},\widehat{B}\right] &= \Phi^{-1}\left(A_t - B_x + [A,B]\right)\Phi = 0, \\ \widehat{A}_t + \widehat{B}_x &= \Phi^{-1}\left(A_t + B_x\right)\Phi = 0, \end{aligned} \tag{18}$$

where we once more took into account (3). Equations (18) imply $Q = 0$.

Thus, given a solution Φ_0 of (6) we obtain a unique solution Γ_0 of linear system (14). Then, we define $\Phi(x,t,y_0) := \Phi_0(x,t)$ and interpret $\Gamma_0(x,y)$ as y-derivative of $\Phi(x,t,y)$ at $y = y_0$. □

We proceed to showing the integrability of deformed chiral model (11) by constructing the Darboux-Bäcklund transformation, following [12]. In what follows we usually suppress the dependence on x, t, y, sometimes also the dependence on λ. We will write, for instance, $D(\lambda)$ or D instead of $D(x,t,y;\lambda)$.

Lemma 2. *The Darboux transformation $\widetilde{\Psi} = D\Psi$ with a canonical normalization (i.e., $D(\infty) = I$) applied to a linear problem $\Psi_x = U(\lambda)\Psi$ preserves the constraint $U(\infty) = 0$.*

Proof. We have $\widetilde{U} = D_x D^{-1} + DUD^{-1}$. Therefore, the canonical normalization implies $\widetilde{U}(\infty) = U(\infty)$. □

Proposition 3. *The Darboux transformation for $GL(n)$-valued linear problem (8) is given by*

$$\widetilde{\Psi}(\lambda) = f(\lambda)\left(I + \frac{\lambda_1 - \mu_1}{\lambda - \lambda_1}P\right)\Psi(\lambda), \tag{19}$$

where $f(\lambda)$ is a constant (i.e., it does not depend on x, t, y), λ_1 and μ_1 are complex parameters, and P is uniquely defined by its kernel $\ker P = \Psi(\lambda_1)V_{\ker}$ and image $\operatorname{Im} P = \Psi(\mu_1)V_{\operatorname{Im}}$, where $V_{\ker}$ and V_{Im} are subspaces of $\mathbb{C}^n$ such that $V_{\ker} \oplus V_{\operatorname{Im}} = \mathbb{C}^n$ and $\Psi(\lambda)$ is a solution of (8) assumed to be known. The corresponding transformation for the chiral field Φ reads

$$\widetilde{\Phi} = f_0\left(I + \frac{\mu_1 - \lambda_1}{\lambda_1}P\right)\Phi, \tag{20}$$

where $f_0 \equiv f(0)$ is a constant. The reduction $SL(n,C)$ consists in choosing

$$f(\lambda) = \left(\frac{\lambda - \lambda_1}{\lambda - \mu_1}\right)^{d/n}, \tag{21}$$

where $d = \dim \operatorname{Im} P$. The reduction $U(n)$ consists in adding the following constraints: $\mu_1 = \overline{\lambda}_1$ and V_{Im} is orthogonal to $V_{\ker}$.

Proof. Formula (19) preserves all poles of the linear problem (8), as shown, for instance, in [12]. There is also another condition to be preserved: all matrices of our linear problem vanish at $\lambda = \infty$. By Lemma 2 this constraint is preserved by any Darboux matrix with the canonical normalization. Formula (20) follows from (19) by substitution $\lambda = 0$. In particular, we have $\Phi = \Psi(0)$. Implications of the reductions $U(n)$ and $SL(n,C)$ are described in [12] in detail. □

Proposition 3 proves integrability of the related nonlinear system. All steps of the proof are exactly the same as in the case of the principal chiral model. Therefore, multisoliton solution can be obtained in the standard way; see [10, 13].

3. $SU(2)$-Valued Lax Triples Linear in λ Are Trivial

Let us consider $SU(2)$-valued Lax representations linear in λ:

$$\begin{aligned} \Psi_x &= (A + \lambda L)\Psi, \\ \Psi_t &= (B + \lambda M)\Psi, \\ \Psi_y &= (C + \lambda N)\Psi, \end{aligned} \tag{22}$$

where A, B, C, L, M, N are $su(2)$-valued functions of x, t, y (without dependence on λ) and $\Psi = \Psi(x,t,y;\lambda) \in SU(2)$. Subscripts x, t, y denote differentiation. Compatibility conditions ($\Psi_{xt} = \Psi_{tx}$, $\Psi_{ty} = \Psi_{yt}$, $\Psi_{xy} = \Psi_{yx}$) read

$$\begin{aligned} [L,M] &= 0, \\ [L,N] &= 0, \\ [M,N] &= 0, \end{aligned} \tag{23}$$

$$\begin{aligned} A_t - B_x + [A,B] &= 0, \\ B_y - C_t + [B,C] &= 0, \\ C_x - A_y + [C,A] &= 0, \end{aligned} \tag{24}$$

$$\begin{aligned} L_t - M_x + [L,B] + [A,M] &= 0, \\ M_y - N_t + [M,C] + [B,N] &= 0, \\ N_x - L_y + [N,A] + [C,L]. \end{aligned} \tag{25}$$

We make use of isomorphism $\mathbb{E}^3 \simeq su(2)$, where by $\mathbb{E}^3$ we mean the three-dimensional Euclidean space with scalar and skew product. In this isomorphism the scalar product is identified with the Killing-Cartan form (without entering into details we denote by $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ one of the orthonormal

frames) and the skew product with the Lie bracket (commutator of matrices). In particular, vanishing commutator of two elements means that they are parallel and so forth.

The first set of compatibility conditions, namely, (23), can be easily solved: L, M, N are parallel. Therefore, there exists a unit matrix $m \in su(2)$ and real functions α, β, γ such that

$$\begin{aligned} L &= \alpha m, \\ M &= \beta m, \\ N &= \gamma m. \end{aligned} \tag{26}$$

Any unit element $m \in su(2)$ can be obtained from any constant unit element, say $\mathbf{e}_3$, by a rotation. In other words, there exists $T \in SU(2)$ such that

$$m = T^{-1}\mathbf{e}_3 T. \tag{27}$$

Substituting (26) into (25) we get

$$\begin{aligned} \alpha_t m + \alpha m_t - \beta_x m - \beta m_x + \alpha [m, B] + \beta [A, m] &= 0, \\ \beta_y m + \beta m_y - \gamma_t m - \gamma m_t + \beta [m, C] + \gamma [B, m] &= 0, \\ \gamma_x m + \gamma m_x - \alpha_y m - \alpha m_y + \gamma [m, A] + \alpha [C, m] &= 0. \end{aligned} \tag{28}$$

Taking into account that any $[m, X]$ is orthogonal to m (for any X) and any derivative of m is orthogonal to m (because m is a unit element), we can decompose (28) into parallel and orthogonal parts. In particular, we have

$$\begin{aligned} \alpha_t &= \beta_x, \\ \beta_y &= \gamma_t, \\ \gamma_x &= \alpha_y. \end{aligned} \tag{29}$$

Hence, there exists a real function $\varphi = \varphi(x, t, y)$ such that

$$\begin{aligned} \alpha &= \varphi_x, \\ \beta &= \varphi_t, \\ \gamma &= \varphi_y. \end{aligned} \tag{30}$$

Formula (27) suggests gauge transformation $\widehat{\Psi} = T\Psi$. Then, using also (26) and (30), we have

$$\begin{aligned} \widehat{\Psi}_x &= \left(\widehat{A} + \lambda\varphi_x \mathbf{e}_3\right)\widehat{\Psi}, \\ \widehat{\Psi}_t &= \left(\widehat{B} + \lambda\varphi_t \mathbf{e}_3\right)\widehat{\Psi}, \\ \widehat{\Psi}_y &= \left(\widehat{C} + \lambda\varphi_y \mathbf{e}_3\right)\widehat{\Psi}, \end{aligned} \tag{31}$$

where

$$\begin{aligned} \widehat{A} &= T_x T^{-1} + TAT^{-1}, \\ \widehat{B} &= T_t T^{-1} + TBT^{-1}, \\ \widehat{C} &= T_y T^{-1} + TCT^{-1}. \end{aligned} \tag{32}$$

Therefore, all $SU(2)$-valued Lax representations linear in λ can be reduced to (31). It turns out that this system can be reduced further. Matrices $\widehat{A}$, $\widehat{B}$, $\widehat{C}$ satisfy compatibility conditions consisting of (24) and

$$\begin{aligned} \left[\mathbf{e}_3, \varphi_t \widehat{A} - \varphi_x \widehat{B}\right] &= 0, \\ \left[\mathbf{e}_3, \varphi_x \widehat{C} - \varphi_y \widehat{A}\right] &= 0, \\ \left[\mathbf{e}_3, \varphi_t \widehat{C} - \varphi_y \widehat{B}\right] &= 0. \end{aligned} \tag{33}$$

Hence, after similar consideration to above, we obtain

$$\begin{aligned} \widehat{A} &= \varphi_x \widehat{Q} + a\mathbf{e}_3, \\ \widehat{B} &= \varphi_t \widehat{Q} + c\mathbf{e}_3, \\ \widehat{C} &= \varphi_y \widehat{Q} + c\mathbf{e}_3, \end{aligned} \tag{34}$$

where $\widehat{Q}$ is a matrix-valued function orthogonal to $\mathbf{e}_3$ and a, b, c are real functions. From (24) we get

$$\begin{aligned} &a_t \mathbf{e}_3 - b_x \mathbf{e}_3 + \varphi_x \widehat{Q}_t - \varphi_t \widehat{Q}_x + \varphi_x b \left[\widehat{Q}, \mathbf{e}_3\right] \\ &\quad + a\varphi_t \left[\mathbf{e}_3, \widehat{Q}\right] = 0, \\ &b_y \mathbf{e}_3 - c_t \mathbf{e}_3 + \varphi_t \widehat{Q}_y - \varphi_y \widehat{Q}_t + \varphi_t c \left[\widehat{Q}, \mathbf{e}_3\right] \\ &\quad + b\varphi_y \left[\mathbf{e}_3, \widehat{Q}\right] = 0, \\ &c_x \mathbf{e}_3 - a_y \mathbf{e}_3 + \varphi_y \widehat{Q}_x - \varphi_x \widehat{Q}_y + \varphi_y a \left[\widehat{Q}, \mathbf{e}_3\right] \\ &\quad + c\varphi_x \left[\mathbf{e}_3, \widehat{Q}\right] = 0. \end{aligned} \tag{35}$$

The components parallel to $\mathbf{e}_3$ yield

$$\begin{aligned} a_t &= b_x, \\ b_y &= c_t, \\ c_y &= a_y, \end{aligned} \tag{36}$$

which means that there exists a real function χ such that

$$\begin{aligned} a &= \chi_x, \\ b &= \chi_t, \\ c &= \chi_y. \end{aligned} \tag{37}$$

The function χ can be eliminated by another gauge transformation:

$$\Phi = e^{-\chi \mathbf{e}_3}\widehat{\Psi}. \tag{38}$$

Then, we obtain the following linear problem:

$$\begin{aligned} \Phi_x &= \varphi_x \left(\lambda \mathbf{e}_3 + Q\right)\Phi, \\ \Phi_t &= \varphi_t \left(\lambda \mathbf{e}_3 + Q\right)\Phi, \\ \Phi_y &= \varphi_y \left(\lambda \mathbf{e}_3 + Q\right)\Phi, \end{aligned} \tag{39}$$

where $Q = e^{-\chi \mathbf{e}_3}\widehat{Q}e^{\chi \mathbf{e}_3}$. The obtained linear problem is trivial because it can be transformed into a single equation by a change of variables. Indeed, taking $\tilde{x} = \varphi(x, y, t)$, $\tilde{t} = t$, and $\tilde{y} = y$ we transform system (39) into

$$\begin{aligned} \Phi_{\tilde{x}} &= (\lambda \mathbf{e}_3 + Q)\,\Phi, \\ \Phi_{\tilde{t}} &= 0, \\ \Phi_{\tilde{y}} &= 0. \end{aligned} \tag{40}$$

Therefore Q can be taken as an arbitrary function of $\tilde{x}$. No differential equations are involved.

4. An Integrable System Associated with Triply Orthogonal Coordinates

Because of isomorphism (1) $SU(2)$-valued Lax representations are a rich source of integrable geometries in $\mathbb{E}^3$; see [5]. In some problems, however, $SU(2)$ assumption is too restrictive. For instance, searching for integrable geometries associated with Lax representations linear in the spectral parameter we obtained a negative result: there are no nontrivial $SU(2)$-cases (see Section 3). In order to show integrability of geometric problems in $\mathbb{E}^3$ one sometimes needs a larger Lie group; see [14–16].

Here we present in more detail the case of a special system of triply orthogonal coordinates. We consider the following Lax representation:

$$\begin{aligned} \Psi_x &= \frac{1}{2}\left(\mathbf{a}_1\lambda + \mathbf{b}_1\right)\mathbf{e}_1\Psi, \\ \Psi_t &= \frac{1}{2}\left(\mathbf{a}_2\lambda + \mathbf{b}_2\right)\mathbf{e}_2\Psi, \\ \Psi_y &= \frac{1}{2}\left(\mathbf{a}_3\lambda + \mathbf{b}_3\right)\mathbf{e}_3\Psi, \end{aligned} \tag{41}$$

where subscripts x, t, y denote differentiation and

$$\begin{aligned} \mathbf{a}_1 &= \alpha_{11}\mathbf{e}_4 + \alpha_{21}\mathbf{e}_5 + \alpha_{31}\mathbf{e}_6, \\ \mathbf{b}_1 &= \beta_{21}\mathbf{e}_2 + \beta_{31}\mathbf{e}_3, \\ \mathbf{a}_2 &= \alpha_{12}\mathbf{e}_4 + \alpha_{22}\mathbf{e}_5 + \alpha_{32}\mathbf{e}_6, \\ \mathbf{b}_2 &= \beta_{12}\mathbf{e}_1 + \beta_{32}\mathbf{e}_3, \\ \mathbf{a}_3 &= \alpha_{13}\mathbf{e}_4 + \alpha_{23}\mathbf{e}_5 + \alpha_{33}\mathbf{e}_6, \\ \mathbf{b}_3 &= \beta_{13}\mathbf{e}_1 + \beta_{23}\mathbf{e}_2, \end{aligned} \tag{42}$$

α_{jk}, β_{jk} are real functions, and $\mathbf{e}_k$ are generators of the Clifford algebra $\mathscr{C}_{6,0}$; that is, they satisfy the following relations:

$$\begin{aligned} \mathbf{e}_k^2 &= 1, \quad (\text{for } k = 1, 2, \ldots, 6), \\ \mathbf{e}_j\mathbf{e}_k &= -\mathbf{e}_k\mathbf{e}_j, \quad (\text{for } j \neq k). \end{aligned} \tag{43}$$

Note that the Lie algebra of linear problem (41) is spanned by bivectors $\mathbf{e}_j\mathbf{e}_k$ which means that this is Spin(6)-valued (double covering of corresponding $SO(6)$ linear problem). Compatibility conditions for (41) read

$$\begin{aligned} \left\langle \mathbf{a}_j \mid \mathbf{a}_k \right\rangle &= 0, \quad (\text{for } j \neq k), \\ \partial_j \alpha_{ik} &= \beta_{jk}\alpha_{ij}, \quad (\text{for } j \neq k), \\ \partial_j \beta_{ik} &= \beta_{ij}\beta_{jk}, \quad (\text{for } i \neq j \neq k \neq i), \end{aligned} \tag{44}$$

$$\partial_k \beta_{kj} + \partial_j \beta_{jk} + \beta_{ik}\beta_{ij} = 0, \quad (\text{for } i \neq j \neq k \neq i), \tag{45}$$

where $\partial_1 \equiv \partial_x$, $\partial_2 = \partial_t$, $\partial_3 = \partial_y$, and

$$\left\langle \mathbf{a}_j \mid \mathbf{a}_k \right\rangle := \sum_{i=1}^{3} \alpha_{ij}\alpha_{ik}. \tag{46}$$

From (44) it follows that $\langle \mathbf{a}_1 \mid \mathbf{a}_1 \rangle$, $\langle \mathbf{a}_2 \mid \mathbf{a}_2 \rangle$, $\langle \mathbf{a}_3 \mid \mathbf{a}_3 \rangle$ are functions of one variable (resp., x, t, or y). These functions can be eliminated by a change of variables. Therefore without loss of the generality we can assume

$$\left\langle \mathbf{a}_1 \mid \mathbf{a}_1 \right\rangle = \left\langle \mathbf{a}_2 \mid \mathbf{a}_2 \right\rangle = \left\langle \mathbf{a}_3 \mid \mathbf{a}_3 \right\rangle = 1. \tag{47}$$

Solutions of the nonlinear system (44) and (45) have a direct connection with the geometry of triply orthogonal systems in $\mathbb{E}^3$. Namely, the metric

$$ds^2 = H_1^2 dx^2 + H_2^2 dt^2 + H_3^2 dy^2 \tag{48}$$

describes a triply orthogonal system of coordinates if and only if Lamé coefficients H_1, H_2, H_3 satisfy the Lamé equations [17, 18]:

$$\begin{aligned} \frac{\partial^2 H_1}{\partial y \partial t} &= \frac{1}{H_2}\frac{\partial H_2}{\partial y}\frac{\partial H_1}{\partial t} + \frac{1}{H_3}\frac{\partial H_3}{\partial t}\frac{\partial H_1}{\partial y}, \\ \frac{\partial^2 H_2}{\partial x \partial y} &= \frac{1}{H_3}\frac{\partial H_3}{\partial x}\frac{\partial H_2}{\partial y} + \frac{1}{H_1}\frac{\partial H_1}{\partial y}\frac{\partial H_2}{\partial x}, \\ \frac{\partial^2 H_3}{\partial t \partial x} &= \frac{1}{H_1}\frac{\partial H_1}{\partial t}\frac{\partial H_3}{\partial x} + \frac{1}{H_2}\frac{\partial H_2}{\partial x}\frac{\partial H_3}{\partial t}, \\ \frac{\partial}{\partial x}\left(\frac{1}{H_1}\frac{\partial H_2}{\partial x}\right) + \frac{\partial}{\partial t}\left(\frac{1}{H_2}\frac{\partial H_1}{\partial t}\right) + \frac{1}{H_3^2}\frac{\partial H_1}{\partial y}\frac{\partial H_2}{\partial y} &= 0, \\ \frac{\partial}{\partial t}\left(\frac{1}{H_2}\frac{\partial H_3}{\partial t}\right) + \frac{\partial}{\partial y}\left(\frac{1}{H_3}\frac{\partial H_2}{\partial y}\right) + \frac{1}{H_1^2}\frac{\partial H_2}{\partial x}\frac{\partial H_3}{\partial x} &= 0, \\ \frac{\partial}{\partial y}\left(\frac{1}{H_3}\frac{\partial H_1}{\partial y}\right) + \frac{\partial}{\partial x}\left(\frac{1}{H_1}\frac{\partial H_3}{\partial x}\right) + \frac{1}{H_2^2}\frac{\partial H_3}{\partial t}\frac{\partial H_1}{\partial t} &= 0. \end{aligned} \tag{49}$$

Defining the so called rotation coefficients [18]

$$\beta_{kj} = \frac{\partial_k H_j}{H_k} \tag{50}$$

one can show that system (45) is equivalent to (49).

The whole system of compatibility conditions, (44), (45), and (47), is equivalent to the Lamé equations (49) with the following constraint:

$$H_1^2 + H_2^2 + H_3^2 = 1. \tag{51}$$

It is sufficient to define (for any fixed j)

$$H_k = \alpha_{jk}. \tag{52}$$

Then, condition (51) follows from the orthogonality of the matrix α_{jk}. The Lamé system with constraint (51) was first considered by Darboux; see [17, book III, chapter X]. Modern approach to the Darboux transformations for this system can be found in [15, 19, 20] (in [20] the constraint (51) is not explicitly presented). Some other reductions of Lamé equations are discussed in [21]. Lamé equations without this constraint are also integrable; see [18, 22] and references quoted therein.

Throughout this section we used a convenient language of Clifford numbers. It is worthwhile noticing that system (41) evaluated at $\lambda = 0$ becomes equivalent to

$$\begin{aligned} \Psi_x &= A\Psi, \\ \Psi_t &= B\Psi, \\ \Psi_y &= C\Psi, \end{aligned} \tag{53}$$

where

$$\begin{aligned} A &= \begin{pmatrix} 0 & -\beta_{21} & -\beta_{31} \\ \beta_{21} & 0 & 0 \\ \beta_{31} & 0 & 0 \end{pmatrix} \longleftrightarrow \frac{1}{2}\beta_{21}\mathbf{e}_2\mathbf{e}_1 + \frac{1}{2}\beta_{31}\mathbf{e}_3\mathbf{e}_1, \\ B &= \begin{pmatrix} 0 & \beta_{12} & 0 \\ -\beta_{12} & 0 & -\beta_{32} \\ 0 & \beta_{32} & 0 \end{pmatrix} \longleftrightarrow \frac{1}{2}\beta_{12}\mathbf{e}_1\mathbf{e}_2 + \frac{1}{2}\beta_{32}\mathbf{e}_3\mathbf{e}_2, \\ C &= \begin{pmatrix} 0 & 0 & \beta_{13} \\ 0 & 0 & \beta_{23} \\ -\beta_{13} & -\beta_{23} & 0 \end{pmatrix} \longleftrightarrow \frac{1}{2}\beta_{13}\mathbf{e}_1\mathbf{e}_3 + \frac{1}{2}\beta_{23}\mathbf{e}_2\mathbf{e}_3. \end{aligned} \tag{54}$$

One can easily check that elements $(1/2)\mathbf{e}_2\mathbf{e}_3$, $(1/2)\mathbf{e}_3\mathbf{e}_1$, and $(1/2)\mathbf{e}_1\mathbf{e}_2$ generate the Lie algebra $su(2)$; compare (1). Compatibility conditions for (53) yield augmented Gauss-Mainardi-Codazzi system [1].

5. Soliton Surfaces

Soliton surfaces approach [5, 23] (see also [24]) can be easily applied to the analysis of deformation of surfaces in $\mathbb{E}^3$. Namely, having an $SU(2)$-valued Lax representation

$$\begin{aligned} \Psi_x &= U(\lambda)\Psi, \\ \Psi_t &= V(\lambda)\Psi, \\ \Psi_y &= W(\lambda)\Psi, \end{aligned} \tag{55}$$

such that $U(0) = A$, $V(0) = B$, $W(0) = C$, we define

$$\mathbf{r} = \Psi^{-1}\Psi_\lambda\big|_{\lambda=0}. \tag{56}$$

This is y-family of surfaces in $su(2) \simeq \mathbb{E}^3$. Soliton surfaces can be easily analysed on the implicit level (fundamental forms), which is one of the important advantages of this approach. Indeed, we compute

$$\begin{aligned} \mathbf{r}_x &= \Psi^{-1}U_\lambda\Psi, \\ \mathbf{r}_t &= \Psi^{-1}V_\lambda\Psi, \\ \mathbf{r}_y &= \Psi^{-1}W_\lambda\Psi, \\ \mathbf{r}_{xt} &= \Psi^{-1}\left(U_{\lambda t} + [U_\lambda, B]\right)\Psi, \text{ etc.} \end{aligned} \tag{57}$$

Therefore, in any particular case all fundamental forms can be calculated without the explicit knowledge of Ψ. Geometric interpretation often helps to understand trivial or degenerated cases. For linear problem (22) we get

$$\begin{aligned} \mathbf{r}_x &= \Psi^{-1}L\Psi, \\ \mathbf{r}_t &= \Psi^{-1}M\Psi, \\ \mathbf{r}_y &= \Psi^{-1}N\Psi. \end{aligned} \tag{58}$$

Therefore (23) simply mean that all tangent vectors are parallel. Hence, system (22) describes a curve rather than a family of deformed surfaces.

In the case of triply orthogonal systems with constraint (51) the soliton surfaces approach shows an unexpected feature: soliton submanifold (56) is immersed in a six-dimensional space. However, by making an appropriate projections on three-dimensional spaces we obtain three different triply orthogonal systems of coordinates in $\mathbb{E}^3$ [15], mutually related by the so called Combescure transformations.

Another advantage of this approach is a unification of several integrable models on the same soliton surface [25]. For instance, in the case of chiral model (4) we easily obtain

$$\begin{aligned} \mathbf{r}_x &= \Phi^{-1}A\Phi, \\ \mathbf{r}_t &= \Phi^{-1}B\Phi, \\ \mathbf{r}_{xt} &= \Phi^{-1}\left(A_t + [A, B]\right)\Phi. \end{aligned} \tag{59}$$

Hence, taking into account (2), we obtain

$$\mathbf{r}_{xt} = \frac{1}{2}\left[\mathbf{r}_x, \mathbf{r}_t\right]. \tag{60}$$

Thus the soliton surface of the principal chiral model is swept out by the motion of a relativistic string model [26]. Similar computation for other pairs of variables yields

$$\begin{aligned} \mathbf{r}_{xy} &= \frac{\kappa}{1-\kappa}\left[\mathbf{r}_x, \mathbf{r}_y\right], \\ \mathbf{r}_{yt} &= \frac{\kappa}{1+\kappa}\left[\mathbf{r}_y, \mathbf{r}_t\right]; \end{aligned} \tag{61}$$

that is, we get relativistic strings, as well.

6. Further Developments

Soliton surfaces approach was generalized on semisimple Lie groups [5] and Spin(p, q) groups [4, 15]. There are also further

developments with more stress on symmetries [24, 27, 28]. Lie point symmetries turned out to be very promising as a tool for inserting the spectral parameter into Gauss-Weingarten equations ("nonparametric linear problems") associated with surfaces immersed in $\mathbb{E}^3$ [6, 7, 29–32]. In other words, it turns out that in some cases the spectral parameter λ is a group parameter.

A natural next step consists in finding some integrable cases by studying Lie point symmetries for special systems of augmented Gauss-Mainardi-Codazzi equations (53). We hope to get new integrable cases. The work in this direction is in progress.

Competing Interests

The authors declare that they have no competing interests.

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