**Research Article**

**Description of the Magnetic Field and Divergence of Multisolenoid Aharonov-Bohm Potential**

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Explicit formulas for the magnetic field and divergence of multisolenoid Aharonov-Bohm potential are obtained; the mathematical essence of this potential is explained. It is shown that the magnetic field and divergence of this potential are very singular generalized functions concentrated at a finite number of thin solenoids. Deficiency index is found for the minimal operator generated by the Aharonov-Bohm differential expression.

1. Introduction

66 years have passed since the publication of Aharonov and Bohm’s “Significance of Electromagnetic Potential in the Quantum Theory” [1], and since then interest in this paper has never faded. According to Web of Science®-Google Scholar, it has been cited 5680 times (as of December 2014)! Note that there are plenty of both supporters and opponents of this work (see, e.g., [2, 3]).

The purpose of our work is to find explicit formulas for the magnetic field and divergence of multisolenoid Aharonov-Bohm potential and to explain the mathematical essence of it. The obtained formulas show (see Theorems 1 and 3) that the magnetic field and divergence of this potential are very singular generalized functions concentrated at a finite number of thin solenoids perpendicular to the plane $x_1Ox_2$.

2. Main Results

Let $\xi_k = (x_1^{(k)}, x_2^{(k)}), k = 1, 2, \ldots, n$, be pairwise distinct points in $R_2$, let $a_k : S_1(0) \rightarrow R_1, k = 1, 2, \ldots, n$, be real, bounded, and measurable functions on the unit circle $S_1(0) \subset R_2$, and $\Omega' = R_2 \setminus \{\xi_k, k = 1, 2, \ldots, n\}$. Define the magnetic Aharonov-Bohm potential as follows:

$$A(x) = \sum_{k=1}^{n} a_k \left( \frac{x - \xi_k}{|x - \xi_k|} \right) \frac{1}{|x - \xi_k|^2} \left( -x_2 + x_2^{(k)} , x_1 - x_1^{(k)} \right),$$

$$x = (x_1, x_2) \in \Omega',$$

where

$$|x - \xi_k| = \sqrt{(x_1 - x_1^{(k)})^2 + (x_2 - x_2^{(k)})^2}. \quad (2)$$

As far as we know, in all the earlier works (except for [4]) dedicated to the Aharonov-Bohm effect (for short, AB effect), the functions $a_k((x - \xi_k)/|x - \xi_k|), k = 1, 2, \ldots, n$, are constants.

The following theorems are true (in case $n = 1$ they were proved in [4]).
Theorem 1. Let the magnetic field $B = \nabla \times A$ be generated by the magnetic Aharonov-Bohm potential (1) in the sense of generalized functions. Then the following equality is true:

$$B = \nabla \times A = \sum_{k=1}^{n} \left[ \int_{-\pi}^{\pi} a_k(\theta) \, d\theta \right] \delta(x - \xi_k),$$

where $\delta(x - \xi_k), k = 1, 2, \ldots, n$, are the Dirac functions and $\nabla = (\partial/\partial x_1, \partial/\partial x_2)$ is the gradient operator.

Proof. Let

$$A(x) = (A_{x_1}, A_{x_2}),$$

where

$$A_{x_1} = \sum_{k=1}^{n} \left( \frac{x - \xi_k}{|x - \xi_k|^2} \right) x_2^{(k)} - x_2,$$

$$A_{x_2} = \sum_{k=1}^{n} \left( \frac{x - \xi_k}{|x - \xi_k|^2} \right) x_1 - x_1^{(k)}.$$  

Then the definition of magnetic field

$$B = \nabla \times A = \frac{\partial A_{x_1}}{\partial x_2} - \frac{\partial A_{x_2}}{\partial x_1}$$

implies that for every function $f(x) \in C^0_0(R_2)$ we have

$$\int_{R_2} B f(x) \, dx$$

$$= \int_{R_2} \left( \frac{\partial A_{x_1}}{\partial x_1} - \frac{\partial A_{x_2}}{\partial x_2} \right) f(x_1, x_2) \, dx_1 dx_2.$$  

Taking into account the identity

$$\left( \frac{\partial A_{x_1}}{\partial x_1} - \frac{\partial A_{x_2}}{\partial x_2} \right) f(x_1, x_2) = \frac{\partial}{\partial x_1} (A_{x_1} f) - \frac{\partial}{\partial x_2} (A_{x_2} f) - A_{x_1} \frac{\partial f}{\partial x_1} + A_{x_2} \frac{\partial f}{\partial x_2}$$

and the Green formula, we rewrite relation (7) as follows:

$$\int_{R_2} B f(x) \, dx$$

$$= \int_{R_2} \left( A_{x_1} \frac{\partial f}{\partial x_2} - A_{x_2} \frac{\partial f}{\partial x_1} \right) dx_1 dx_2.$$  

Hence, by virtue of (5), we get

$$\int_{R_2} B f(x) \, dx$$

$$= \int_{R_2} \sum_{k=1}^{n} \left[ \int_{R_2} a_k \left( \frac{x - \xi_k}{|x - \xi_k|^2} \right) \left[ x_2^{(k)} \frac{\partial f}{\partial x_2} - x_2 \frac{\partial f}{\partial x_1} \right] \, dx_1 dx_2 \right]$$

$$= \sum_{k=1}^{n} \left[ \int_{R_2} \frac{a_k}{|x - \xi_k|^2} \left[ x_2^{(k)} \frac{\partial f}{\partial x_2} - x_2 \frac{\partial f}{\partial x_1} \right] \, dx_1 dx_2 \right]$$

where

$$J_k(f) = \int_{R_2} \frac{a_k}{|x - \xi_k|^2} \left[ x_2^{(k)} \frac{\partial f}{\partial x_2} - x_2 \frac{\partial f}{\partial x_1} \right] \, dx_1 dx_2,$$

$$k = 1, 2, \ldots, n.$$  

Using the transformation of plane into itself defined by the formulas

$$t_1 = x_1 - x_1^{(k)},$$

$$t_2 = x_2 - x_2^{(k)},$$

and considering the equalities

$$\frac{\partial f}{\partial x_1} = \frac{\partial f(t_1 + x_1^{(k)}, t_2 + x_2^{(k)})}{\partial t_1},$$

$$\frac{\partial f}{\partial x_2} = \frac{\partial f(t_1 + x_1^{(k)}, t_2 + x_2^{(k)})}{\partial t_2},$$

we arrive at the following formula:

$$J_k(f) = \int_{R_2} \frac{a_k}{|t|^2} \left[ \frac{t_1}{|t|^2} \frac{\partial f(t_1 + x_1^{(k)}, t_2 + x_2^{(k)})}{\partial t_1} + \frac{t_2}{|t|^2} \frac{\partial f(t_1 + x_1^{(k)}, t_2 + x_2^{(k)})}{\partial t_2} \right] \, dt_1 dt_2,$$

$$k = 1, 2, \ldots, n.$$
After transition to polar coordinates
\[ t_1 = r \cos \theta, \]
\[ t_2 = r \sin \theta, \]
\[ r > 0, \quad -\pi < \theta \leq \pi \quad (t = r \left( \cos \theta, \sin \theta \right)), \] (15)

and using the equalities
\[
\frac{\partial f}{\partial t_1} = \frac{\partial}{\partial r} \left( r \cos \theta + x^{(k)}_1, r \sin \theta + x^{(k)}_2 \right) \cos \theta
\]
\[ = \frac{\partial f}{\partial r} \left( r \cos \theta + x^{(k)}_1, r \sin \theta + x^{(k)}_2 \right) \cos \theta
\]
\[
\frac{\partial}{\partial r} \left( r \cos \theta + x^{(k)}_1, r \sin \theta + x^{(k)}_2 \right) \cos \theta
\]
we get

\[ J_k(f) = \int_{0}^{+\infty} \int_{-\pi}^{\pi} a_k(\cos \theta, \sin \theta) \cos \theta \left\{ \frac{\partial f}{\partial r} \left( r \cos \theta + x^{(k)}_1, r \sin \theta + x^{(k)}_2 \right) \right\} \] (17)

Taking into account \( f(x) \in C^\infty_0(R_2) \) and denoting \( a_k(\cos \theta, \sin \theta) \equiv a_k(\theta) \), from (17) we have

\[ J_k(f) = -f(x^{(k)}_1, x^{(k)}_2) \int_{-\pi}^{\pi} a_k(\theta) \, d\theta \]
\[ = -f(\xi_k) \int_{-\pi}^{\pi} a_k(\theta) \, d\theta. \] (18)

The Dirac function \( \delta(x - \xi_k) \) acts as follows:
\[ (\delta(x - \xi_k), f(x)) = f(\xi_k). \] (19)

Then the functional defined by the right-hand side of (18) is a generalized function. Thus, formula (18) can be rewritten in the following way:

\[ J_k(f) = -\int_{-\pi}^{\pi} a_k(\theta) \, d\theta \delta(x - \xi_k), f(x) \]
\[ = -(\int_{-\pi}^{\pi} a_k(\theta) \, d\theta \delta(x - \xi_k), f(x)), \] (20)

Due to (20), equality (10) has the following form:

\[ (B, f(x)) \equiv \int_{R_2} B f(x) \, dx \]
\[ = \sum_{k=1}^{n} \left( \int_{-\pi}^{\pi} a_k(\theta) \, d\theta \delta(x - \xi_k), f(x) \right) \]
\[ = \left( \sum_{k=1}^{n} \int_{-\pi}^{\pi} a_k(\theta) \, d\theta \delta(x - \xi_k), f(x) \right). \] (21)

Consequently, we have

\[ B = \nabla \times A = \sum_{k=1}^{n} \int_{-\pi}^{\pi} a_k(\theta) \, d\theta \delta(x - \xi_k). \] (22)

The theorem is proved. □

Remark 2. The formula

\[ B = \sum_{k=1}^{n} \int_{-\pi}^{\pi} a_k(\theta) \, d\theta \delta(x - \xi_k) \] (23)
implies that if the condition

$$\int_{-\pi}^{\pi} a_k(\theta) \, d\theta = 0 \quad (24)$$

holds for every $k$ from $\{1, 2, \ldots, n\}$, then the AB effect is absent because the total magnetic flux of the magnetic field $B$ passing through the closed contour that covers all the points $\xi_k = (x_1^{(k)}, x_2^{(k)})$, $k = 1, 2, \ldots, n$, is equal to zero.

The conditions for both the presence and absence of the AB effect in multiply connected domains are thoroughly studied in [3, 5].

**Theorem 3.** Let the divergence $\nabla \cdot A$ be generated by the magnetic Aharonov-Bohm potential (1) in the sense of generalized functions. Then the following equality is true:

$$\nabla \cdot A = \sum_{k=1}^{n} V.p. \left\{ \frac{1}{|x - \xi_k|^2} \left[ -\frac{\partial a_k \left( \frac{x_1 - x_1^{(k)}}{|x - \xi_k|}, \frac{x_2 - x_2^{(k)}}{|x - \xi_k|} \right)}{\partial \left( \frac{x_1 - x_1^{(k)}}{|x - \xi_k|} \right)} \frac{x_2 - x_2^{(k)}}{|x - \xi_k|} \right] + \frac{\partial a_k \left( \frac{x_1 - x_1^{(k)}}{|x - \xi_k|}, \frac{x_2 - x_2^{(k)}}{|x - \xi_k|} \right)}{\partial \left( \frac{x_2 - x_2^{(k)}}{|x - \xi_k|} \right)} \frac{x_1 - x_1^{(k)}}{|x - \xi_k|} \right\}, \quad k = 1, 2, \ldots, n \quad (25)$$

where

$$V.p. \left\{ \frac{1}{|x - \xi_k|^2} \left[ -\frac{\partial a_k \left( \frac{x_1 - x_1^{(k)}}{|x - \xi_k|}, \frac{x_2 - x_2^{(k)}}{|x - \xi_k|} \right)}{\partial \left( \frac{x_1 - x_1^{(k)}}{|x - \xi_k|} \right)} \frac{x_2 - x_2^{(k)}}{|x - \xi_k|} \right] + \frac{\partial a_k \left( \frac{x_1 - x_1^{(k)}}{|x - \xi_k|}, \frac{x_2 - x_2^{(k)}}{|x - \xi_k|} \right)}{\partial \left( \frac{x_2 - x_2^{(k)}}{|x - \xi_k|} \right)} \frac{x_1 - x_1^{(k)}}{|x - \xi_k|} \right\}, \quad k = 1, 2, \ldots, n \quad (26)$$

are singular generalized functions; the letters $V.p.$ mean “Cauchy principal value.”

**Proof.** Let $f(x) \in C_0^\infty(R_2)$. Then, by the definition of the derivative of generalized function, using formulas (5), we have

$$\nabla \cdot A = \int_{R_2} \left( \frac{\partial A_{x_1}}{\partial x_1} + \frac{\partial A_{x_2}}{\partial x_2} \right) f(x) \, dx = -\int_{R_2} \left( A_{x_1} \frac{\partial f(x)}{\partial x_1} + A_{x_2} \frac{\partial f(x)}{\partial x_2} \right) dx$$

$$= -\sum_{k=1}^{n} \lim_{\delta \to 0} \int_{|x-\xi_k|>\delta} a_k \left( \frac{x_1 - x_1^{(k)}}{|x - \xi_k|}, \frac{x_2 - x_2^{(k)}}{|x - \xi_k|} \right) \left[ -\frac{x_2 + x_2^{(k)} \frac{\partial f(x_1, x_2)}{\partial x_1}}{|x - \xi_k|^2} + \frac{x_1 - x_1^{(k)} \frac{\partial f(x_1, x_2)}{\partial x_2}}{|x - \xi_k|^2} \right] dx_1 \, dx_2 \quad (27)$$

where

$$I_{k,\delta}(f) = \int_{|x-\xi_k|>\delta} a_k \left( \frac{x_1 - x_1^{(k)}}{|x - \xi_k|}, \frac{x_2 + x_2^{(k)} \frac{\partial f(x_1, x_2)}{\partial x_1}}{|x - \xi_k|^2} + \frac{x_1 - x_1^{(k)} \frac{\partial f(x_1, x_2)}{\partial x_2}}{|x - \xi_k|^2} \right) dx_1 \, dx_2, \quad k = 1, 2, \ldots, n \quad (28)$$

Using substitutions (12) and (15) and formulas (13) and (16), we obtain
\[ I_{k,\delta}(f) = \int_{|t| \geq \delta} a_k \left( \frac{t}{|t|} \right) \left[ \frac{-t_2}{|t|^2} \frac{\partial f}{\partial t_1} \left( t_1 + x_1^{(k)}, t_2 + x_2^{(k)} \right) + \frac{t_1}{|t|^2} \frac{\partial f}{\partial t_2} \left( t_1 + x_1^{(k)}, t_2 + x_2^{(k)} \right) \right] dt_1 dt_2 = \int_\delta^\infty \int_{-\pi}^\pi a_k(\cos \theta, \sin \theta) \]

\[ \cdot \left[ -\sin \theta \left[ \frac{\partial f(\cos \theta + x_1^{(k)}, \sin \theta + x_2^{(k)})}{\partial \theta} \cos \theta - \frac{\partial f(\cos \theta + x_1^{(k)}, \sin \theta + x_2^{(k)})}{\partial \theta} \sin \theta \right] + \cos \theta \right] \]

\[ \left[ \frac{\partial f(\cos \theta + x_1^{(k)}, \sin \theta + x_2^{(k)})}{\partial \theta} \sin \theta + \frac{\partial f(\cos \theta + x_1^{(k)}, \sin \theta + x_2^{(k)})}{\partial \theta} \cos \theta \right] \right] dr d\theta \]

\[ = \int_\delta^\infty \int_{-\pi}^\pi a_k(\theta) \frac{\partial f(\cos \theta + x_1^{(k)}, \sin \theta + x_2^{(k)})}{\partial \theta} \frac{1}{r} dr d\theta = \int_\delta^\infty \frac{1}{r} \int_{-\pi}^\pi a_k(\theta) \frac{\partial f(\cos \theta + x_1^{(k)}, \sin \theta + x_2^{(k)})}{\partial \theta} d\theta d\theta, \]

\[ = -\int_\delta^\infty \frac{1}{r} \int_{-\pi}^\pi a_k'(\theta) f(\cos \theta + x_1^{(k)}, \sin \theta + x_2^{(k)}) d\theta d\theta = \int_\delta^\infty \frac{1}{r} \int_{-\pi}^\pi a_k(\theta) \frac{\partial f(\cos \theta + x_1^{(k)}, \sin \theta + x_2^{(k)})}{\partial \theta} d\theta, \]

\( k = 1, 2, \ldots, n. \)

Now, to express \( a_k'(\theta) \) in Cartesian coordinates \( x_1 \) and \( x_2 \), we put

\[ M_k(x_1, x_2) \equiv a_k \left( \frac{x_1 - x_1^{(k)}}{|x - \xi_k|}, \frac{x_2 - x_2^{(k)}}{|x - \xi_k|} \right) \]

\[ = a_k(\cos \theta, \sin \theta) = a_k(\theta), \quad k = 1, 2, \ldots, n. \)

Having solved the system of equations

\[ \frac{\partial M_k(x_1, x_2)}{\partial x_1} = \frac{\partial M_k}{\partial \theta} \cos \theta - \frac{\partial M_k}{\partial \theta} \sin \theta, \quad k = 1, 2, \ldots, n, \]

\[ \frac{\partial M_k(x_1, x_2)}{\partial x_2} = \frac{\partial M_k}{\partial \theta} \sin \theta + \frac{\partial M_k}{\partial \theta} \cos \theta, \quad k = 1, 2, \ldots, n, \]

\[ a_k'(\theta) = \left[ \frac{\partial a_k \left( \frac{x_1 - x_1^{(k)}}{|x - \xi_k|}, \frac{x_2 - x_2^{(k)}}{|x - \xi_k|} \right)}{\partial \left( \frac{x_1 - x_1^{(k)}}{|x - \xi_k|} \right)} \left( \frac{-\left( x_1 - x_1^{(k)} \right) \left( x_2 - x_2^{(k)} \right)}{|x - \xi_k|^3} \right) \right] \]

\[ + \left[ \frac{\partial a_k \left( \frac{x_1 - x_1^{(k)}}{|x - \xi_k|}, \frac{x_2 - x_2^{(k)}}{|x - \xi_k|} \right)}{\partial \left( \frac{x_2 - x_2^{(k)}}{|x - \xi_k|} \right)} \right] \left( \frac{1}{|x - \xi_k|} - \frac{\left( x_2 - x_2^{(k)} \right)^2}{|x - \xi_k|^3} \right) \]

\[ - \left[ \frac{\partial a_k \left( \frac{x_1 - x_1^{(k)}}{|x - \xi_k|}, \frac{x_2 - x_2^{(k)}}{|x - \xi_k|} \right)}{\partial \left( \frac{x_1 - x_1^{(k)}}{|x - \xi_k|} \right)} \right] \left( \frac{1}{|x - \xi_k|} - \frac{\left( x_1 - x_1^{(k)} \right)^2}{|x - \xi_k|^3} \right) \]

\[ + \left[ \frac{\partial a_k \left( \frac{x_1 - x_1^{(k)}}{|x - \xi_k|}, \frac{x_2 - x_2^{(k)}}{|x - \xi_k|} \right)}{\partial \left( \frac{x_2 - x_2^{(k)}}{|x - \xi_k|} \right)} \right] \left( \frac{-\left( x_1 - x_1^{(k)} \right) \left( x_2 - x_2^{(k)} \right)}{|x - \xi_k|^3} \right) \]

Differentiating the composite function \( M_k(x_1, x_2) \) in \( x_1 \) and \( x_2 \) and using formula (32), we find

\[ a_k'(\theta) = \frac{\partial M_k}{\partial x_1} \cos \theta - \frac{\partial M_k}{\partial x_1} \sin \theta = r \left( \frac{\partial M_k(x_1, x_2)}{\partial x_1} \cos \theta - \frac{\partial M_k(x_1, x_2)}{\partial x_2} \sin \theta \right) \]
\[
\frac{\partial a_k}{\partial \left( \frac{x_1 - x_1^{(k)}}{|x - \xi_k|} \right)} + \frac{\partial a_k}{\partial \left( \frac{x_2 - x_2^{(k)}}{|x - \xi_k|} \right)} \cdot \frac{x_1 - x_1^{(k)}}{|x - \xi_k|}, \quad k = 1, 2, \ldots, n.
\]

Passing to the limit in (29) as \( \delta \to 0 \) and taking into account (33), we obtain

\[
\lim_{0<\delta \to 0} I_{k,\delta} (f) = -V_p \int_{R_2} \left\{ \frac{1}{|x - \xi_k|^2} \left[ -\frac{\partial a_k}{\partial \left( \frac{x_1 - x_1^{(k)}}{|x - \xi_k|} \right)} + \frac{\partial a_k}{\partial \left( \frac{x_2 - x_2^{(k)}}{|x - \xi_k|} \right)} \right] f(x) \, dx \right\}, \quad k = 1, 2, \ldots, n.
\]

It is seen from (27) and (34) that the following equality is true for every \( f(x) \in C_0^\infty (R_2) \):

\[
\begin{align*}
\text{(div}A, f(x)) & = \sum_{k=1}^n V_p \int_{R_1} \left\{ \frac{1}{|x - \xi_k|^2} \left[ -\frac{\partial a_k}{\partial \left( \frac{x_1 - x_1^{(k)}}{|x - \xi_k|} \right)} + \frac{\partial a_k}{\partial \left( \frac{x_2 - x_2^{(k)}}{|x - \xi_k|} \right)} \right] f(x) \, dx \right\}, \\
& = \sum_{k=1}^n V_p \int_{R_1} \left[ \frac{1}{|x - \xi_k|^2} \right] \left\{ \frac{\partial a_k}{\partial \left( \frac{x_1 - x_1^{(k)}}{|x - \xi_k|} \right)} + \frac{\partial a_k}{\partial \left( \frac{x_2 - x_2^{(k)}}{|x - \xi_k|} \right)} \right\} f(x) \, dx.
\end{align*}
\]

Thus, the following equality is true in the sense of generalized functions:

\[
\text{div}A = \sum_{k=1}^n V_p \left[ \frac{1}{|x - \xi_k|^2} \left[ \frac{\partial a_k}{\partial \left( \frac{x_1 - x_1^{(k)}}{|x - \xi_k|} \right)} + \frac{\partial a_k}{\partial \left( \frac{x_2 - x_2^{(k)}}{|x - \xi_k|} \right)} \right] \right].
\]

The theorem is proved. \( \square \)

Screening every thin solenoid \( \xi_k = (x_1^{(k)}, x_2^{(k)}, x_3) \) \( (k = 1, 2, \ldots, n) \), with the use of Dirac function \( \delta(x - \xi_k) \) \( (k = 1, 2, \ldots, n) \), we obtain a multcenter Schrödinger operator

\[
(i\mathbf{\nabla} + A(x))^2 - b_1 \delta (x - \xi_1) - b_2 \delta (x - \xi_2) - \cdots - b_n \delta (x - \xi_n), \quad (37)
\]

with the magnetic Aharonov-Bohm potential of type (1), where \( b_k \)'s \( (k = 1, 2, \ldots, n) \) are real numbers.

Consider in \( L_2(R_2) \) the symmetric operator \( H_0 \) with the domain \( D(H_0) = C_0^\infty (R_2 \setminus \{ \xi_1, \xi_2, \ldots, \xi_n \}) \) \( (C_0^\infty (\Omega') \) is the totality of all infinitely differentiable finite functions in \( \Omega' \), which acts as follows:

\[
H_0 \psi (x) = (i\mathbf{\nabla} + A(x))^2 \psi (x), \quad \psi (x) \in C_0^\infty (R_2 \setminus \{ \xi_1, \xi_2, \ldots, \xi_n \}).
\]
We denote by $H$ the closure of the operator $H_0$. Let
\begin{equation}
\int_{-\pi}^{\pi} a_k(\theta)d\theta = \bar{a}_k + \alpha_k, \quad k = 1, 2, \ldots, n, \tag{39}
\end{equation}
where $\bar{a}_k$ is the integral part and $\alpha_k$ is the fractional part of the number $\int_{-\pi}^{\pi} a_k(\theta)d\theta$. Obviously, $0 \leq \alpha_k < 1$, $k = 1, 2, \ldots, n$. Without loss of generality, we will assume that there exists an integer $l \leq n$ such that
\begin{equation}
0 < \alpha_j < 1, \quad \text{if } j = 1, 2, \ldots, l, \\
\alpha_j = 0, \quad \text{if } j = l + 1, l + 2, \ldots, n. \tag{40}
\end{equation}

**Theorem 4.** (i) The domain $D(H_0^*)$ of the conjugate operator $H_0^*$ coincides with the set
\begin{equation}
D(H_0^*) = \left \{ \psi(x) : \psi(x) \in L_2(R_2), \right. \\
\left. \nabla^2_{\text{loc}}(\Omega'), (i\nabla + A(x))^2 \psi(x) \in L_2(R_2) \right \}, \tag{41}
\end{equation}
where $W^2_{\text{loc}}(\Omega')$ is a local second-order Sobolev space.

(ii) Deficiency index of the operator $H$ is $(2l, 2l)$, where $l$ is an integer ($l \leq n$) defined in (40).

**Proof.** (i) As the domain of the operator $H_0$ is dense in $L_2(R_2)$, it has a conjugate operator $H_0^*$. The domain of this conjugate operator $D(H_0^*)$ is the totality of all $\psi(x)$ from $L_2(R_2)$ for which there exist $u(x) \in L_2(R_2)$ such that
\begin{equation}
(H_0\psi(x), \psi(x)) = (\varphi(x), u(x)), \tag{42}
\end{equation}
for every $\varphi(x) \in D(H_0)$, and $H_0\psi(x) = u(x)$. From
\begin{equation}
(H_0\phi(x), \psi(x)) = (\varphi(x), H_0^* \psi(x)), \tag{43}
\end{equation}
it follows that $u(x) = (i\nabla + A(x))^2 \psi(x)$ in the sense of generalized functions in $C_0^\infty(\Omega')$. Hence, in view of the ellipticity of the operator $(i\nabla + A(x))^2$, we have $\psi(x) \in W^2_{\text{loc}}(\Omega')$ (see [6]).

(ii) Considering the notations
\begin{equation}
A_1(x) = \sum_{k=1}^l a_k \left( \frac{x - \xi_k}{|x - \xi_k|} \right)^2 (x_1 - x_1^{(k)}), \tag{44}
\end{equation}
in (1), we rewrite the potential $A(x)$ in the form of the sum of two summands:
\begin{equation}
A(x) = A_1(x) + A_2(x). \tag{45}
\end{equation}

Now we introduce the magnetic $l$-flux potential
\begin{equation}
B(x) = \sum_{k=1}^l \frac{a_k}{|x - \xi_k|^2} (x_2 + x_2^{(k)}, x_1 - x_1^{(k)}), \tag{46}
\end{equation}
where $\alpha_k$ is the fractional part of the number $\int_{-\pi}^{\pi} a_k(\theta)d\theta$.

It is proved in [7] that the minimal operator $H_{0,B}$ generated by the differential expression $(i\nabla + B(x))^2$ has the deficiency index $(2l, 2l)$. It follows from the results of [7, 8] that $A_1(x) \sim B(x)$ and $A_2(x) \sim 0$; that is, the pairs of potentials $(A_1(x), B(x))$ and $(A_2(x), 0)$ are gauge equivalent. Consequently, the assertion (ii) of the theorem follows from the gauge equivalence of the potentials $A(x)$ and $B(x)$. The theorem is proved.

**Remark 5.** The assertions of Theorem 4 stay true if the Aharonov-Bohm solenoids lie in a homogeneous magnetic field of intensity $\gamma$, that is, for potentials of the form
\begin{equation}
A(x) + \left( \frac{\gamma}{2} x_2, \frac{\gamma}{2} x_1 \right). \tag{47}
\end{equation}

Now let us make a few concluding remarks about the mathematical justification for the AB effect. Proceeding from Berezin and Faddeev’s idea (see [9]), we arrive at the conclusion that the rigorous mathematical justification for the Aharonov-Bohm effect is that the pure Aharonov-Bohm operator $H_{AB}$ lies among the self-adjoint extensions of the operator $H_0$; that is,
\begin{equation}
H_0 \subset H_{AB} \subset H_0^*. \tag{48}
\end{equation}

For local and nonlocal $\delta$-interactions without magnetic field this idea was confirmed in many works (see, e.g., [10–13]), while for the Aharonov-Bohm operator it was confirmed in [7, 8, 14]. So the following question remains open for the potential of form (1): which of the self-adjoint extensions of the operator $H_0$ corresponds to the pure Aharonov-Bohm operator $H_{AB}^*$?

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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