Research Article
Firm Growth Function and Extended-Gibrat’s Property

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We analytically show that the logarithmic average sales of firms first follow power-law growth and subsequently follow exponential growth, if the growth-rate distributions of the sales obey the extended-Gibrat’s property and Gibrat’s law. Here, the extended-Gibrat’s property and Gibrat’s law are statistically observed in short-term data, which denote the dependence of the growth-rate distributions on the initial values. In the derivation, we analytically show that the parameter of the extended-Gibrat’s property is identical to the power-law growth exponent and that it also decides the parameter of the exponential growth. By employing around one million bits of exhaustive sales data of Japanese firms in the ORBIS database, we confirmed our analytic results.

1. Introduction

As in natural science, statistical laws are also frequently observed in social science. In econophysics, the statistical laws observed in the behavior of people and firms and their universality have been thoroughly investigated [1, 2]. The statistical laws observed in various social quantities at a point in time are especially well known. One of the main arguments is power-law distribution [3–23]. The power-law distributions of firm-size variables $x_T$ in calendar year $T$ (e.g., sales, assets, and number of employees) over size threshold $x_{th}$ have been observed in a number of years and countries as follows:

$$P(x_T) \propto x_T^{-\mu-1} \quad \text{for} \quad x_T > x_{th}. \quad (1)$$

Here, $P(x_T)$ is the probability density function (PDF) of $x_T$. Exponent $\mu$ is called Pareto’s index [3]. At the same time, mid-sized variables under size threshold $x_{th}$ have also frequently been described by log-normal distribution.

Short-term statistical laws have also been investigated, which were observed in firm-size variables at two successive points in time $(x_T, x_{T+1})$: (quasi-)inverse symmetry and (non-)Gibrat’s law. Inverse symmetry denotes that the system is static and symmetric under the time reversal exchange of variables $x_T \leftrightarrow x_{T+1}$ [24, 25]. Quasi-inverse symmetry means that the system is quasi-static and can be expressed as symmetric under the exchange of variables $x_T \leftrightarrow ax_{T+1}^\theta$, where $a$ and $\theta$ are parameters [26, 27]. Gibrat’s law suggests that conditional PDF $Q(R \mid x_T)$ of the growth rate defined by $R = x_{T+1}/x_T$ is independent of initial value $x_T$ [24, 25]. This law is observed in the large-scale range over size threshold $x_{th}$. Non-Gibrat’s law, which reflects the dependence of growth-rate distribution $Q(R \mid x_T)$ on initial value $x_T$, is observed in the mid-scale range under $x_{th}$ [28–31].

The statistical laws, which are observed in firm-size variables $x_T$ at a point in time, are related to the statistical laws observed in them at two successive points in time $(x_T, x_{T+1})$. Fujiwara et al. showed that the power law (1) observed in large-sized variables is derived from Gibrat’s law [32, 33] under inverse symmetry [24, 25]. The log-normal distribution observed in mid-sized variables is inferred from non-Gibrat’s law under inverse symmetry [28–31]. Quasi-statically varying power-law and log-normal distributions are derived from Gibrat’s and non-Gibrat’s laws under quasi-inverse symmetry, respectively [26, 27].
At the same time, interesting studies, related to long-term statistical laws, have been reported in empirical data analyses of the network of firms. Miura et al. showed that the number of business connections of firms exponentially increases as they age [34]. Mizuno et al. reported that the sales of firms depend on the number of business connections [35]. Such observations suggest that the sales of firms exponentially grow with age. In this context, the emergence of an early power-law growth is natural for firms that started small. Small firms must grow rapidly to reach the trajectory of exponential growth. This result is probably related to Luttmer’s report that the number of employees $L$ grows rapidly in the beginning but soon loses momentum [36]. This can be interpreted as power-law growth. Besides firm-size variables, Petersena et al. observed power-law growth in the progress of researchers defined by the number of publications of scientific papers [37].

In this study, we propose a law of the newly formulated firm growth derived from short-term statistical laws. If there are laws in firm growth [38], they are long-term statistical laws that are different from statistical laws in the short-term. However, these laws might be related. We verify this conjecture in the rest of our paper, which is organized as follows. The law of firm growth must be related to non-Gibrat’s and Gibrat’s laws that describe the change of the growth-rate distributions that are dependent on the initial values. In the next section, we review Gibrat’s law and introduce an extended-Gibrat’s property instead of non-Gibrat’s law. We also analytically derive firm growth that first obeys the power-law function and subsequently follows an exponential function. After that, using our database, we demonstrate that firm growth is analytically derived and observed in the empirical data. The last section concludes this paper.

2. Results

2.1. Gibrat’s Law and Extended-Gibrat’s Property. Gibrat’s law is represented using conditional PDF $Q(R | x_T)$ as follows:

$$Q(R | x_T) = Q(R) \text{ for } x_T > x_{th}. \quad (2)$$

Here, $Q(R | x_T)$ does not depend on initial value $x_T$. This Gibrat’s law is observed in a range over size threshold $x_{th}$. On the other hand, under threshold $x_{th}$, $Q(R | x_T)$ does depend on $x_T$. For the case of sales [31], the positive growth-rate distributions gradually decrease as $x_T$ increases, and the negative growth-rate distributions hardly change. Using conditional PDF $q(r | x_T)$ of logarithmic growth rate $r = \log_10 R$ which is related to $Q(R | x_T)$ by $q(r | x_T) = \ln 10 Q(R | x_T)$, this property can be expressed as

$$q(r | x_T) = \frac{f(r)}{x_T^{-\gamma}} \text{ for } r > 0, \quad (3)$$

$$q(r | x_T) = g(r) \text{ for } r < 0, \quad (4)$$

where $\gamma$ is a constant. In this paper, we call (3) and (4) the extended-Gibrat’s property, which is different from the previously examined non-Gibrat’s law [30, 31]. Later, we confirm the extended-Gibrat’s property with empirical data.

2.2. Analytical Derivation of Firm Growth. Next, we analytically derive firm growth from the extended-Gibrat’s property ((3) and (4)) and from Gibrat’s law (2). To estimate firm growth, it is convenient to identify a starting point in time for every firm. We consider a firm’s incorporation as its starting point and consider its age from its year of foundation. For simplicity, we examine the average values instead of the distribution of variables and denote the logarithmic average value of the firm-size variable at firm age $t$ as $x_T$. The foundation year is $t = 1$.

We define the growth rate from $x_T$ to $x_{T+1}$ as

$$R(x_T) = \frac{x_{T+1}}{x_T}. \quad (5)$$

The dependence in growth rate $R$ on $x_T$ is significant in the extended-Gibrat’s property. We hypothesize that Gibrat’s law and the extended-Gibrat’s property, which are observed in two successive calendar years $(T, T + 1)$, are also valid in two successive firm ages $(t, t + 1)$. Therefore, in the following discussions, calendar year $T$ in $Q(R | x_T)$ can be replaced by firm age $t$.

In extended-Gibrat’s property range $x_{th} < x_T$, using (3) and (4), the average value of $r$ is expressed as

$$r(x_T) = \frac{1}{x_T^{1/\gamma}} \int_{0}^{x_T} dr q(r) + \int_{-\infty}^{0} dr g(r) \quad (6)$$

$$= \frac{A}{x_T^{1/\gamma}} + B,$$

where integrations $\int_{0}^{x_T} dr q(r)$ and $\int_{-\infty}^{0} dr g(r)$ are assumed to converge and are denoted as $A$ and $B$, respectively. In (6), the dependence of $r(x_T)$ on $x_T$ is important. When the second term is negligible compared with the first term in (6), $r(x_T)$ is approximated by

$$\log_{10} r(x_T) = -\frac{1}{\gamma} \log_{10} x_T + \log_{10} A + \frac{1}{\gamma} \log_{10} D. \quad (7)$$

Here, $\log_{10} A = \log_{10} y + (1/\gamma) \log_{10} D$. Equation (7) can be rewritten as follows:

$$r(x_T) = y \left( \frac{D}{x_T} \right)^{1/\gamma}. \quad (8)$$

By combining (5) and (8), we obtain the following recurrence formula:

$$x_{T+1} = x_T 10^{y(D/x_T)^{1/\gamma}}. \quad (9)$$

When $t$ is sufficiently larger than 1, (9) has a solution:

$$x_T = (\ln 10)^y D^{1/y}. \quad (10)$$

We used Maclaurin expansion as $\ln(1 + 1/t) \sim 1/t$. 
Advances in Mathematical Physics

Figure 1: Conditional PDFs $q(r \mid x_T)$ of logarithmical sales growth rate $r = \log_{10} x_{T+1}/x_T$ in logarithmically equal-sized bins $x_T \in [10^{0.5(n-1)}, 10^{0.5n}]$ ((a) $n = 1, 2, \ldots, 6$; (b) $n = 7, 8, \ldots, 12$) of Japanese firms in ORBIS database. Here, $x_T$ and $x_{T+1}$ are sales in consecutive years $T = 2007$ in thousands of US dollars.

In Gibrat's law range $x_t > x_{th}$, the average value of $r$ is a constant: $\gamma (D/x_{th})^{1/\gamma}$. Therefore, the recurrence formula takes the following form:

$$x_{t+1} = x_t C,$$

where $C = 10^{(D/x_{th})^{1/\gamma}}$ is a constant. Equation (11) has a solution:

$$x_t = C^t \exp[\beta t].$$

In (12), $C^t$ is a constant and $\beta$ is given by

$$\beta = \gamma \left( \frac{D}{x_{th}} \right)^{1/\gamma} \ln 10.$$

Consequently, using the extended-Gibrat's property and Gibrat's law, we show that the growth of the logarithmic average of sales $x_t$ first follows a power-law function when $t \gg 1$ under $x_{th}$ and next follows an exponential function above $x_{th}$. In this discussion, the parameter of the extended-Gibrat's property $\gamma$ is identified by the power-law exponent, and $\gamma$ also decides the parameter of exponential growth $\beta$.

2.3. Data Analysis. We employ the ORBIS database, provided by Bureau van Dijk [39], which contains around 150 million pieces of firm-size data from all over the world. In the database, we analyze around one million pieces of Japanese firm-size data for 2007 and 2008. Since the number of active firms in Japan is estimated to be around one million [40], this database is exhaustive. We denote the sales data of Japanese firms from 2007 and 2008 as $(x_T, x_{T+1})$ with $T = 2007$.

First, we confirm Gibrat's law (2) and the extended-Gibrat's property ((3) and (4)). Figure 1 depicts the conditional PDFs $q(r \mid x_T)$ of logarithmical sales growth $r$ in logarithmically equal-sized bins $x_T \in [10^{0.5(n-1)}, 10^{0.5n}]$ ($n = 1, 2, \ldots, 12$) with $\gamma = 3.3$ in Figure 1(a).

Figure 2: Conditional PDFs $q(r \mid x_T)$ times $x_T^{1/\gamma}$ with $\gamma = 3.3$ in Figure 1(a).
In the analytical derivation of firm growth, we assumed that the shapes of Figures 1, 2, and 3 resemble other years that are not $T = 2007$ and that Gibrat’s law and the extended-Gibrat’s property are valid in two successive firm ages $(t, t+1)$. Confirming this assumption in the examined ORBIS database is difficult, because it does not include long-term data. For the same reason, the growth of firms cannot be observed directly by tracing the history of the sales of each firm. However, the database contains not only firm-size data but also the incorporation years of firms. Therefore, the growth of the average value of sales $x_t$ (10) and (12)) that are analytically derived can be approximately observed under the above assumption.

Third, using these data, we can observe the firm growth derived in the previous subsection. In Figure 4(a), by classifying the sales of Japanese firms in 2007 that are included in the database into age-range bins with a 1-year width, the logarithmic averages of sales $x_t$ in each bin $(t = 1, 2, \ldots, 50)$ are depicted by circles in a single logarithmic plot. Similarly, the logarithmic averages of sales in 2008 in Japan are also depicted by crosses. In this figure, to adjust the growth in 2007 to the growth in 2008, sales $x_t$ in 2007 are divided by the total amount of sales in 2007 and multiplied by the total sales in 2008. In Figure 4(b), the logarithmic averages are plotted in a log-log scale. In Figure 4, the dashed line is an optimal power-law function, and the dotted line is an optimal exponential function. These figures show that, under $x_{th} \sim 10^3$, $x_t$ is well fitted by the power-law function (10). At the same time, over $x_{th} \sim 10^3$, $x_t$ is well fitted by an exponential function (12). Using the least square method, the parameters of exponential growth $\beta$ and power-law growth $\gamma$ are estimated by $\beta = 0.022 \pm 0.006$ and $\gamma = 0.89 \pm 0.05$, respectively. These parameters that are estimated by long-term data are not close to the parameters of the short-term data. This discrepancy probably comes from the breakdown of the assumption that the growth rate of sales with respect to firm age was always distributed in a similar shape between 2007 and 2008. At the same time, in Figure 4, the power-law growth (10) is not smoothly continuous to the exponential growth (12) because of the distortion of sales from ages 15 to 20. This corresponds to Japan’s bubble economy from 1987 to 1992.

3. Discussion

In this study, we analytically showed that the logarithmic average sales of firms first follow a power-law growth and subsequently an exponential growth, if the growth-rate distributions of the sales obey the extended-Gibrat’s property and Gibrat’s law. Here, the extended-Gibrat’s property and Gibrat’s law are statistically observed in the short-term data for two successive years, which denote the dependence of the growth-rate distributions on the initial values. By employing the exhaustive sales data of Japanese firms in 2007 and 2008 that are contained in the ORBIS database, first, we confirmed the extended-Gibrat’s property and Gibrat’s law.
At the same time, we checked the consistency of the analytic derivation of firm growth by comparing the parameters of the extended-Gibrat’s property, Gibrat’s law, the power-law growth, and the exponential growth in short-term data. After that, we investigated the average firm growth of sales in 2007 and 2008 from Japan with respect to firm ages using the firm incorporation years in the database. We observed early power-law growth and subsequent exponential growth.

In the analytic derivation of firm growth, we did not assume inverse symmetry [24, 25] or non-Gibrat’s law [28–31]. Instead, we proposed an extended-Gibrat’s property under size threshold $x_{th}$, which is important to power-law growth. Gibrat’s law over $x_{th}$ is also important to exponential growth. Similar growths are known as the S-curve of a product. In S-curves, expansion periods correspond to power-law growth and later maturity periods correspond to exponential growth. The average growth rate of sales definitely exceeds 1 even in a Gibrat’s law range where $x_t > x_{th}$. This feature is different from the growth of a product, which follows an S-curve.

It is fascinating that the extended-Gibrat’s property and Gibrat’s law, which are observed in short-term statistical data, can explain both power-law and exponential growth, which are long-term statistical laws. The boundary between Gibrat’s and the extended-Gibrat’s ranges presented in this study $x_{th}$ is determined in all kinds of industries and firms in Japan. Perhaps it differs with respect to kinds of industries and other factors. In this case, mega firms might exist, which grow under a power law beyond average boundary $x_{th}$.

The discussions in this paper, constructed by analyzing sales data, are applicable to other firm-size variables, such as assets, number of employees, and positive profits. The results obtained here can also be applied to GDP growth analyses. The growth-rate distributions of GDP, which follow the extended-Gibrat’s property, decide a GDP’s power-law growth and exponential growth. This is not only an intriguing theme but also a significant one for such advanced countries that are suffering from sluggish GDP growth as Japan and the United States.

Competing Interests

The authors declare that they have no competing interests.

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