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Research Article

A Lipschitz Stability Estimate for the Inverse Source Problem and the Numerical Scheme

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We consider the inverse source problem for heat equation, where the source term has the form $f(t)\phi(x)$. We give a numerical algorithm to compute unknown source term $f(t)$. Also, we give a stability estimate in the case that $f(t)$ is a piecewise constant function.

1. Introduction to the Problem

The inverse source problems in heat equation are solving the heat equation with unknown heat sources. They are well known to be ill-posed. There were many studies to identify different types of heat sources. For example, Cannon and DuChateau [1] estimated the nonlinear temperature dependent heat source. In [2, 3], the method of fundamental solutions has been presented for an inverse time-dependent heat source problem. In [4–10], several numerical schemes have been proposed to determine a space-dependent heat source. In [11–13], these numerical methods were considered to solve some two- and three-dimensional inverse heat source problems. In Savateev [14], Trong et al. [15], and Shidfar et al. [16], the authors considered the source as a function of both space and time, but it was additive or separable.

Throughout this paper, we will assume that Ω is an open set in R^n. We consider the following initial-boundary value problem:

$$\begin{aligned} u_t &= \Delta u + f(t)\phi(x) \quad (x,t) \in \Omega \times (0,T), \\ u(x,0) &= 0 \quad x \in \Omega, \\ u(x,t) &= 0 \quad (x,t) \in \partial\Omega \times (0,T). \end{aligned} \tag{1}$$

Problems of the above equations include inverse problems, heat conduction processes, hydrology, material sciences, and heat transfer problems. In the context of heat conduction and diffusion, u represents either temperature or concentration. In this system, $\phi(x)$ is spatial density and unknown $f(t)$ is interpreted as either a heat or a material source, respectively, in a chemical or a biochemical application.

When $f(t) > 0$, it means that term $f(t)\phi(x)$ is a source. We assume that ϕ is a given function and satisfies the following condition:

$$\begin{gathered} \phi \geq 0, \quad \not\equiv 0 \text{ in } \Omega \\ \phi \text{ has compact support in } \Omega. \end{gathered} \tag{2}$$

Our problem is to derive $f(t)$ and a conditional stability in the determination of $f(t)$, $0 \leq t \leq T$, from the observation

$$u(x_0,t), \quad 0 < t < T, \tag{3}$$

where $x_0 \in \Omega/\overline{\operatorname{supp}\phi}$.

2. Some Existing Results

To derive the stability estimate, we need the following inequalities, which can be found in [17]. Here, we just list the result.

Theorem 1. *Let $p \geq 1$, $\delta > 0$, $0 \leq \alpha < T$, and $f, g \in L^\infty(0,T)$ satisfy*

$$0 \leq f, g \leq M < \infty, \quad 0 < t < T. \tag{4}$$

Then,

$$\|f\|_{L^p(\alpha,T)}\|g\|_{L^p(0,\delta)} \le M^{(2p-2)/p}\left(\int_\alpha^{T+\delta}\left(\int_\alpha^t f(s)\,g(t-s)\,ds\right)dt\right)^{1/p}. \quad (5)$$

In particular, for

$$(f*g)(t)=\int_0^t f(t-s)\,g(s)\,ds,\quad 0<t<T \quad (6)$$

and for $\alpha=0$, one has

$$\|f\|_{L^p(\alpha,T)}\|g\|_{L^p(0,\delta)} \le M^{(2p-2)/p}\|f*g\|_{L^1(0,T+\delta)}^{1/p}. \quad (7)$$

In [17], by using the above theorem, a Hölder estimate was obtained for the case that $\Omega=R^n$ and $f\in U$, where

$$\mathcal{U}=\left\{f\in C[0,T];\|f\|_{C[0,T]} \le M, f \text{ changes the signs at most } N\text{-times}\right\}. \quad (8)$$

Theorem 2. *Let ϕ satisfy (2), and set $\mathcal{U}$ to be defined as above definition.*

Then, for arbitrarily given $\delta>0$, there exists a constant $C=C(x_0,\phi,T,p,\delta,\mathcal{U})>0$ such that

$$\|f\|_{L^p(0,T)}\le C\|u(x_0,\cdot)\|_{L^1(0,T+\delta)}^{1/p^N}, \quad (9)$$

for any $f\in\mathcal{U}$.

Corollary 3. *In Theorem 2, if one replaces $\mathcal{U}$ by*

$$\mathcal{P}_N=\left\{f;\ f \text{ is a polynomial whose order is at most } N,\ \|f\|_{C[0,T]}\le M\right\}, \quad (10)$$

also the same result can be obtained.

But we are more interested in the case that $f(t)$ is a piecewise constant function. In that case, $f(t)$ can have infinity number of zeros. For that case, we obtained the similar result, and we will claim the result and the proof in the next section.

3. The Estimate

For problem (1), we can see that solution $u(x,t)$ can be expressed in the following form:

$$u(x,t)=\int_0^t\int_\Omega K(x,y,t-s)\,f(s)\,\phi(y)\,dy\,ds,\quad x\in\Omega,\ t>0, \quad (11)$$

where $K(x,y,t)$ is the fundamental function for problem (1). See [18].

Also from [18], we can see that fundamental solution $K(x,y,t)$ satisfies the following conditions:

$$K(x,y,t)>0,\quad t>0. \quad (12)$$

Therefore, setting

$$\mu_{x_0}(t)=\int_\Omega K(x_0,y,t)\,\phi(y)\,dy,\quad t>0, \quad (13)$$

we have

$$u(x_0,t)=\int_0^t \mu_{x_0}(t-s)\,f(s)\,ds,\quad 0<t<T, \quad (14)$$

which is a Volterra integral equation of the first kind with respect to f.

Here, we consider the case that $f\in F=\{f \text{ is a piecewise constant function in } [0,T]\}$.

Theorem 4 (uniqueness). *For problem (1), given $u(x_0,t)$, then $f\in F$ is uniquely determined except for finite point in $(0,T)$.*

Proof. Suppose that there are two functions $f_1(t), f_2(t)\in F$ satisfying (1). We assume that the discontinuous points of f_1 and f_2 are $0=t_0^1<t_1^1<\cdots<t_{I_1}^1=T$ and $0=t_0^2<t_1^2<\cdots<t_{I_2}^2=T$, respectively, and $f_1(t)=f_j^1,\ t\in(t_{j-1}^1,t_j^1]$ and $f_2(t)=f_j^2,\ t\in(t_{j-1}^2,t_j^2]$.

After some resorting procedure, we get points $0=\tau_0<\tau_1<\cdots<\tau_{I_3}=T$; then, for $0<t<\tau_1$,

$$\int_0^t \mu_{x_0}(t-s)\left(f_1(s)-f_2(s)\right)ds=u(x_0,t)-u(x_0,t)=0 \quad (15)$$

$$\left(f_1^1-f_1^2\right)\int_0^t \mu_{x_0}(t-s)\,ds=0$$

and $\int_0^t \mu_{x_0}(t-s)ds>0,\ 0<t<\tau_1$, so we get $f_1(t)=f_2(t)$, $0<t<\tau_1$.

For $\tau_1<t<\tau_2$,

$$\int_0^t \mu_{x_0}(t-s)\left(f_1(s)-f_2(s)\right)ds=\int_{\tau_1}^t \mu_{x_0}(t-s)\left(f_1(s)-f_2(s)\right)ds=0, \quad (16)$$

therefore, $f_1(t)=f_2(t),\ t\in(\tau_1,\tau_2)$.

Repeating the above procedure, we can get $f_1(t)=f_2(t),\ t\in(\tau_{j-1},\tau_j),\ j=1,2,\ldots,I_3$.

Thus, end the proof. □

Suppose that f changes its values at $0=t_0<t_1<t_2<\cdots<t_I=T$, $I\le N$, and $f(t)=f_j,\ t\in[t_{j-1},t_j),\ j=1,2,\ldots,I$. Here, we assume that there exists a constant $c>0$, such that $t(i)-t(i-1)\ge c,\ i=1,2,\ldots,I$.

Theorem 5. *If ϕ satisfies (2), $x_0\in\Omega/\overline{\operatorname{supp}\phi}$.*

Then, for $p > 1$, there exists a constant $C = C(x_0, \phi, T, p, F, N) > 0$ such that

$$\|f\|_{L^p(0,T)} \le C \|u(x_0, \cdot)\|_{L^p(0,T)}, \tag{17}$$

for any $f \in F$. Here, N is the number of intervals in $[0, T]$ that $f \neq 0$.

Proof. Since $\mu_{x_0}(t)$ is positive and bounded and (13), we can see that

$$u(x_0, t) = f_1 \int_0^t \mu_{x_0}(t-s)\, ds, \quad 0 \le t \le t_1. \tag{18}$$

Choosing L_p norm from 0 to t_1, we obtain

$$\|u\|_{L_p(0,t_1)} = |f_1| \left\| \int_0^t \mu_{x_0}(t-s)\, ds \right\|_{L_p(0,t_1)}. \tag{19}$$

Thus, we can see that

$$|f_1| \le C_1 \|u\|_{L_p(0,t_1)}, \tag{20}$$

where $C_1 > (\| \int_0^t \mu_{x_0}(t-s) ds\|_{L_p(0,t_1)})^{-1}$ is a constant.

For $t_1 \le t \le t_2$, we can see that

$$\begin{aligned} u(x_0, t) &= \int_0^t \mu_{x_0}(t-s) f(s)\, ds \\ &= f_1 \int_0^{t_1} \mu_{x_0}(t-s)\, ds + f_2 \int_{t_1}^t \mu_{x_0}(t-s)\, ds, \\ u(x_0, t) - f_1 \int_0^{t_1} \mu_{x_0}(t-s)\, ds &= f_2 \int_{t_1}^t \mu_{x_0}(t-s)\, ds. \end{aligned} \tag{21}$$

Then, we can obtain

$$\begin{aligned} &|f_2| \cdot \left\| \int_{t_1}^t \mu_{x_0}(t-s)\, ds \right\|_{L_p(t_1,t_2)} \\ &\quad = \left\| u(x_0, t) - f_1 \int_0^{t_1} \mu_{x_0}(t-s)\, ds \right\|_{L_p(t_1,t_2)} \\ &\quad \le \|u(x_0, t)\|_{L_p(t_1,t_2)} \\ &\qquad + |f_1| \left\| \int_0^{t_1} \mu_{x_0}(t-s)\, ds \right\|_{L_p(t_1,t_2)} \\ &\quad \le \|u(x_0, t)\|_{L_p(t_1,t_2)} + C_2 \|u(x_0, t)\|_{L_p(0,t_1)} \\ &\quad \le C_3 \|u(x_0, t)\|_{L_p(0,t_2)}. \end{aligned} \tag{22}$$

Then

$$|f_2| \le C \|u(x_0, t)\|_{L_p(0,t_2)}. \tag{23}$$

Therefore, in the case of $t_i - t_{i-1} \ge c > 0$, $i = 1, 2, \ldots, I$ and (20), we get

$$\|f\|_{L_p(0,t_2)} \le C \|u(x_0, t)\|_{L_p(0,t_2)}. \tag{24}$$

Continuing this argument until $t_i = T$, we get the proof of this theorem. □

4. Numerical Scheme

For problem (1), if $u(x_0, t)$ is given, we need to reconstruct function $f(t)$.

Consider an auxiliary problem:

$$\begin{aligned} u_t &= \Delta u + \phi(x) \quad (x, t) \in \Omega \times (0, T), \\ u(x, 0) &= 0 \quad x \in \Omega, \\ u(x, t) &= 0 \quad (x, t) \in \partial\Omega \times (0, T). \end{aligned} \tag{25}$$

Actually, we can get the solution of problem (25) and suppose that $\tilde{u}$ is the solution. Then,

$$\tilde{u}(x, t) = \int_0^t \int_\Omega K(x, y, t-s) \phi(y)\, dy\, ds. \tag{26}$$

Given data $u(x_0, t_i)$, $i = 0, 1, 2, \ldots, N$, where $0 = t_0 < t_1 < \cdots < t_N = T$, then, for every t_i, we have

$$u(x_0, t_i) = \int_0^{t_i} \int_\Omega K(x_0, y, t_i - s) f(s) \phi(y)\, dy\, ds, \quad i = 0, 1, 2, \ldots, N. \tag{27}$$

We suppose that $f(t) = f_i$, $t \in [t_{i-1}, t_i]$, $i = 1, 2, \ldots, N$; then

$$\begin{aligned} u(x_0, t_i) &= \int_0^{t_i} \int_\Omega K(x_0, y, t_i - s) f(s) \phi(y)\, dy\, ds \\ &= \sum_{j=1}^i f_j \int_{t_{j-1}}^{t_j} \int_\Omega K(x_0, y, t_i - s) \phi(y)\, dy\, ds \\ &= f_1 \int_0^{t_i} \int_\Omega K(x_0, y, t_i - s) \phi(y)\, dy\, ds \\ &\quad + \sum_{j=2}^i (f_j - f_1) \int_{t_{j-1}}^{t_j} \int_\Omega K(x_0, y, t_i - s) \phi(y)\, dy\, ds \\ &= f_1 \tilde{u}(x_0, t_i) + (f_2 - f_1) \\ &\quad \cdot \int_{t_1}^{t_i} \int_\Omega K(x_0, y, t_i - s) \phi(y)\, dy\, ds \\ &\quad + \sum_{j=3}^i \left(f_j - f_1 - (f_2 - f_1)\right) \\ &\quad \cdot \int_{t_{j-1}}^{t_j} \int_\Omega K(x_0, y, t_i - s) \phi(y)\, dy\, ds = f_1 \tilde{u}(x_0, t_i) \\ &\quad + (f_2 - f_1) \\ &\quad \cdot \int_0^{t_i - t_1} \int_\Omega K(x_0, y, t_i - t_1 - s) \phi(y)\, dy\, ds \\ &\quad + \sum_{j=3}^i (f_j - f_2) \int_{t_{j-1}}^{t_j} \int_\Omega K(x_0, y, t_i - s) \phi(y)\, dy\, ds \\ &= \cdots = f_1 \tilde{u}(x_0, t_i) \\ &\quad + \sum_{j=2}^i (f_j - f_{j-1}) \tilde{u}(x_0, t_i - t_{j-1}). \end{aligned} \tag{28}$$

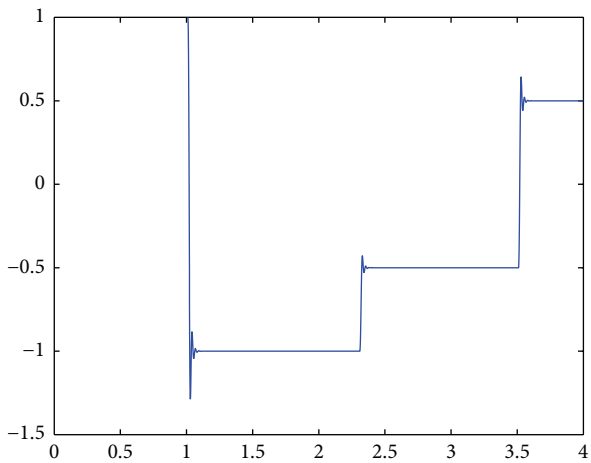

FIGURE 1: Numerical result.

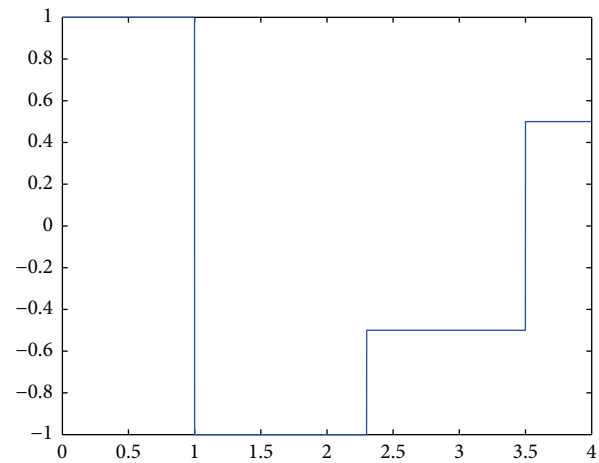

FIGURE 2: Exact solution.

Thus, we can use the above formula to get the discrete data for function $f(t)$.

Example 6. We choose $f(t)$ as the following function:

$$f(t) = \begin{cases} 1, & t \in [0,1) \\ -1, & t \in [1,2.3) \\ -0.5, & t \in [2.3,3.5) \\ 0.5, & t \in [3.5,4]. \end{cases} \tag{29}$$

Choose $\Omega = (0,2)$, $x_0 = 1.5$, $\phi(x) = 1$, $x \in (0,1)$, and $\phi(x) \equiv 0$ outside interval $(0,1)$. Given data $u(x_0,t_i)$, $i = 1,2,\ldots,N$. Then, we can get the numerical approximation of $f(t)$ using the following formula:

$$u(x_0,t_i) = f_1 \tilde{u}(x_0,t_i) + \sum_{j=2}^{i} (f_j - f_1)\tilde{u}(x_0,t_i - t_{j-1}), \quad i = 1,2,\ldots,N. \tag{30}$$

And the numerical result is as shown in Figure 1.

Figure 1 is the numerical result and Figure 2 is the exact function.

Figure 3 shows the error between the numerical result and exact solution.

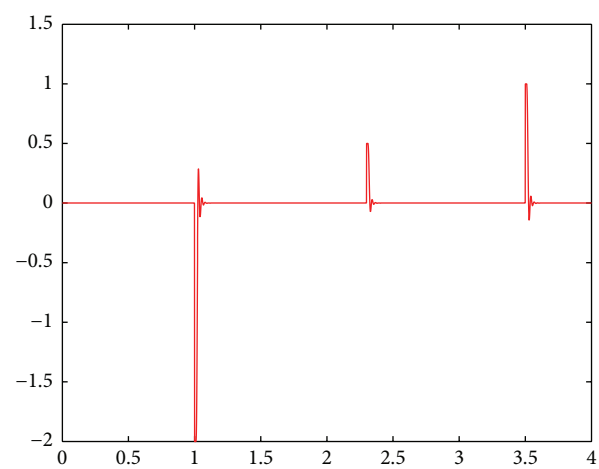

FIGURE 3: The error between numerical result and exact solution.

Example 7. Now, we will give a numerical result for the 2D case. Suppose the problem is also (1), and the domain $\Omega = (-1,1) \times (-1,1)$, and

$$\phi(x) = \begin{cases} 1, & |x| \le 0.5 \\ 0, & x \in \Omega, \ |x| > 0.5 \end{cases}$$

$$f(t) = \begin{cases} 1, & t \in [0,0.8) \\ 0, & t \in [0.8,2.2) \\ -2, & t \in [2.2,3]. \end{cases} \tag{31}$$

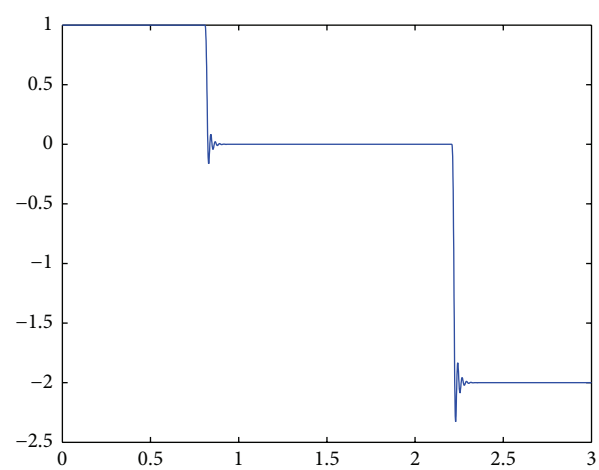

FIGURE 4: Numerical result.

Let $x_0 = (0.7, 0.8)$ and $T = 3$, and the numerical result is as shown in Figure 4.

Figure 4 is the numerical result and Figure 5 is the exact function.

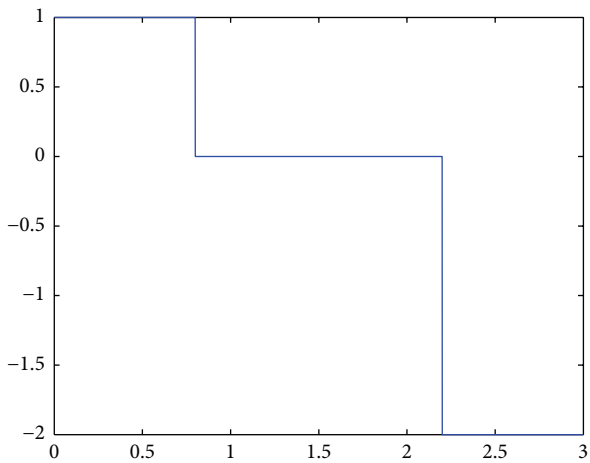

Figure 5: Exact solution.

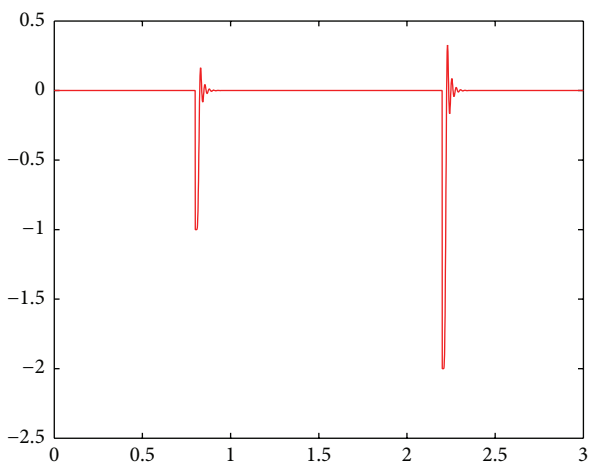

Figure 6: The error between numerical result and exact solution.

Figure 6 is the error between exact f and the numerical result.

Competing Interests

The author declares that there are no competing interests regarding the publication of this paper.

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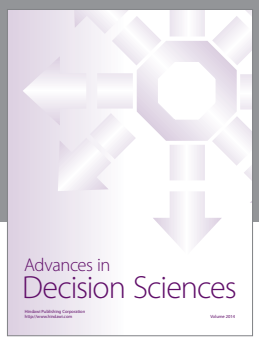

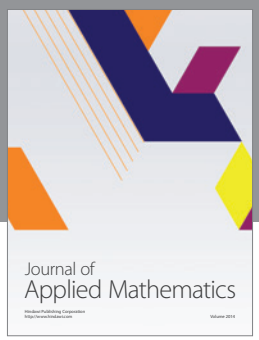

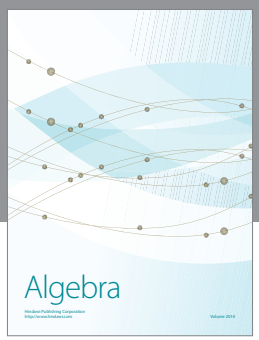

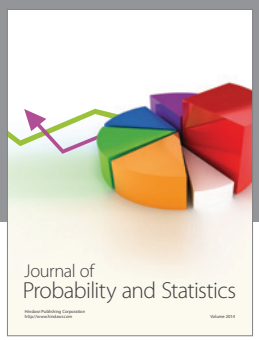

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