Research Article

Third-Order Conditional Lie–Bäcklund Symmetries of Nonlinear Reaction-Diffusion Equations

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The third-order conditional Lie–Bäcklund symmetries of nonlinear reaction-diffusion equations are constructed due to the method of linear determining equations. As a consequence, the exact solutions of the resulting equations are derived due to the compatibility of the governing equations and the admitted differential constraints, which are resting on the characteristic of the admitted conditional Lie–Bäcklund symmetries to be zero.

1. Introduction

The conditional Lie–Bäcklund symmetry (CLBS) was independently proposed by Zhdanov [1] and Fokas and Liu [2]. Exact solutions possessing the form of nonlinear separation of variables [3] and exact solutions defined on the invariant subspaces [4] for nonlinear heat conductivity equations were constructed due to the corresponding symmetry reductions in [1], which cannot be obtained within the framework of Lie's classical symmetry [5], conditional symmetry [6, 7], and Lie–Bäcklund symmetry [8]. Multishock and multisoliton solutions of nonintegrable equations were obtained by using CLBS in [2], which are regarded as remnants of integrability for the considered nonintegrable equations.

As a consequence, CLBSs were applied to a number of particular subclasses of evolution equations [9, 10]. It has been noted that CLBS is very effective in studying classifications and reductions of diffusion equations [11–16]. CLBS can also be used to consider the reductions of initial value problems for nonlinear evolution equations [17–19]. CLBS for evolution systems were studied in [20, 21].

The procedure for determining whether or not a given Lie–Bäcklund symmetry is conditionally invariant of the considered equations is straightforward; however, the determination of the most general CLBS admitted by a given differential equation is a very difficult, if not impossible, problem. CLBSs are generalizations of conditional symmetries, in the same

way that Lie–Bäcklund symmetries are generalizations of Lie's classical symmetry. Therefore the procedure for computing CLBSs is about the same as that for the conditional symmetries. We need to determine the form of CLBS presumably. It is proved that CLBSs related to separation of variables [13], sign-invariants [14], and invariant subspaces [15, 16] are powerful to study different types of diffusion equations. Are there other choices of CLBS which may yield new interesting results for these considered equations?

Once the CLBS and the governing equation are determined, exact solutions of the discussed equations can be easily constructed due to the compatibility of the considered equations and the additional differential constraints obtained by setting the characteristic of the corresponding CLBS to be zero. Thus, CLBS can be reformulated by using the technicalities of the method of differential constraints.

The key idea of the method of differential constraint was proposed by Yanenko in [22]. The general formulation of this method requires that the original system of partial differential equations and the additional differential constraints satisfy some conditions of compatibility. The problem of finding all differential constraints compatible with certain equations can be more complicated than the investigation of solving the original equations. Hence, it is better to content oneself with finding constraints in some fixed classes, and these classes must be chosen using additional considerations.

In practice, differential constraints related to separation of variables [23] and invariant subspaces [24] are powerful to seek for reductions of different types of evolution equations. Olver and Rosenau [25, 26], Olver [27], Kaptsov [28], and Levi and Winternitz [29] showed that many reductions including Bluman and Cole's nonclassical method [6], partial invariance method [30], Clarkson and Kruskal's direct method [31], and Galaktionov's higher-order direct method [3] can be understood within the framework of differential constraints.

B-determining equations [32] and linear determining equations [33, 34] were both proposed to construct differential constraints. The method of linear determining equations with free parameters is much easier to illustrate than the one of B-determining equations with undetermined functions. The linear determining equations are more general than the classical determining equations for Lie's generators [30].

In this paper, we will consider the general third-order CLBS with the characteristic

$$\eta = u_3 + g(t, x, u, u_1, u_2) \quad (1)$$

of the nonlinear reaction-diffusion equations

$$u_t = (u^k u_x)_x + Q(u) \quad (2)$$

by using the linear determining equations. Equation (2) is widely used as mathematical models of heat conduction and diffusion, filtration of gases and fluids in porous media, process of chemical kinetics, and biological process. It is noted that $u_j = \partial^j u / \partial x^j$ ($j = 1, 2, 3$) in (1).

There are numerous works aimed to study symmetry classifications and reductions of (2). CLBS was also used to consider the variant forms of (2). Three types of CLBS were applied to study such equations. The CLBS with the characteristic

$$\eta = [f(u)]_{xt} \quad (3)$$

was presented to study functionally separable solutions

$$f(u) = \phi(t) + \psi(x) \quad (4)$$

of (2) in [13]. The CLBS with the characteristic

$$\eta = u_2 + H(u)u_1^2 + G(u)u_1 + F(u) \quad (5)$$

was proposed to study the variant forms of (2) in [11, 12], which can give symmetry interpretation for first-order sign-invariants of the considered equations. The CLBS with the characteristic

$$\begin{aligned} \eta = & [f(u)]_n + a_1(x)[f(u)]_{n-1} + a_2(x)[f(u)]_{n-2} \\ & + \cdots + a_n(x)f(u) \end{aligned} \quad (6)$$

was proposed to study the various forms of (2) in [15, 16]. The corresponding linearized form

$$\sigma = u_n + a_1(x)u_{n-1} + a_2(x)u_{n-2} + \cdots + a_n(x)u \quad (7)$$

of (6) is related to the invariant subspace

$$W = L\{f_1(x), f_2(x), \dots, f_n(x)\}, \quad (8)$$

which is the right solution space of the linear ODE $\sigma = 0$.

The forms of CLBS in [11–16] are all determined in advance. Thus the corresponding results are of course restricted to this ansatz. The third-order differential constraints (1) of the nonlinear diffusion equation $u_t = (u^k u_x)_x$ are considered in [33]. The differential constraints

$$\eta = u_n + g(t, x, u, u_1, u_2, \dots, u_{n-1}) = 0 \quad (9)$$

with $n = 2, 3$ of (2) are discussed in [34]. However, the study there was worked out for only several special cases. Here we will extend the work there to develop a complete classification for third-order CLBS with the form (1) of (2) by applying the method of linear determining equations. New types of CLBS which are not included in the form of (3), (5), and (6) will be derived and therefore a huge number of new exact solutions of (2) will be constructed due to the compatibility of the governing equations and corresponding resulting differential constraints.

2. General Statement

Consider a nonlinear evolution equation

$$u_t = F(t, x, u, u_1, u_2, \dots, u_k) \quad (10)$$

with Lie–Bäcklund vector field (LBVF)

$$\begin{aligned} V = & \eta \frac{\partial}{\partial u} + D_x \eta \frac{\partial}{\partial u_1} + D_t \eta \frac{\partial}{\partial u_t} + D_x^2 \eta \frac{\partial}{\partial u_2} \\ & + D_x D_t \eta \frac{\partial}{\partial u_{1t}} + \cdots, \end{aligned} \quad (11)$$

where $\eta = \eta(x, t, u, u_1, u_t, u_{1t}, u_{1t}, \dots)$ is the characteristic of LBVF (11) and the total differentiation operators, respectively, denote

$$\begin{aligned} D_t = & \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{1t} \frac{\partial}{\partial u_1} + \cdots, \\ D_x = & \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + u_{1t} \frac{\partial}{\partial u_t} + u_2 \frac{\partial}{\partial u_1} + \cdots. \end{aligned} \quad (12)$$

Definition 1 (see [5]). The evolutionary vector field (11) is said to be a Lie–Bäcklund symmetry of the evolution equation (10) if the condition

$$V(u_t - F)|_M = 0 \quad (13)$$

holds, where M denotes the set of all differential consequences of (10).

Definition 2 (see [1, 2]). The evolutionary vector field (11) is said to be a CLBS of (10) if the following condition

$$V(u_t - F)|_{M \cap L_x} = 0 \quad (14)$$

holds, where L_x denotes the set of all differential consequences of equation $\eta = 0$ with respect to x .

It is noted that we can exclude all derivatives with respect to t on the set of (10) and thus obtain the vector field (11) with η of the form

$$\eta = \eta(t, x, u, u_1, u_2, \dots, u_n). \quad (15)$$

A direct computation of (14) yields

$$D_t \eta|_{M \cap L_x} = 0. \quad (16)$$

On the other hand, we can derive that

$$D_t \eta|_{M \cap L_x} = D_t \eta|_{M_x \cap L_x}, \quad (17)$$

where M_x denotes the set of all differential consequences of (10) with respect to x . Thus, we have that the condition

$$D_t \eta|_{M_x \cap L_x} = 0 \quad (18)$$

holds, which is exactly the sufficient condition [28, 35] for which $\eta = 0$ is a differential constraint of the evolution equation (10). Thus $\eta = 0$ defines an invariant manifold of the equation (10), which is right corresponding to the symmetry reductions due to the compatibility of $\eta = 0$ and the considered (10). There is an equivalence relation between CLBS and differential constraint for the evolution equation (10).

Kaptsov [33, 34] proposed a method for finding differential constraints, which uses linear determining equations. The definition of linear determining equation was first given in [33], which is listed as follows.

Definition 3 (see [33, 34]). A linear determining equation corresponding to differential constraints $\eta(t, x, u, u_1, u_2, \dots, u_n) = 0$ of the evolution equation (10) is an equation of the form

$$D_t \eta|_{M_x} = \sum_{i=0}^n \sum_{j=0}^i b_{ij} D_x^{i-j} (F_{u_{n-j}}) D_x^{n-i} (\eta), \quad (19)$$

where b_{ij} are some constants.

3. CLBS of (2)

As stated in [34], the linear determining equation for the differential constraints $\eta = 0$ of (2) is right

$$\begin{aligned} D_t \eta|_{E_x} &= u^k D_x^2 \eta + b_1 k u^{k-1} u_1 D_x \eta \\ &+ [b_3 k u^{k-1} u_2 + b_2 k (k-1) u^{k-2} u_1^2 + b_4 f_u] \eta, \end{aligned} \quad (20)$$

where $b_1, b_2, b_3, b_4 \in \mathbb{R}$ and E_x is the differential consequences of (2) with respect to x . It is easy to see that (2) admits LBVF (11) if the condition (20) holds.

Substituting (1) into (20), the left-hand side and the right-hand side are, respectively, simplified as

$$\begin{aligned} D_t \eta|_{E_x} &= u^k u_5 + u^{k-1} (u g_{u_2} + 5k u_1) u_4 + [u^k g_{u_1} \\ &+ 4k u^{k-1} u_1 g_{u_2} + 10k u^{k-1} u_2 + 10k(k-1) u^{k-2} u_1^2 \\ &+ Q_u] u_3 + [k(k-1)(k-2) u^{k-3} u_1^4 + 3k u^{k-1} u_2^2 \\ &+ 6k(k-1) u^{k-2} u_2 u_1^2 + u_1^2 Q_{uu} + u_2 Q_u] g_{u_2} \\ &+ [u_1 Q_u + k(k-1) u^{k-2} u_1^3 + 3k u^{k-1} u_2 u_1] g_{u_1} \\ &+ (u^k u_2 + k u^{k-1} u_1^2 + Q_u) g_u + g_t + u_1^3 Q_{uuu} \\ &+ 3u_2 u_1 Q_{uu} + k(k-1)(k-2)(k-3) u^{k-4} u_1^5 \\ &+ 10k(k-1)(k-2) u^{k-3} u_2 u_1^3 + 15k(k-1) \\ &\cdot u^{k-2} u_2^2 u_1, \end{aligned} \quad (21)$$

$$\begin{aligned} u^k D_x^2 \eta &+ b_1 k u^{k-1} u_1 D_x \eta + [b_3 k u^{k-1} u_2 \\ &+ b_2 k(k-1) u^{k-2} u_1^2 + b_4 f_u] \eta = u^k u_5 + u^{k-1} (u g_{u_2} \\ &+ b_1 k u_1) u_4 + u^k g_{u_2 u_2} u_3^2 + [b_4 Q_u \\ &+ u^k (2u_1 g_{uu_2} + 2g_{xu_2} + 2u_2 g_{u_1 u_2} + g_{u_1}) \\ &+ b_1 k u^{k-1} u_1 g_{u_2} + b_2 k(k-1) u^{k-2} u_1^2 + b_3 k u^{k-1} u_2] \\ &\cdot u_3 + u^k (u_2^2 g_{u_1 u_1} + 2u_1 u_2 g_{uu_1} + u_1^2 g_{uu} + 2u_2 g_{xu_1} \\ &+ 2u_1 g_{xu} + g_{xx} + u_2 g_u) + b_1 k u^{k-1} u_1 (u_2 g_{u_1} \\ &+ u_1 g_u + g_x) + [b_3 k u^{k-1} u_2 + b_2 k(k-1) u^{k-2} u_1^2 \\ &+ b_4 Q_u] g, \end{aligned}$$

which can be regarded as two polynomials with respect to u_5, u_4 , and u_3 . It is noted that the subscripts here denote the differentiation with respect to the indicated variables.

Considering the coefficients of u_4 and u_3^2 , we obtain

$$b_1 = 5, \quad (22)$$

$$g_{u_2 u_2} = 0.$$

So g can be represented by

$$g = h(t, x, u, u_1) u_2 + f(t, x, u, u_1). \quad (23)$$

Here h and f are both functions of t, x, u , and u_1 . Collecting the coefficients of u_3 , we have

$$\begin{aligned} &[2u^k h_{u_1} + (b_3 - 10) k u^{k-1}] u_2 + 2u^k (u_1 h_u + h_x) \\ &+ k u^{k-1} u_1 h + k(k-1)(b_2 - 10) u^{k-2} u_1^2 \\ &+ (b_4 - 1) Q_u = 0. \end{aligned} \quad (24)$$

Thus, we know that h must satisfy

$$2u^k h_{u_1} + (b_3 - 10)ku^{k-1} = 0, \quad (25)$$

$$2u^k (u_1 h_u + h_x) + ku^{k-1} u_1 h + k(k-1)(b_2 - 10)u^{k-2} u_1^2 + (b_4 - 1)Q_u = 0. \quad (26)$$

Solving (25), we obtain

$$h = -\frac{k(b_3 - 10)}{2u}u_1 + a(t, x, u) \quad (27)$$

and $a(t, x, u)$ is the function of t , x , and u . As a consequence, (26) can be simplified as

$$k\left(5k - kb_2 + \frac{1}{2}kb_3 + b_2 - b_3\right)u^{k-2}u_1^2 - (2ua_u + ku^{k-1}a)u_1 - 2ua_x + (1 - b_4)Q_u = 0, \quad (28)$$

which implies

$$\begin{aligned} k\left(5k - kb_2 + \frac{1}{2}kb_3 + b_2 - b_3\right) &= 0, \\ 2ua_u + ku^{k-1}a &= 0, \\ -2ua_x + (1 - b_4)Q_u &= 0. \end{aligned} \quad (29)$$

Solving the linear ordinary differential equation about a , we get $a = C(t, x)u^{-k/2}$. The simplification of the above equation about Q gives

$$(1 - b_4)Q_u - 2u^{k/2}C_x = 0. \quad (30)$$

We know that $C_x = d_1$ must hold and then the above equations about Q become

$$(1 - b_4)Q_u - 2d_1u^{k/2} = 0. \quad (31)$$

Solving this ordinary differential equation, we know that there are three subcases for further discussion, which are listed as

(1)

$$\begin{aligned} b_4 &= 1, \\ d_1 &= 0; \end{aligned} \quad (32)$$

(2)

$$\begin{aligned} k &= -2, \\ Q &= \frac{2d_1 \ln(u)}{1 - b_4} + d_3; \end{aligned} \quad (33)$$

(3)

$$Q = \frac{2d_1 u^{1+k/2}}{(1 - b_4)(1 + k/2)} + d_3, \quad b_4 \neq 1, \quad k \neq -2. \quad (34)$$

For these three different cases, a further identical vanishing of (20) will finally determine the form of g in (1). The detailed procedure is omitted here. We just present the corresponding results in Table 1, where d_i ($i = 1, 2, 3, 4, 5$) are arbitrary constants.

4. Exact Solutions of (2)

In this section, we construct exact solutions of (2) due to the compatibility of differential constraints $\eta = 0$ and the governing equation. The equivalence relation between CLBSs and differential constraints for the evolution equation (10) is first rigorously proved in [36], where it is also stated that finding of the whole set of CLBSs of an evolution equation is a no-go problem. In fact, the general forms CLBSs η of (2) are completely determined by solving the linear system of determining equations about η , which are listed in Table 1. Solving $\eta = 0$, we can obtain u as a function of x with t -dependent undetermined functions. Substituting the form of u into (2) will yield that the t -dependent coefficients in (2) satisfy three-dimensional dynamical system. Here we give several examples to illustrate the reduction procedure.

Example 1. Equation

$$u_t = (u^k u_x)_x + d_1 u + d_2 u^{1-k} + \frac{(k+1)d_3}{k^2} u^{1+k} \quad (35)$$

admits the CLBS

$$\eta = u_3 + \frac{3(k-1)}{u}u_1 u_2 + \frac{(k-1)(k-2)}{u^2}u_1^3 + d_3 u_1. \quad (36)$$

The corresponding solutions are given as follows.

(i) For $d_3 > 0$,

$u(x, t)$

$$= \left[\alpha(t) + \beta(t) \sin\left(\sqrt{d_3}x\right) + \gamma(t) \cos\left(\sqrt{d_3}x\right) \right]^{1/k}, \quad (37)$$

where $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ satisfy three-dimensional dynamical system

$$\begin{aligned} \alpha' &= \frac{(k+1)d_3}{k}\alpha^2 + \frac{d_3}{k}\beta^2 + \frac{d_3}{k}\gamma^2 + kd_1\alpha + kd_2, \\ \beta' &= \frac{(k+2)d_3}{k}\alpha\beta + kd_1\beta, \\ \gamma' &= \frac{(k+2)d_3}{k}\alpha\gamma + kd_1\gamma. \end{aligned} \quad (38)$$

(ii) For $d_3 < 0$,

$$\begin{aligned} u(x, t) &= \left[\alpha(t) + \beta(t) \sinh\left(\sqrt{-d_3}x\right) \right. \\ &\quad \left. + \gamma(t) \cosh\left(\sqrt{-d_3}x\right) \right]^{1/k}, \end{aligned} \quad (39)$$

where $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ satisfy three-dimensional dynamical system

$$\begin{aligned} \alpha' &= \frac{(k+1)d_3}{k}\alpha^2 - \frac{d_3}{k}\beta^2 + \frac{d_3}{k}\gamma^2 + kd_1\alpha + kd_2, \\ \beta' &= \frac{(k+2)d_3}{k}\alpha\beta + kd_1\beta, \\ \gamma' &= \frac{(k+2)d_3}{k}\alpha\gamma + kd_1\gamma. \end{aligned} \quad (40)$$

TABLE 1: CLBS (1) of (2).

Number	Equation (2)	CLBS (1)	Remarks
1	$u_t = (u^k u_x)_x + d_1 u + d_2 u^{1+k}$	$\eta = u_3 + \frac{3k-1}{u} u_1 u_2 + \frac{k(k-2)}{u^2} u_1^3 + d_2 k u_1$	$b_2 = \frac{7k^2 - 3k - 2}{k(k-1)}, b_3 = \frac{4k+2}{k}, b_4 = 1$
2	$u_t = (u^k u_x)_x$	$\eta = u_3 + \frac{3k-1}{u} u_1 u_2 + \frac{k(k-2)}{u^2} u_1^3$	$b_2 = \frac{7k^2 - 3k - 2}{k(k-1)}, b_3 = \frac{4k+2}{k}$
3	$u_t = (u^k u_x)_x + d_1 u + d_2 u^{1-k} + \frac{(k+1)d_3}{k^2} u^{1+k}$	$\eta = u_3 + \frac{3(k-1)}{u} u_1 u_2 + \frac{(k-1)(k-2)}{u^2} u_1^3 + d_3 u_1$	$b_2 = \frac{7k+6}{k}, b_3 = \frac{4k+6}{k}, b_4 = 1$
4	$u_t = (u^{-2} u_x)_x$	$\eta = u_3 + \left(-\frac{10}{u} u_1 + d_1 u \right) u_2 + \frac{15}{u^2} u_1^3 - 3d_1 u_1^2 + d_2 u^2 u_1 + d_3 u^4 + (d_4 x + d_5) u^5$	$b_2 = \frac{10}{3}, b_3 = 0$
5	$u_t = (u^{-2} u_x)_x + d_1 u$	$\eta = u_3 + \left[-\frac{10}{u} u_1 + d_2 \exp(-d_1 t) u \right] u_2 + \frac{15}{u^2} u_1^3 - 3d_2 \exp(-d_1 t) u_1^2 + d_3 \exp(-2d_1 t) u^2 u_1 + (d_4 x + d_5) \exp(-4d_1 t) u^5 + d_6 \exp(-3d_1 t) u^4$	$b_2 = \frac{10}{3}, b_3 = 0, b_4 = 1$
6	$u_t = (u^{-2} u_x)_x$	$\eta = u_3 + \left[-\frac{9}{u} u_1 + (d_1 t + d_2) u \right] u_2 + \frac{12}{u^2} u_1^3 - 2(d_1 t + d_2) u_1^2 + (d_3 x + d_4) u^3 u_1 + \frac{1}{2} (2d_3 - d_1) u^4$	$b_2 = 4, b_3 = 1$
7	$u_t = (u^{-2} u_x)_x + d_1 u$	$\eta = u_3 + \left[-\frac{9}{u} u_1 + (d_2 \exp(-d_1 t) + d_3 \exp(-3d_1 t)) u \right] u_2 + \frac{12}{u^2} u_1^3 - 2[d_2 \exp(-d_1 t) + d_3 \exp(-3d_1 t)] u_1^2 + (d_4 x + d_5) \exp(-3d_1 t) u^3 u_1 + (d_3 d_1 + d_4) \exp(-3d_1 t) u^4$	$b_2 = 4, b_3 = 1, b_4 = 1$
8	$u_t = \left(u^{(-1/2)u_x} \right)_x + d_1 u + d_2 u^{1/2}$	$\eta = u_3 - \frac{5}{2u} u_1 u_2 + \frac{5}{4u^2} u_1^3 - \frac{1}{2} d_2 u_1 + d_3 \exp\left(\frac{1}{2} d_1 t\right) u^{1/2}$	$b_2 = \frac{5}{3}, b_3 = 0, b_4 = 1$
9	$u_t = \left(u^{(-1/2)u_x} \right)_x + d_1 u$	$\eta = u_3 - \frac{5}{2u} u_1 u_2 + \frac{5}{4u^2} u_1^3 + d_2 \exp\left(-\frac{3}{2} d_1 t\right) u^{5/2} + d_3 \exp\left(\frac{1}{2} d_1 t\right) u^{1/2}$	$b_2 = \frac{5}{3}, b_3 = 0, b_4 = 1$
10	$u_t = \left(u^{-1/2} u_x \right)_x$	$\eta = u_3 - \frac{5}{2u} u_1 u_2 + \frac{5}{4u^2} u_1^3 + d_1 u^{5/2} + d_2 u^{1/2}$	$b_2 = \frac{5}{3}, b_3 = 0$

TABLE I: Continued.

Number	Equation (2)	CLBS (1)	Remarks
11	$u_t = (u^{-3/2}u_x)_x + d_1u + d_2u^{5/2}$	$\eta = u_3 - \frac{15}{2u}u_1u_2 + \frac{35}{4u^2}u_1^3 + d_3\exp\left(-\frac{3}{2}d_1t\right)u^{5/2}$	$b_2 = 3, b_3 = 0, b_4 = 1$
12	$u_t = (u^{-3/2}u_x)_x + d_1u$	$\eta = u_3 - \frac{15}{2u}u_1u_2 + \frac{35}{4u^2}u_1^3 + d_2\exp\left(-\frac{3}{2}d_1t\right)u^{5/2} + d_3\exp\left(-\frac{7}{2}d_1t\right)u^{9/2}$	$b_2 = 3, b_3 = 0, b_4 = 1$
13	$u_t = (u^{-3/2}u_x)_x$	$\eta = u_3 - \frac{15}{2u}u_1u_2 + \frac{35}{4u^2}u_1^3 + d_1u^{5/2} + d_2u^{9/2}$	$b_2 = 3, b_3 = 0$
14	$u_t = (u^{(-4/3)u_x})_x + d_1u + d_2u^{5/3} + d_3u^{-1/3}$	$\eta = u_3 - \frac{5}{u}u_1u_2 + \frac{40}{9u^2}u_1^3 - \frac{4}{3}d_3u_1$	$b_2 = \frac{65}{14}, b_3 = \frac{5}{2}, b_4 = 1$
15	$u_t = (u^{-1}u_x)_x$	$\eta = u_3 - \frac{6}{u}u_1u_2 + \frac{6}{u^2}u_1^3 + d_1u_1$	$b_2 = 1, b_3 = -2$
16	$u_t = (u^{-1}u_x)_x + d_1u$	$\eta = u_3 - \frac{4}{u}u_1u_2 + \frac{3}{u^2}u_1^3 + d_2\exp(-2d_1t)u^2u_1$	$b_2 = 4, b_3 = 2, b_4 = 1$
17	$u_t = (u^{-1}u_x)_x + d_1$	$\eta = u_3 - \frac{4}{u}u_1u_2 + \frac{3}{u^2}u_1^3 - d_1u_1$	$b_2 = 4, b_3 = 2$
18	$u_t = (u^{-1}u_x)_x$	$\eta = u_3 - \frac{4}{u}u_1u_2 + \frac{3}{u^2}u_1^3 + d_1u^2u_1$	$b_2 = 4, b_3 = 2$
19	$u_t = (uu_x)_x + d_1u^2 + d_2u + d_3$	$\eta = u_3 + \frac{1}{2}d_1u_1$	$b_3 = 10, b_4 = 1$
20	$u_t = (uu_x)_x + d_1$	$\eta = u_3$	$b_3 = 10$

(iii) For $d_3 = 0$,

$$u(x, t) = [\alpha(t) + \beta(t)x + \gamma(t)x^2]^{1/k}, \quad (41)$$

where $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ satisfy three-dimensional dynamical system

$$\begin{aligned} \alpha' &= \frac{1}{k}\beta^2 + 2\alpha\gamma + kd_1\alpha + kd_2, \\ \beta' &= \frac{2(k+2)}{k}\beta\gamma + kd_1\beta, \\ \gamma' &= \frac{2(k+2)}{k}\gamma^2 + kd_1\gamma. \end{aligned} \quad (42)$$

Example 2. Equation

$$u_t = (u^{-4/3}u_x)_x + d_1u + d_2u^{5/3} \quad (43)$$

admits the CLBS

$$\eta = u_3 - \frac{5}{u}u_1u_2 + \frac{40}{9u^2}u_1^3. \quad (44)$$

The corresponding solutions of (43) are given by

$$u(x, t) = [\alpha(t) + \beta(t)x + \gamma(t)x^2]^{-3/2}, \quad (45)$$

where $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ satisfy three-dimensional dynamical system

$$\begin{aligned} \alpha' &= 2\alpha^2\gamma - \frac{1}{2}\alpha\beta^2 - \frac{2}{3}d_1\alpha - \frac{2}{3}d_2, \\ \beta' &= -\frac{1}{2}\beta^3 + 2\alpha\beta\gamma - \frac{2}{3}d_1\beta, \\ \gamma' &= 2\alpha\gamma^2 - \frac{1}{2}\beta^2\gamma - \frac{2}{3}d_1\gamma. \end{aligned} \quad (46)$$

Example 3. Equation

$$u_t = (u^{-1}u_x)_x + d_1u^2 + d_2u \quad (47)$$

admits the CLBS

$$\eta = u_3 - \frac{6}{u}u_1u_2 + \frac{6}{u^2}u_1^3. \quad (48)$$

The corresponding solutions are given by

$$u(x, t) = -\frac{\beta(t)}{\alpha(t) - (x + c_1)^2}, \quad (49)$$

where $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ are listed as follows.

(i) For $d_2 \neq 0$,

$$\begin{aligned} \alpha(t) &= \left[\frac{2d_1 \ln(c_2d_2 \exp(d_2t) - 2)}{d_2^4c_2^2} + \frac{d_1 \exp(d_2t)}{d_2^3c_2} \right. \\ &\quad \left. + c_3 \right] [c_2d_2 - 2 \exp(-d_2t)], \end{aligned} \quad (50)$$

$$\beta(t) = c_2 \exp(d_1t) - \frac{2}{d_2}.$$

(ii) For $d_2 = 0$,

$$\begin{aligned} \alpha(t) &= \left[\frac{1}{2}d_1 \ln(2t + c_2) + c_3 \right] (2t + c_2)^2, \\ \beta(t) &= 2t + c_2. \end{aligned} \quad (51)$$

Example 4. Equation

$$u_t = (u^{-1}u_x)_x \quad (52)$$

admits the CLBS

$$\eta = u_3 - \frac{4}{u}u_1u_2 + \frac{3}{u^2}u_1^3 + d_1u^2u_1. \quad (53)$$

The corresponding solutions are listed as follows.

(i) For $d_1 > 0$,

$$u(x, t) = \frac{-4d_1c_1^2 \exp\left[-\sqrt{c_1^2/d_1}(x + \gamma(t))\right]}{-\exp\left[-2\sqrt{c_1^2/d_1}(x + \gamma(t))\right] + 4d_1^3 \exp\left[-\sqrt{c_1^2/d_1}(x + \gamma(t))\right] \beta(t) - 4d_1^4 \alpha(t)}, \quad (54)$$

where the unknown functions about t are, respectively, given as

$$\alpha(t) = c_1^2 [1 + \tan^2(c_1t + c_2)],$$

$$\beta(t) = -\frac{c_1}{d_1} \tan(c_1t + c_2),$$

$$\gamma(t) = c_3 - \frac{1}{2} \sqrt{\frac{d_1}{c_1^2}} \ln[1 + \tan^2(c_1t + c_2)].$$

(55)

(ii) For $d_1 < 0$,

$$u(x, t) = \frac{4d_1 c_1^2 \exp \left[-\sqrt{-c_1^2/d_1} (x + \gamma(t)) \right]}{-\exp \left[-2\sqrt{-c_1^2/d_1} (x + \gamma(t)) \right] + 4d_1^3 \exp \left[-\sqrt{-c_1^2/d_1} (x + \gamma(t)) \right] \beta(t) - 4d_1^4 \alpha(t)}. \quad (56)$$

The unknown functions in the solutions are, respectively, given as

$$\begin{aligned} \alpha(t) &= c_1^2 \left[\tanh^2(c_1 t + c_2) - 1 \right], \\ \beta(t) &= \frac{c_1}{d_1} \tanh(c_1 t + c_2), \\ \gamma(t) &= c_3 + \frac{1}{2} \sqrt{-\frac{d_1}{c_1^2}} \ln \left[\tanh^2(c_1 t + c_2) - 1 \right]. \end{aligned} \quad (57)$$

Exact solutions of the fast diffusion equation (52) are investigated from the symmetry point of view in many literatures. The widest and most arranged collection of such solutions is given in [37]. The second-order CLBS and corresponding symmetry reductions for (52) are referred to [12]. The solutions constructed here are not equivalent to the ones in [12, 37].

5. Conclusions

In this paper, the linear determining equations with parameters are applied to construct CLBSs of the nonlinear reaction-diffusion equation (2). The method generalizes the determining equations used in the search for admissible Lie operators. The procedure for the derivation of exact solutions to the evolution equation (10) is sketched as below. Firstly, finding solutions of the linear determining equations (20), we will obtain the explicit form of CLBSs (1) for (2). Solving the obtained differential constraints, we obtain u including three arbitrary functions depending on t . Consequently, substituting the form of u into (10) will lead to three-dimensional dynamical system for the undetermined functions about t .

We are mainly concerned with third-order CLBSs of (2) in the present paper. The corresponding exact solutions are constructed due to the resulting symmetry reductions. Naturally, a question will arise: how complicated can the CLBS admitted by the evolution equation (10) be? In other words, what is the vast value of n in the CLBS (9) of (10)? For linear case (7), the question is answered by Svirshchevskii in [38] due to the dimension estimate of invariant subspace admitted by the nonlinear operator F , where $F[u] = F(x, t, u, u_1, \dots, u_k)$. Does the resulting relation $n \leq 2k + 1$ there is still effective for the nonlinear case? This will be involved in our future study.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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