Research Article

The Perturbed Riemann Problem for Special Keyfitz-Kranzer System with Three Piecewise Constant States

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The Riemann problem for a special Keyfitz-Kranzer system is investigated and then seven different Riemann solutions are constructed. When the initial data are chosen as three piecewise constant states under suitable assumptions, the global solutions to the perturbed Riemann problem are constructed explicitly by studying all occurring wave interactions in detail. Furthermore, the stabilities of solutions are obtained under the specific small perturbations of Riemann initial data.

1. Introduction

The Keyfitz-Kranzer system is a pair of conservation laws in the following form [1]:

$$\begin{aligned} u_t + (\phi u)_x &= 0, \\ v_t + (\phi v)_x &= 0, \end{aligned} \quad (1)$$

in which $\phi = \phi(u, v)$ is a given function. System (1) arises as a model for the stretched elastic string to describe the propagation of forward longitudinal and transverse waves [1, 2]. It is also used to illustrate certain features of the solar wind in magnetohydrodynamics [3, 4]. Furthermore, it is usually assumed that ϕ is a function of r , where $r = \sqrt{u^2 + v^2}$, which is called the symmetric Keyfitz-Kranzer system and thus widely investigated such as in [3, 5]. Thus, system (1) was also taken as an example for a nonstrictly hyperbolic system of conservation laws. In addition, it should be pointed out that the nonsymmetric Keyfitz-Kranzer system

$$\begin{aligned} \rho_t + (\rho \phi(\rho, u_1, u_2, \dots, u_n))_x &= 0, \\ (\rho u_i)_t + (\rho u_i \phi(\rho, u_1, u_2, \dots, u_n))_x &= 0, \\ i &= 1, 2, \dots, n, \end{aligned} \quad (2)$$

has also been proposed and widely investigated in [6–8]; also see [9, 10] about the Riemann problem for some special forms of system (2).

Recently, it has been assumed in [11], where ϕ is a function of $r = au + bv$, that system (1) can be simplified into the form

$$\begin{aligned} r_t + (r\phi(r))_x &= 0, \\ u_t + (u\phi(r))_x &= 0. \end{aligned} \quad (3)$$

One may also see [12] where ϕ is a function of $r = u/v$. It is easily shown that system (3) owns two eigenvalues $\lambda_1 = \phi$ and $\lambda_2 = \phi + r\phi_r$; thus system (3) is also nonstrictly hyperbolic. More precisely, the characteristic field associated with λ_1 is linearly degenerate and the characteristic field associated with λ_2 is genuinely nonlinear when $r(r\phi)_{rr} \neq 0$ or otherwise linearly degenerate when $r(r\phi)_{rr} = 0$. In order to ensure $r(r\phi)_{rr} \neq 0$, the assumptions $\phi_r > 0$, $(r\phi)_{rr} > 0$, and $\phi(0) = 0$ were made and then the delta shock wave was captured in [11]. In the present paper, we give up the assumption $\phi_r > 0$ and want to discover some new interesting nonlinear phenomena. More precisely, we take the detailed example $\phi(r) = (c_1 r + c_2/r)r$, such that system (3) is simplified into

$$\begin{aligned} r_t + \left(\left(c_1 r + \frac{c_2}{r} \right) r \right)_x &= 0, \\ u_t + \left(\left(c_1 r + \frac{c_2}{r} \right) u \right)_x &= 0, \end{aligned} \quad (4)$$

in which c_1 and c_2 are two given positive constants, which enable us to deal with system (4) in completely explicit forms.

It is easily shown that system (4) has two eigenvalues $\lambda_1 = 2c_1 r$ and $\lambda_2 = c_1 r + c_2/r$; thus we can deduce that $\lambda_1 > \lambda_2$ for $r > \sqrt{c_2/c_1}$ and otherwise $\lambda_1 < \lambda_2$ for $0 < r < \sqrt{c_2/c_1}$. It is easy to see that the special form (4) of the Keyfitz-Kranzer system (1) is also nonstrictly hyperbolic in the quarter (r, u) phase plane. One of the main features of system (4) lies in that the shock curves have the same representation with the rarefaction ones for the λ_2 -characteristic family in the quarter (r, u) phase plane thanks to the special form of system (4), which belongs to the so-called Temple class for some hyperbolic systems of conservation laws [13, 14]. Compared with general hyperbolic systems of conservation laws, the well-posed result for Temple class may be achieved in a more general sense of initial data.

In this paper, we want to construct the global solutions to the particular Cauchy problem for system (4) in fully explicit forms when the three piecewise constant states are taken for the initial conditions as follows:

$$(r, u)(x, 0) = \begin{cases} (r_-, u_-), & -\infty < x < 0, \\ (r_m, u_m), & 0 < x < x_0, \\ (r_+, u_+), & x_0 < x < +\infty, \end{cases} \quad (5)$$

in which $x_0 > 0$ is arbitrarily small. This type of initial data (5) has been widely used to study the wave interaction problem [15–17] for different hyperbolic systems of conservation laws. It is worthwhile to notice that the initial data (5) may be regarded as a special small perturbation of the corresponding Riemann initial data

$$(r, u)(x, 0) = \begin{cases} (r_-, u_-), & -\infty < x < 0, \\ (r_+, u_+), & 0 < x < +\infty. \end{cases} \quad (6)$$

Thus, the particular Cauchy problem (4) and (5) is usually called the perturbed Riemann problem (or the double Riemann problems) in literature.

The first task of this paper is to construct the solutions to the Riemann problem (4) and (6) when the Riemann initial data (6) lie in the quarter (r, u) phase plane. More precisely, we find seven different combinations for the solutions to the Riemann problem (4) and (6) according to the choices of Riemann initial data (6). In particular, some composite waves are needed be introduced in the constructions of Riemann solutions. The second task of this paper is to construct the global solutions to the perturbed Riemann problem (4) and (5), which is essential to study various possible interactions of elementary waves for system (4). For simplicity, we restrict ourselves to consider the situation that the condition $r_-, r_m, r_+ > \sqrt{c_2/c_1}$ is satisfied for the initial data (5). Under this assumption, the global solutions to the perturbed Riemann problem (4) and (5) are constructed in fully explicit forms by studying all the wave interactions appearing in the construction processes of solutions. Furthermore, we can see that the solutions of the Riemann problem (4) and (6) are stable under the particular small

perturbation (5) of Riemann initial data (6) when the limit $x_0 \rightarrow 0$ is taken in the solutions. In addition, it should be pointed out that the wave interaction problem for the Temple class has been widely investigated recently, such as for the Aw-Rascle model [18, 19], the Chaplygin gas model [20–23], and the chromatography model [24–28].

The paper is organized in the following way. In Section 2, the Riemann problem (4) and (6) is investigated and the Riemann solutions are constructed for seven different cases. In Section 3, under the suitable assumptions, the global solutions to the perturbed Riemann problem (4) and (5) are constructed in fully explicit forms by investigating all appearing wave interactions when the initial data are taken to be three piecewise constant states. In the end, the stability of Riemann solutions is analyzed with respect to the specific small perturbations (5) of the Riemann initial data (6).

2. The Riemann Problem for the Special Keyfitz-Kranzer System (4)

In this section, we are devoted to investigating the Riemann problem for system (4) associated with the Riemann initial data (6). Let us see, for example, [1, 3, 11, 12] about the related Riemann problem for system (4). We can also refer to [15, 29] for the general knowledge about the Riemann problem for hyperbolic systems of conservation laws.

By a simple calculation, it is obtained that there are two eigenvalues for system (4) as follows:

$$\begin{aligned} \lambda_1 &= 2c_1 r, \\ \lambda_2 &= c_1 r + \frac{c_2}{r}. \end{aligned} \quad (7)$$

We have $\lambda_1 > \lambda_2$ for $r > \sqrt{c_2/c_1}$; otherwise we have $\lambda_1 < \lambda_2$ for $0 < r < \sqrt{c_2/c_1}$. Thus, system (4) is nonstrictly hyperbolic in the quarter (r, u) phase plane ($r > 0, u > 0$). The corresponding right eigenvectors for system (4) are given, respectively, by

$$\begin{aligned} \vec{r}_1 &= (r, u)^T, \\ \vec{r}_2 &= (0, 1)^T. \end{aligned} \quad (8)$$

It is easy to get $\nabla \lambda_1 \cdot \vec{r}_1 = 2c_1 r$ and $\nabla \lambda_2 \cdot \vec{r}_2 = 0$, in which $\nabla = (\partial/\partial r, \partial/\partial u)$. The characteristic field is genuinely nonlinear for λ_1 provided that $r > 0$ and is always linear degeneracy for λ_2 . The waves associated with the first characteristic field will be either shock waves (denoted by S) or rarefaction waves (denoted by R), which are determined by the choice of initial data. The waves associated with the second characteristic field are always contact discontinuities (denoted by J).

For the Riemann problem (4) and (6), it is invariant under uniform stretching of coordinates: $(x, t) \rightarrow (\alpha x, \alpha t)$ with $\alpha > 0$, such that we may consider solutions in the self-similar form

$$(r, u)(x, t) = (r, u)(\xi), \quad \xi = \frac{x}{t}. \quad (9)$$

By carrying out the self-similar transformation $\xi = x/t$, the Riemann problem (4) and (6) may be reformulated as the ordinary different equations

$$\begin{aligned} -\xi r_\xi + 2c_1 r r_\xi &= 0, \\ \left(c_1 r + \frac{c_2}{r} - \xi\right) u_\xi + \left(c_1 u - \frac{c_2 u}{r^2}\right) r_\xi &= 0, \end{aligned} \quad (10)$$

associated with the boundary values at infinity $(r, u)(\pm\infty) = (r_\pm, u_\pm)$.

For smooth solutions, (10) is equivalent to

$$\begin{pmatrix} 2c_1 r - \xi & 0 \\ c_1 u - \frac{c_2 u}{r^2} & c_1 r + \frac{c_2}{r} - \xi \end{pmatrix} \begin{pmatrix} dr \\ du \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (11)$$

It means either constant state $(r, u) = \text{const}$ or singular solution, which is a wave of the first characteristic family,

$$\begin{aligned} \xi &= c_1 r + \frac{c_2}{r}, \\ dr &= 0, \end{aligned} \quad (12)$$

or a wave of the second characteristic family,

$$\begin{aligned} \xi &= 2c_1 r, \\ \left(c_1 u - \frac{c_2 u}{r^2}\right) dr + \left(c_1 r + \frac{c_2}{r} - 2c_1 r\right) du &= 0. \end{aligned} \quad (13)$$

For the given left state (r_-, u_-) , by integrating (12), the contact discontinuity curves in the quarter (r, u) phase plane, which are the state set that can be connected on the right by a contact discontinuity, are as follows:

$$J : \begin{cases} \xi = c_1 r + \frac{c_2}{r}, \\ r = r_-. \end{cases} \quad (14)$$

Similarly, by integrating (13), the rarefaction wave curves in the quarter (r, u) phase plane, which are the state set that can be connected on the right by a rarefaction wave, are as follows:

$$R : \begin{cases} \xi = 2c_1 r, \\ \frac{u}{r} = \frac{u_-}{r_-}, \\ r > r_-. \end{cases} \quad (15)$$

On the other hand, for a bounded discontinuity at $\xi = \sigma$, the Rankine-Hugoniot conditions read

$$\begin{aligned} -\sigma [r] + \left[\left(c_1 r + \frac{c_2}{r}\right) r\right] &= 0, \\ -\sigma [u] + \left[\left(c_1 r + \frac{c_2}{r}\right) u\right] &= 0, \end{aligned} \quad (16)$$

in which $\sigma = dx/dt$ is the propagation speed of the discontinuity and $[r] = r_+ - r_-$ is the jump across the discontinuity with $r_- = r(x(t) - 0, t)$ and $r_+ = r(x(t) + 0, t)$. When the two states $(r_\pm, u_\pm)$ can be connected by only one

discontinuous wave directly, by solving (16), one can see if it is a contact discontinuity corresponding to a wave of the first characteristic family; then they should satisfy

$$J : \begin{cases} \sigma = c_1 r_- + \frac{c_2}{r_-} = c_1 r_+ + \frac{c_2}{r_+}, \\ r_- = r_+. \end{cases} \quad (17)$$

Otherwise, if it is a shock wave corresponding to a wave of the second characteristic family, then they should satisfy

$$S : \begin{cases} \sigma = c_1 (r_- + r_+), \\ \frac{u_-}{r_-} = \frac{u_+}{r_+}, \\ r_+ < r_-, \end{cases} \quad (18)$$

in which the entropy condition is taken into account.

Using these elementary waves, we are now in a position to construct the solutions to the Riemann problem (4) and (6) by the analysis method in the quarter (r, u) phase plane. It can be seen from (14), (15), (17), and (18) that if the Riemann initial data (6) satisfy $r_- = r_+$ or $u_-/r_- = u_+/r_+$, then the Riemann solution contains only a single wave. For the other situations, we can construct the following seven different combinations for the solutions to the Riemann problem (4) and (6) according to the choices of Riemann initial data (6).

(1) When $0 < r_+ < r_- < \sqrt{c_2/c_1}$, the solution can be expressed by the symbol $S + J$ (see Figure 1), which is given by

$$(r, u)(x, t) = \begin{cases} (r_-, u_-), & x < \sigma t, \\ \left(r_+, \frac{r_+ u_-}{r_-}\right), & \sigma t < x < \lambda_2(r_+) t, \\ (r_+, u_+), & x > \lambda_2(r_+) t. \end{cases} \quad (19)$$

The symbol $S + J$ is used to stand for a shock wave S followed by a contact discontinuity J . In what follows, similar symbols are also used and not explained again without confusion. Here and below, the symbol $\sigma = c_1(r_- + r_+)$ is used to indicate the propagation speed of the shock wave S and $\lambda_2(r_+) = c_1 r_+ + c_2/r_+$ is used to indicate the propagation speed of the contact discontinuity.

(2) When $0 < r_- < r_+ < \sqrt{c_2/c_1}$, the solution is $R + J$ (see Figure 2) in the form

$$(r, u)(x, t) = \begin{cases} (r_-, u_-), & x < \lambda_1(r_-) t, \\ \left(\frac{x}{2c_1 t}, \frac{u_- x}{2c_1 r_- t}\right), & \lambda_1(r_-) t < x < \lambda_1(r_+) t, \\ \left(r_+, \frac{r_+ u_-}{r_-}\right), & \lambda_1(r_+) t < x < \lambda_2(r_+) t, \\ (r_+, u_+), & x > \lambda_2(r_+) t. \end{cases} \quad (20)$$

FIGURE 1: The Riemann solution of (4) and (6) is shown for $0 < r_+ < r_- < \sqrt{c_2/c_1}$. FIGURE 2: The Riemann solution of (4) and (6) is shown for $0 < r_- < r_+ < \sqrt{c_2/c_1}$.

(3) When $r_- > r_+ > \sqrt{c_2/c_1}$, the solution is $J + S$ (see Figure 3) given by

$$(r, u)(x, t) = \begin{cases} (r_-, u_-), & x < \lambda_2(r_-)t, \\ \left(r_-, \frac{r_-u_+}{r_+}\right), & \lambda_2(r_-)t < x < \sigma t, \\ (r_+, u_+), & x > \sigma t, \end{cases} \quad (21)$$

in which the propagation speed of the contact discontinuity is $\lambda_2(r_-) = c_1 r_- + c_2/r_-$ and the propagation speed of the shock wave is also $\sigma = c_1(r_- + r_+)$.

(4) When $r_+ > r_- > \sqrt{c_2/c_1}$, the solution is $J + R$ (see Figure 4) in the form

$$(r, u)(x, t) = \begin{cases} (r_-, u_-), & x < \lambda_2(r_-)t, \\ \left(r_-, \frac{r_-u_+}{r_+}\right), & \lambda_2(r_-)t < x < \lambda_1(r_-)t, \\ \left(\frac{x}{2c_1t}, \frac{u_+x}{2c_1r_+t}\right), & \lambda_1(r_-)t < x < \lambda_1(r_+)t, \\ (r_+, u_+), & x > \lambda_1(r_+)t. \end{cases} \quad (22)$$

(5) When $r_+ > \sqrt{c_2/c_1} > r_- > 0$, the solution can be expressed by a composite wave $R_1 + J + R_2$ (see Figure 5), which is given by

$$(r, u)(x, t) = \begin{cases} (r_-, u_-), & x < \lambda_1(r_-)t, \\ \left(\frac{x}{2c_1t}, \frac{u_-x}{2c_1r_-t}\right), & \lambda_1(r_-)t < x < 2\sqrt{c_1c_2}t, \\ \left(\frac{x}{2c_1t}, \frac{u_+x}{2c_1r_+t}\right), & 2\sqrt{c_1c_2}t < x < \lambda_1(r_+)t, \\ (r_+, u_+), & x > \lambda_1(r_+)t. \end{cases} \quad (23)$$

(6) When $0 < r_+ < \sqrt{c_2/c_1} < r_-$ and $r_-r_+ < c_2/c_1$, we can also construct the solution $S + J$ (see Figure 6) in the form

$$(r, u)(x, t) = \begin{cases} (r_-, u_-), & x < \sigma t, \\ \left(r_+, \frac{r_+u_-}{r_-}\right), & \sigma t < x < \lambda_2(r_+)t, \\ (r_+, u_+), & x > \lambda_2(r_+)t. \end{cases} \quad (24)$$

FIGURE 3: The Riemann solution of (4) and (6) is shown for $r_- > r_+ > \sqrt{c_2/c_1}$. FIGURE 4: The Riemann solution of (4) and (6) is shown for $r_+ > r_- > \sqrt{c_2/c_1}$. FIGURE 5: The Riemann solution of (4) and (6) is shown for $r_+ > \sqrt{c_2/c_1} > r_- > 0$. FIGURE 6: The Riemann solution of (4) and (6) is shown when $0 < r_+ < \sqrt{c_2/c_1} < r_-$ and $r_- r_+ < c_2/c_1$ are satisfied.

FIGURE 7: The Riemann solution of (4) and (6) is shown when $0 < r_+ < \sqrt{c_2/c_1} < r_-$ and $r_- r_+ > c_2/c_1$ are satisfied.

It can be calculated from (7) that

$$\begin{aligned}\lambda_1(r_-, u_-) &= 2c_1 r_-, \\ \lambda_2(r_-, u_-) &= c_1 r_- + \frac{c_2}{r_-}, \\ \lambda_1\left(r_+, \frac{r_- u_-}{r_+}\right) &= 2c_1 r_+, \\ \lambda_2\left(r_+, \frac{r_- u_-}{r_+}\right) &= c_1 r_+ + \frac{c_2}{r_+}.\end{aligned}\quad (25)$$

In view of $r_+ < \sqrt{c_2/c_1} < r_-$ and $r_- r_+ < c_2/c_1$, it can be concluded that the entropy condition should obey

$$\begin{aligned}\lambda_1(r_-, u_-) &> \lambda_2(r_-, u_-) > \sigma, \\ \lambda_1\left(r_+, \frac{r_- u_-}{r_+}\right) &< \sigma < \lambda_2\left(r_+, \frac{r_- u_-}{r_+}\right),\end{aligned}\quad (26)$$

which implies that the shock wave is a 1-shock wave.

(7) When $0 < r_+ < \sqrt{c_2/c_1} < r_-$ and $r_- r_+ > c_2/c_1$, the solution is $J + S$ (see Figure 7) which is given by

$$(r, u)(x, t) = \begin{cases} (r_-, u_-), & x < \lambda_2(r_-)t, \\ \left(r_+, \frac{r_- u_-}{r_+}\right), & \lambda_2(r_-)t < x < \sigma t, \\ (r_+, u_+), & x > \sigma t. \end{cases}\quad (27)$$

It follows from (7) that

$$\begin{aligned}\lambda_1(r_+, u_+) &= 2c_1 r_+, \\ \lambda_2(r_+, u_+) &= c_1 r_+ + \frac{c_2}{r_+}, \\ \lambda_1\left(r_-, \frac{r_- u_-}{r_+}\right) &= 2c_1 r_-, \\ \lambda_2\left(r_-, \frac{r_- u_-}{r_+}\right) &= c_1 r_- + \frac{c_2}{r_-}.\end{aligned}\quad (28)$$

Based on $r_+ < \sqrt{c_2/c_1} < r_-$ and $r_- r_+ > c_2/c_1$, it can be checked that the entropy condition should satisfy

$$\begin{aligned}\lambda_1(r_+, u_+) &< \lambda_2(r_+, u_+) < \sigma, \\ \lambda_2\left(r_-, \frac{r_- u_-}{r_+}\right) &< \sigma < \lambda_1\left(r_-, \frac{r_- u_-}{r_+}\right),\end{aligned}\quad (29)$$

which implies that the shock wave is a 2-shock wave.

It can be concluded from the above discussions that the solutions to the Riemann problem (4) and (6) can be constructed completely when the Riemann initial data (6) lie in the quarter (r, u) phase plane. It is obvious to see that if we give up the assumptions $\phi_r > 0$, $(r\phi)_{rr} > 0$, and $\phi(0) = 0$ in system (3), then the Riemann solutions are also changed and different from each other.

3. Construction of Global Solutions to the Perturbed Riemann Problem (4) and (5)

In this section, we are dedicated to the constructions of global solutions to the perturbed Riemann problem (4) and (5). In this paper, for simplicity, we restrict ourselves only to consider the situation that the condition $r_-, r_m, r_+ > \sqrt{c_2/c_1}$ is satisfied for the initial data (5). Under the above assumption, the wave interaction problems are investigated by employing the method of characteristics and then the global solutions to the perturbed Riemann problem (4) and (5) are constructed in fully explicit forms.

Case 1 ($J+S$ and $J+S$). We first consider the situation that each of the Riemann solutions originating from the initial points $(0, 0)$ and $(x_0, 0)$ is a contact discontinuity followed by a shock wave, respectively. This case arises when the condition $\sqrt{c_2/c_1} < r_+ < r_m < r_-$ is satisfied. In this case, when the time t is sufficiently small, the solution to the perturbed Riemann problem (4) and (5) can be represented as (see Figure 8)

$$\begin{aligned}(r_-, u_-) &+ J_1 + (r_1, u_1) + S_2 + (r_m, u_m) + J_3 + (r_2, u_2) \\ &+ S_4 + (r_+, u_+).\end{aligned}\quad (30)$$

FIGURE 8: The interaction between $J + S$ and $J + S$ is shown when $r_- > r_m > r_+ > \sqrt{c_2/c_1}$.

The intermediate states (r_1, u_1) and (r_2, u_2) may be given, respectively, by

$$\begin{aligned} (r_1, u_1) &= \left(r_-, \frac{r_- u_m}{r_m} \right), \\ (r_2, u_2) &= \left(r_m, \frac{r_m u_+}{r_+} \right). \end{aligned} \quad (31)$$

First of all, we have the following lemma to describe the interaction between the shock wave S_2 and the contact discontinuity J_3 .

Lemma 1. *The shock wave S_2 keeps up with the contact discontinuity J_3 in finite time and then the interaction between S_2 and J_3 happens. After the interaction, on the one hand, the contact discontinuity J_5 when across the shock wave S_2 is parallel to the contact discontinuity J_1 . On the other hand, the shock wave S_6 cannot change its direction when it passes through the contact discontinuity J_3 .*

Proof. The speeds of S_2 and J_3 can be calculated, respectively, by

$$\begin{aligned} \sigma_2 &= c_1 (r_- + r_m), \\ \tau_3 &= c_1 r_m + \frac{c_2}{r_m}. \end{aligned} \quad (32)$$

We can easily get

$$\sigma_2 - \tau_3 = c_1 r_- - \frac{c_2}{r_m} > 0, \quad (33)$$

which means that $\sigma_2 > \tau_3$. That is to say, the shock wave S_2 meets J_3 in finite time. The intersection (x_1, t_1) is determined by

$$\begin{aligned} x_1 &= \sigma_2 t_1, \\ x_1 - x_0 &= \tau_3 t_1, \end{aligned} \quad (34)$$

which enables us to gain

$$(x_1, t_1) = \left(\frac{c_1 (r_- + r_m) r_m x_0}{c_1 r_- r_m - c_2}, \frac{r_m x_0}{c_1 r_- r_m - c_2} \right). \quad (35)$$

The interaction between S_2 and J_3 happens at the point (x_1, t_1) , such that we can get a new Riemann problem with the initial data (r_1, u_1) and (r_2, u_2) again. It can be deduced from $r_1 = r_- > r_m = r_2 > \sqrt{c_2/c_1}$ that the Riemann solution is a new contact discontinuity denoted by J_5 followed by a new shock wave denoted by S_6 originating from the point (x_1, t_1) . The intermediate state (r_3, u_3) between J_5 and S_6 is given by

$$(r_3, u_3) = \left(r_-, \frac{r_- u_+}{r_+} \right). \quad (36)$$

The propagation speeds of J_5 and S_6 can be computed, respectively, by

$$\tau_5 = c_1 r_1 + \frac{c_2}{r_1} = c_1 r_- + \frac{c_2}{r_-}, \quad (37)$$

$$\sigma_6 = c_1 (r_3 + r_2) = c_1 (r_- + r_m). \quad (38)$$

That is to say, the contact discontinuities J_5 and J_1 are parallel and the shock wave cannot change its direction when it passes through the contact discontinuity. $\square$

Consequently, we consider the coalescence of two shock waves S_4 and S_6 belonging to the same family.

Lemma 2. *The two shock waves S_4 and S_6 merge into a new shock wave denoted by S_7 .*

Proof. The propagation speed of S_4 is

$$\sigma_4 = c_1 (r_m + r_+), \quad (39)$$

which together with (38) yields

$$\sigma_6 - \sigma_4 = c_1 (r_- - r_+) > 0, \quad (40)$$

which means that S_6 catches up with S_4 in finite time. The intersection (x_2, t_2) is determined by

$$\begin{aligned} x_2 &= \sigma_6 t_2, \\ x_2 - x_0 &= \sigma_4 t_2, \end{aligned} \quad (41)$$

which yields

$$(x_2, t_2) = \left(\frac{(r_- + r_m) x_0}{r_- - r_+}, \frac{x_0}{c_1 (r_- - r_+)} \right). \quad (42)$$

We can see from the (r, u) phase plane in Figure 8 that the two states (r_3, u_3) and (r_+, u_+) can be connected by a shock wave directly, for the reason that it follows from (36) that $u_3/r_3 = u_+/r_+$. It means that the two shock waves S_4 and S_6 merge into a new shock wave which is denoted by S_7 whose propagation speed is

$$\sigma_7 = c_1 (r_- + r_+). \quad (43)$$

It is easy to get $\sigma_4 < \sigma_7 < \sigma_6$. In other words, the propagation speed of S_7 is between those of S_4 and S_6 . The proof is completed. $\square$

Case 2 ($J + S$ and $J + R$). In this case, we investigate the situation that the Riemann solution is a contact discontinuity followed by a shock wave originating from the point $(0, 0)$ and a contact discontinuity followed by a rarefaction wave originating from the point $(x_0, 0)$ when the initial data (5) need to satisfy both $\sqrt{c_2/c_1} < r_m < r_-$ and $\sqrt{c_2/c_1} < r_m < r_+$. When t is small enough, the solution of the perturbed Riemann problem (4) and (5) may be expressed in the symbol (see Figures 9 and 10):

$$(r_-, u_-) + J_1 + (r_1, u_1) + S_2 + (r_m, u_m) + J_3 + (r_2, u_2) + R_4 + (r_+, u_+). \quad (44)$$

The intermediate states (r_1, u_1) , (r_2, u_2) , and (r_3, u_3) have the same expressions as those in Case 1. As in Case 1, S_2 collides with J_3 at the point (x_1, t_1) which is given by the same formula (35). Similarly, a new local Riemann problem is generated at the point (x_1, t_1) with the left state (r_1, u_1) and the right state (r_2, u_2) , which gives rise to a contact discontinuity denoted by J_5 and a shock wave denoted by S_6 . The propagation speeds of J_5 and S_6 can also be calculated by the formulae (37) and (38), such that we have $\tau_1 = \tau_5$ and $\sigma_6 = \sigma_2$. Now, we need to deal with the situation that the shock wave S_6 penetrates the rarefaction wave R_4 .

Lemma 3. *The shock wave S_6 overtakes the wave back of the rarefaction wave R_4 in finite time. More precisely, if $r_+ < r_-$, then S_6 is able to cancel the whole R_4 thoroughly (see Figure 9). Otherwise, if $r_+ > r_-$, then S_6 passes through R_4 incompletely and has the line $x - x_0 = 2c_1 r_- t$ in R_4 as its asymptote (see Figure 10).*

Proof. The propagation speeds of S_6 and the wave back of R_4 are calculated, respectively, by

$$\begin{aligned} \sigma_6 &= c_1 (r_- + r_m), \\ \xi_4(r_2, u_2) &= 2c_1 r_2 = 2c_1 r_m, \end{aligned} \quad (45)$$

such that we have

$$\sigma_6 - \xi_4(r_2, u_2) = c_1 (r_- - r_m) > 0. \quad (46)$$

It means that S_6 overtakes the wave back of R_4 in finite time. The intersection (x_2, t_2) is computed by

$$\begin{aligned} x_2 &= \sigma_6 t_2 = c_1 (r_- + r_m) t_2, \\ x_2 - x_0 &= \xi_4(r_2, u_2) t_2 = 2c_1 r_m t_2, \end{aligned} \quad (47)$$

which enables us to have

$$(x_2, t_2) = \left(\frac{(r_- + r_m)x_0}{r_- - r_m}, \frac{x_0}{c_1 (r_- - r_m)} \right). \quad (48)$$

After the time t_2 , S_6 begins to penetrate R_4 with a varying propagation speed since it has a rarefaction wave on its right. During the process of penetration, the shock wave is denoted by S_7 which can be calculated by

$$\begin{aligned} \frac{dx}{dt} &= c_1 (r + r_3), \\ \frac{x - x_0}{t} &= 2c_1 r, \\ x(t_2) &= x_2, \end{aligned} \quad (49)$$

in which we have $r_3 = r_-$ and r varies from r_m to r_+ . We can get

$$x = 2c_1 r_- t - 2\sqrt{c_1 x_0 (r_- - r_m)} t + x_0, \quad (50)$$

and by differentiating (50) with respect to t twice, we have

$$\frac{d^2 x}{dt^2} = \frac{1}{2} \sqrt{\frac{c_1 x_0 (r_- - r_m)}{t^3}} > 0. \quad (51)$$

It means that S_7 begins to accelerate during the process of penetration. Now, there exist two possible situations to happen which depend on the relation between r_- and r_+ .

(1) If $r_+ < r_-$, then S_7 is able to cancel R_4 completely and ends at the point (x_3, t_3) (see Figure 9) which is given by

$$\begin{aligned} x_3 - x_0 &= 2c_1 r_+ t_3, \\ x_3 - x_0 &= 2c_1 r_- t_3 - 2\sqrt{c_1 x_0 (r_- - r_m)} t_3, \end{aligned} \quad (52)$$

which enables us to get

$$(x_3, t_3) = \left(\frac{x_0 (r_-^2 + r_+^2 - 2r_+ r_m)}{(r_- - r_+)^2}, \frac{x_0 (r_- - r_m)}{c_1 (r_- - r_+)^2} \right). \quad (53)$$

The shock wave is denoted by S_8 when it penetrates R_4 completely, whose propagation speed is

$$\sigma_8 = c_1 (r_+ + r_-). \quad (54)$$

(2) Otherwise, if $r_+ > r_-$, then S_7 cannot cancel R_4 completely and has the line $x - x_0 = 2c_1 r_- t$ in R_4 as its asymptote (see Figure 10). $\square$

Case 3 ($J + R$ and $J + S$). In this case, we consider that the Riemann solution originating from the point $(0, 0)$ is a contact discontinuity followed by a rarefaction wave and the Riemann solution originating from the point $(x_0, 0)$ is a contact discontinuity followed by a shock wave when the initial data (5) need to satisfy both $\sqrt{c_2/c_1} < r_- < r_m$ and $\sqrt{c_2/c_1} < r_+ < r_m$ (see Figures 11 and 12). When t is small enough, the

FIGURE 9: The interaction between $J + S$ and $J + R$ is shown when $r_- > r_+ > r_m > \sqrt{c_2/c_1}$. FIGURE 10: The interaction between $J + S$ and $J + R$ is shown when $r_+ > r_- > r_m > \sqrt{c_2/c_1}$. FIGURE 11: The interaction between $J + R$ and $J + S$ is shown when $\sqrt{c_2/c_1} < r_+ < r_- < r_m$. FIGURE 12: The interaction between $J + R$ and $J + S$ is shown when $\sqrt{c_2/c_1} < r_- < r_+ < r_m$.

solution of the Riemann problem (4) and (5) may be displayed in the symbol form (see Figure 11):

$$(r_-, u_-) + J_1 + (r_1, u_1) + R_2 + (r_m, u_m) + J_3 + (r_2, u_2) + S_4 + (r_+, u_+). \quad (55)$$

The intermediate states (r_1, u_1) and (r_2, u_2) are given, respectively, by

$$(r_1, u_1) = \left(r_-, \frac{r_- u_m}{r_m} \right), \quad (56)$$

$$(r_2, u_2) = \left(r_m, \frac{r_m u_+}{r_+} \right).$$

Lemma 4. *The wave front of R_2 collides with the contact discontinuity J_3 in finite time. Consequently, on the one hand, the contact discontinuity penetrates the rarefaction wave completely. On the other hand, the rarefaction wave also passes through the contact discontinuity without changing its direction.*

Proof. The propagation speeds of the wave front R_2 and J_3 can be computed, respectively, by

$$\xi_2(r_m, u_m) = 2c_1 r_m, \quad (57)$$

$$\tau_3 = c_1 r_m + \frac{c_2}{r_m}.$$

We can get

$$\xi_2(r_m, u_m) - \tau_3 = c_1 r_m - \frac{c_2}{r_m} > 0. \quad (58)$$

It means that the wave front of R_2 overtakes J_3 at a point which can be calculated by

$$x_1 = \xi_2(r_m, u_m) t_1, \quad (59)$$

$$x_1 - x_0 = \tau_3 t_1,$$

which yields

$$(x_1, t_1) = \left(\frac{2c_1 r_m^2 x_0}{c_1 r_m^2 - c_2}, \frac{x_0 r_m}{c_1 r_m^2 - c_2} \right). \quad (60)$$

After the time t_1 , the contact discontinuity J_3 begins to penetrate the rarefaction wave R_2 . During the process of penetration, it is denoted by J_5 which can be calculated by

$$\frac{dx}{dt} = c_1 r + \frac{c_2}{r}, \quad (61)$$

$$\frac{x}{t} = 2c_1 r,$$

$$x(t_1) = x_1,$$

in which r varies from r_m to r_- . We can get

$$x = 2\sqrt{c_1 c_2 t^2 + c_1 r_m x_0 t}. \quad (62)$$

By differentiating (62) with respect to t twice, we have

$$\frac{d^2 x}{dt^2} = -\frac{1}{2} (c_1 c_2 t^2 + c_1 r_m x_0 t)^{-3/2} (c_1 r_m x_0)^2 < 0. \quad (63)$$

It means that J_5 begins to decelerate during the process of penetration. It is clear that J_5 is able to cancel R_2 completely and ends at the point (x_2, t_2) which is given by

$$x_2 = 2c_1 r_- t_2, \quad (64)$$

$$x_2 = 2\sqrt{c_1 c_2 t_2^2 + c_1 r_m x_0 t_2},$$

such that we have

$$(x_2, t_2) = \left(\frac{2c_1 r_- r_m x_0}{c_1 r_-^2 - c_2}, \frac{x_0 r_m}{c_1 r_-^2 - c_2} \right). \quad (65)$$

At the same time, the rarefaction wave R_2 also passes through the contact discontinuity J_5 without changing its direction and is denoted by R_7 after penetration, for the reason that the state variable r cannot change and only the state variable u changes during the process of penetration.

The intermediate state (r_3, u_3) between J_6 and R_2 can also be calculated by (36). The states (r_1, u_1) and (r_3, u_3) can be connected directly by a contact discontinuity denoted with J_6 for the reason that $r_1 = r_3 = r_-$. It is clear to see that the two contact discontinuities J_6 and J_1 are parallel to each other. $\square$

Next we consider the situation that the shock wave S_4 penetrates the rarefaction wave R_5 and have the following lemma to describe the result.

Lemma 5. *The shock wave S_4 can overtake the wave front of the rarefaction wave R_7 in finite time. More precisely, if $r_+ < r_3$, then S_4 can cancel the whole R_7 thoroughly (see Figure 11). Otherwise, if $r_+ > r_3$, then S_4 penetrates R_7 incompletely and finally has the line $x = 2c_1 r_+ t$ in R_7 as its asymptote (see Figure 12).*

Proof. The propagation speeds of S_4 and those of the wave front in the rarefaction wave R_7 can be computed, respectively, by

$$\sigma_4 = c_1 (r_m + r_+), \quad (66)$$

$$\xi_7(r_m, u_m) = 2c_1 r_m.$$

It is easy to calculate

$$\xi_7(r_m, u_m) - \sigma_4 = 2c_1 r_m - c_1 (r_m + r_+) > 0, \quad (67)$$

which implies that the wave front of R_7 overtakes S_4 in finite time. The intersection (x_3, t_3) is computed by

$$x_3 = \xi_7(r_m, u_m) t_3 = 2c_1 r_m t_3, \quad (68)$$

$$x_3 - x_0 = \sigma_4 (r_2, u_2) t_3 = c_1 (r_m + r_+) t_3,$$

such that we have

$$(x_3, t_3) = \left(\frac{2r_m x_0}{r_m - r_+}, \frac{x_0}{c_1 (r_m - r_+)} \right). \quad (69)$$

FIGURE 13: The interaction between $J + R$ and $J + R$ is shown when $\sqrt{c_2/c_1} < r_+ < r_m < r_+$.

After the time t_3 , S_4 begins to penetrate R_7 with a varying speed since it has a rarefaction wave on its right. During the process of penetration, the shock wave is denoted by S_8 which can be calculated by

$$\begin{aligned}\frac{dx}{dt} &= c_1(r_+ + r), \\ \frac{x}{t} &= 2c_1r, \\ x(t_3) &= x_3,\end{aligned}\quad (70)$$

in which r varies from $r_2 (=r_m)$ to $r_3 (=r_-)$. With the similar calculation as before, we can get

$$x = 2c_1r_+t + 2\sqrt{c_1x_0(r_m - r_+)}t. \quad (71)$$

By differentiating (71) with respect to t twice, we have

$$\begin{aligned}\frac{d^2x}{dt^2} &= -\frac{1}{2}(c_1x_0(r_m - r_+))^2(c_1x_0(r_m - r_+)t)^{-3/2} \\ &< 0.\end{aligned}\quad (72)$$

It means that S_8 begins to reduce the speed during the process of penetration.

As before, there also exist two possible situations to happen which depend on the relation between r_- and r_+ .

(1) If $r_+ < r_-$, then S_8 can cancel R_7 completely and end at the point (x_4, t_4) which is computed by

$$\begin{aligned}x_4 &= 2c_1r_mt_4, \\ x_4 &= 2c_1r_+t_4 + 2\sqrt{c_1x_0(r_m - r_+)}t_4,\end{aligned}\quad (73)$$

such that we get

$$(x_4, t_4) = \left(\frac{2r_mx_0}{r_m - r_+}, \frac{x_0}{c_1(r_m - r_+)} \right). \quad (74)$$

The shock wave is denoted by S_9 when it penetrates R_7 , whose propagation speed is $\sigma_9 = c_1(r_- + r_+)$.

(2) If $r_+ > r_3$, then S_8 cannot cancel R_7 completely and finally has the line $x = 2c_1r_+t$ in R_7 as its asymptote. $\square$

Case 4 ($J + R$ and $J + R$). Finally, we consider the situation that both the Riemann solutions originating from the points $(0, 0)$ and $(x_0, 0)$ are a contact discontinuity followed by a rarefaction wave when the initial data (5) meet $\sqrt{c_2/c_1} < r_+ < r_m < r_+$. When the time t is small enough, the solution to the perturbed Riemann problem (4) and (5) may be represented succinctly as (see Figure 13)

$$\begin{aligned}(r_-, u_-) &+ J_1 + (r_1, u_1) + R_2 + (r_m, u_m) + J_3 + (r_2, u_2) \\ &+ R_4 + (r_+, u_+).\end{aligned}\quad (75)$$

The calculation is similar to that in Case 3. Thus, we omit the detail and only draw Figure 13 to illustrate this case.

Up to now, the discussion for all kinds of interactions has been accomplished when the condition $r_-, r_m, r_+ > \sqrt{c_2/c_1}$ is required. The global solutions to the perturbed Riemann problem (4) and (5) are constructed completely in explicit forms when $r_-, r_m, r_+ > \sqrt{c_2/c_1}$. Thus, we can summarize the main results in this section by the theorem below.

Theorem 6. Under the assumption that the initial data (5) need to satisfy $r_-, r_m, r_+ > \sqrt{c_2/c_1}$, the large-time asymptotic states of the global solutions to the perturbed Riemann problem (4) and (5) are identical with the corresponding ones to the Riemann problem (4) and (6). In other words, the solutions to the Riemann problem (4) and (6) can be obtained if the limits $x_0 \rightarrow 0$ of the global solutions to the perturbed Riemann problem (4) and (5) are taken when $r_-, r_m, r_+ > \sqrt{c_2/c_1}$. Thus, it turns out that the Riemann solutions are stable with respect to the specific small perturbation (5) of Riemann initial data (6) when $r_-, r_m, r_+ > \sqrt{c_2/c_1}$.

In fact, if the initial data (5) satisfy $0 < r_-, r_m, r_+ < \sqrt{c_2/c_1}$, the situation can be dealt with similarly. For the other situations, the wave interactions are very complicated for the perturbed Riemann problem (4) and (5) and thus left for our future work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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