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Research Article

$L^1(\mathbb{R})$-Nonlinear Stability of Nonlocalized Modulated Periodic Reaction-Diffusion Waves

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Assuming spectral stability conditions of periodic reaction-diffusion waves $\bar{u}(x)$, we consider $L^1(\mathbb{R})$-nonlinear stability of modulated periodic reaction-diffusion waves, that is, modulational stability, under localized small initial perturbations with nonlocalized initial modulations. $L^p(\mathbb{R})$-nonlinear stability of such waves has been studied in Johnson et al. (2013) for $p \geq 2$ by using Hausdorff-Young inequality. In this note, by using the pointwise estimates obtained in Jung, (2012) and Jung and Zumbrun (2016), we extend $L^p(\mathbb{R})$-nonlinear stability ($p \geq 2$) in Johnson et al. (2013) to $L^1(\mathbb{R})$-nonlinear stability. More precisely, we obtain $L^1(\mathbb{R})$-estimates of modulated perturbations $\tilde{u}(x - \psi(x,t), t) - \bar{u}(x)$ of $\bar{u}$ with a phase function $\psi(x,t)$ under small initial perturbations consisting of localized initial perturbations $\tilde{u}(x - h_0(x), 0) - \bar{u}(x)$ and nonlocalized initial modulations $h_0(x) = \psi(x, 0)$.

1. Introduction

Many evolutionary PDEs possess spatially periodic traveling waves and their stability has been widely studied in recent years. In this paper, by using the results of [1–3], we consider $L^1(\mathbb{R})$-nonlinear modulational stability of spatially periodic traveling waves in a system of reaction-diffusion equations:

$$u_t = u_{xx} + f(u), \tag{1}$$

with $x \in \mathbb{R}$, $t \geq 0$, and $u \in \mathbb{R}^n$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is sufficiently smooth. Suppose that $u(x,t) = \bar{u}(x - ct)$ is a spatially 1-periodic traveling wave of (1) with a wave speed c. By substituting $u(x,t) = \bar{u}(x - ct)$ into (1), one can say that $\bar{u}(x)$ is a stationary 1-periodic solution of

$$u_t = u_{xx} + cu_x + f(u). \tag{2}$$

Indeed, $\bar{u}(x)$ is a 1-periodic profile of traveling wave ODEs:

$$0 = u_{xx} + cu_x + f(u). \tag{3}$$

Compared with other types of traveling wave solutions such as front or pulse, the main difficulty of study of stability of periodic traveling waves is that the linearized operator considered on the whole line has only essential spectrum; that is, the spectrum is continuous up to zero (see Section 1.1 for details). This gives no spectral gap from the origin. The spectral gap plays a very important role in the study of (linear and nonlinear) stability because it gives exponential decay of the linearized operator. This is the reason why stability of the periodic traveling waves has been an open problem for a long time.

However, in the late 1990s, nonlinear stability of bifurcating periodic traveling waves of Swift-Hohenberg equation with respect to the localized perturbation has been studied in [4, 5] by using renormalization techniques. Stable diffusive mixing of periodic reaction-diffusion waves has been obtained in [6] based on a nonlinear decomposition of phase and amplitude variables and renormalization techniques. Johnson, Zumbrun, and their collaborators also showed $L^p(\mathbb{R})$ ($p \geq 2$) nonlinear modulational stability of periodic traveling waves of systems of reaction-diffusion equations and of conservation under both localized and nonlocalized perturbations in [1, 7–9]. By using pointwise linear estimates together with a nonlinear iteration scheme developed by Johnson-Zumbrun, pointwise nonlinear stability of such waves has been also studied in [2, 3, 10]. For other related works on modulated periodic traveling waves, we refer readers to [11–13].

To begin with, we first review the concept of stability of $\bar{u}(x)$ of (2). Roughly speaking, we say that $\bar{u}(x)$ is (boundedly) stable if any other solutions $\tilde{u}(x,t)$ of (2) which are initially near $\bar{u}(x)$ stay near $\bar{u}(x)$ for all $t \geq 0$. More precisely, we understand the following (bounded) stability definition.

Definition 1. Let X_1 and X_2 be Banach spaces and suppose that $\bar{u}(x)$ is a stationary solution of nonlinear partial differential equation of the form

$$u_t = \mathscr{F}(u), \quad u = u(x,t), \tag{4}$$

where $\mathscr{F}$ includes all differential, linear, and nonlinear terms. Then we say $\bar{u}(x)$ is (boundedly) stable if, for all $\varepsilon > 0$, there is $\delta > 0$ such that, for all $\tilde{u}_0(x)$ with $\|\tilde{u}_0(x) - \bar{u}(x)\|_{X_1} < \delta$, the corresponding solutions $\tilde{u}(x,t)$ (with $\tilde{u}(x,0) = \tilde{u}_0(x)$) of (4) satisfy $\|\tilde{u}(x,t) - \bar{u}(x)\|_{X_2} < \varepsilon$ for all $t \geq 0$. In particular, if $\tilde{u}(x,t)$ converge to $\bar{u}(x)$ in X_2 as t tends to infinity, then we say that $\bar{u}(x)$ is asymptotically stable.

Here, we call $\tilde{u}(x,t) - \bar{u}(x)$ as a perturbation of $\bar{u}(x)$. Finally, in order to study stability of $\bar{u}(x)$, we need to estimate perturbations $\tilde{u}(x,t) - \bar{u}(x)$ in X_2 under initial perturbations $\tilde{u}_0(x) - \bar{u}(x)$ in X_1.

In this paper, we study modulational stability by estimating modulated perturbations $v(x,t) := \tilde{u}(x - \psi(x,t), t) - \bar{u}(x)$ in $L^1(\mathbb{R})$ with an appropriate modulation $\psi(x,t)$ with $\psi(x,0) = h_0(x) \notin L^1(\mathbb{R})$ when the initial perturbation $\tilde{u}_0(x) - \bar{u}(x)$ is not localized. To understand intuitively, suppose that $\tilde{u}_0(x) \sim \bar{u}(x + h_0(x))$ for some nonlocalized $h_0(x)$ with $h_{\pm\infty} = \lim_{x\to\pm\infty} h_0 \neq 0$; that is, nearby solutions $\tilde{u}_0(x)$ are obtained by shifting $\bar{u}$ slightly out of phase at $\pm\infty$. Then the nonlocalized data $\tilde{u}_0(x) - \bar{u}(x)$ might be approximated by a localized initial modulational perturbation $\tilde{u}_0(x - h_0) - \bar{u}(x)$ plus a nonlocalized initial modulation $\bar{u}' h_0$. Thus, our main investigation is that we choose an appropriate nonlocalized modulation $\psi(x,t)$ with $\psi(x,0) = h_0(x)$ such that the modulated perturbation $\tilde{u}(x - \psi(x,t), t) - \bar{u}(x)$ remains small in $L^1(\mathbb{R})$ for all time, showing modulational stability of $\bar{u}$, when initial modulated perturbations $\tilde{u}_0(x - h_0(x)) - \bar{u}(x)$ and $\partial_x h_0(x)$ are sufficiently small.

For $2 \leq p \leq \infty$, $L^p(\mathbb{R})$-estimates of such nonlocalized modulated perturbations have been already established in [1] and [9] for systems of reaction-diffusion equations and of conservation laws, respectively, by using the generalized Hausdorff-Young inequality

$$\begin{aligned} \|u\|_{L^p(\mathbb{R})} &\leq \|\check{u}\|_{L^q([-\pi,\pi];L^p([0,1]))} \\ &:= \left(\int_{-\pi}^{\pi} \|\check{u}(\xi,\cdot)\|^q_{L^p([0,1])} \, d\xi \right)^{1/q}, \end{aligned} \tag{5}$$

for $q \leq 2 \leq p$ and $1/p + 1/q = 1$, where $\check{u}(\xi,\cdot)$ is a Bloch transform of u defined below in (12). This is the reason why their stability analysis has been restricted to $p \geq 2$. In this paper, we extend their $L^p(\mathbb{R})$-stability results ($p \geq 2$) to $L^1(\mathbb{R})$-stability by using pointwise estimate of linear behaviors under localized data $v_0(x) := \tilde{u}_0(x - h_0(x)) - \bar{u}(x)$ and nonlocalized modulational data $\bar{u}' h_0(x)$ established in [2, 3].

1.1. Preliminaries. In order to study stability of $\bar{u}$, the spectral information of the linearization of (2) around $\bar{u}$ is required; so we first linearize the PDE (2) about $\bar{u}(x)$. In order to see the importance of linearization about $\bar{u}$, we consider again the general PDE of form (4). By setting perturbations of $\bar{u}$ as $v(x,t) = \tilde{u}(x,t) - \bar{u}(x)$, we have

$$\begin{aligned} v_t &= (\tilde{u} - \bar{u})_t = \tilde{u}_t = \mathscr{F}(\tilde{u}) = \mathscr{F}(v + \bar{u}) \\ &= \mathscr{F}(\bar{u}) + dF(\bar{u})\, v + O\left(|v|^2\right) \\ &= dF(\bar{u})\, v + O\left(|v|^2\right). \end{aligned} \tag{6}$$

Here, $v_t = dF(\bar{u})v$ is referred to as the linearization of (4) about $\bar{u}$; so, from this linearization, we obtain the linear perturbation equation.

We now linearize (2) around $\bar{u}$ and consider the eigenvalue problem of the form

$$\lambda v = Lv := v_{xx} + c v_x + df(\bar{u})\, v, \tag{7}$$

with 1-periodic coefficients. We consider this linear operator L on $L^2(\mathbb{R})$ with densely defined domain $H^2(\mathbb{R})$. In this section, we recall Bloch analysis [1, 3, 7] which is a key idea of spectral analysis of linear operators with periodic coefficients. If we apply Floquet theory [14] to the first-order ODE system obtained by (7), $\lambda \in \mathbb{C}$ lies in $L^2(\mathbb{R})$-spectrum of L if and only if v has the form

$$v(x) = e^{i\xi x} w(\xi, x), \tag{8}$$

for some Floquet exponent $\xi \in [-\pi, \pi)$ and 1-periodic function w in x. It means that, recalling the linear operator L acts on the whole line $\mathbb{R}$, there is no $L^2(\mathbb{R})$-eigenfunction of L; so $L^2(\mathbb{R})$-spectrum of L introduced in (7) is entirely essential; that is, there is no isolated eigenvalue (point spectrum). That is why the spectrum is continuous up to zero without spectral gap.

Inserting $v(x) = e^{i\xi x} w(\xi, x)$ into (7) yields one-parameter family of Bloch operators, for each $\xi \in [-\pi, \pi)$:

$$L_\xi := e^{-i\xi x} L e^{i\xi x} = \left(\partial_x + i\xi\right)^2 + c\left(\partial_x + i\xi\right) + df(\bar{u}), \tag{9}$$

operating on $L^2_{\text{per}}([0,1])$ with densely defined domain $H^2_{\text{per}}([0,1])$. That is, for any 1-periodic function f,

$$L\left(e^{i\xi x} f\right) = e^{i\xi x} L_\xi f. \tag{10}$$

Here, noting first that $H^2_{\text{per}}([0,1])$ is compactly embedded into $L^2_{\text{per}}([0,1])$, L_ξ has only point spectrum in $L^2_{\text{per}}([0,1])$ with eigenfunctions $w(\xi, x)$; in fact, $L^2(\mathbb{R})$-spectrum of L is given by the continuous union of these isolated eigenvalues of L_ξ for all $\xi \in [-\pi, \pi)$.

Continuing with this setup, we recall the inverse Bloch-Fourier representation. Applying the inverse Fourier transform, we have, for any $g \in L^2(\mathbb{R})$, that

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\hat{g}(\xi)\,d\xi = \frac{1}{2\pi}\sum_{j\in\mathbb{Z}}\int_{-\pi}^{\pi} e^{i(\xi+2\pi j)x}\hat{g}(\xi+2\pi j)\,d\xi; \tag{11}$$

so, for any $g \in L^2(\mathbb{R})$,

$$g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\xi x}\check{g}(\xi, x)\,d\xi, \tag{12}$$

where $\check{g}(\xi, x) = \sum_{j\in\mathbb{Z}} e^{i2\pi jx}\hat{g}(\xi + 2\pi j)$ is referred to as the Bloch transform and $\hat{g}(\cdot)$ denotes the Fourier transform. Furthermore, noting that $\check{g}(\xi, x)$ is 1-periodic in x, (10) and (12) give us the formula of periodic coefficient solution operator e^{Lt} for L:

$$S(t)\,g(x) := e^{Lt}g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\xi x}e^{L_\xi t}\check{g}(\xi, x)\,d\xi. \tag{13}$$

As a starting point, we used this formula to estimate linear behaviors of L in terms of the localized and nonlocalized data in [2] and [3], respectively. Indeed, pointwise estimates on Green function of L, by plugging $g(x)$ into the Dirac delta function $\delta_y(x)$, have been obtained in [2] to estimate $S(t)v_0$ for a localized data v_0. Moreover, in [3], we established the pointwise linear behavior of $S(t)$ on nonlocalized data $\bar{u}h_0$.

1.2. Spectral Stability. With these preparations, following [4, 5], we now state the spectral stability assumptions.

(D1) $L^2(\mathbb{R})$-spectrum of $L \subset \{\lambda \in \mathbb{C} : R(\lambda) < 0\} \cup \{0\}$.

(D2) $\lambda = 0$ is a simple eigenvalue of L_0.

(D3) There exists a constant $\theta > 0$ such that $R\sigma(L_\xi) \le -\theta|\xi|^2$ for all $\xi \in [-\pi, \pi)$.

From (D1), we see that the origin is the only neutral spectrum of L. By differentiating the traveling wave ODE (3), we obtain $L\bar{u}'(x) = 0$; so 0 is an eigenvalue of L_0 because $\bar{u}' \in L^2_{\mathrm{per}}([0,1])$. The second assumption (D2) implies that, for sufficiently small $|\xi|$, an eigenvalue $\lambda(\xi)$ of L_ξ bifurcating from 0 is analytic in ξ. If we use Taylor series expansion with respect to ξ, by (D3) and the complex symmetry $\lambda(-\xi) = \bar{\lambda}(\xi)$, $\lambda(\xi)$ can be written as

$$\lambda(\xi) = -ia\xi - b\xi^2 + O\left(|\xi|^3\right), \quad \text{for sufficiently small } |\xi|, \tag{14}$$

where $a \in \mathbb{R}$ and $b > 0$. Moreover, by the perturbation theory, the corresponding right and left eigenfunctions of L_ξ, denoted by $q(x,\xi)$ and $\tilde{q}(x,\xi)$, respectively, are also analytic in ξ for sufficiently small $|\xi|$. In particular, (D2) implies that one can take $q(x,0) = \bar{u}'(x)$ because $L_0\bar{u}'(x) = 0$. Moreover, condition (D3) was verified by direct numerical Evans function analysis in [12].

2. Main Result

For the nonlocalized initial modulation $h_0(x)$, we set $h_{\pm\infty}$ as a piecewise defined function

$$h_{\pm\infty}(x) = \begin{cases} h_{-\infty}, & x \le 0, \\ h_{+\infty}, & x > 0, \end{cases} \tag{15}$$

where $h_{+\infty} = \lim_{x\to\infty}h_0(x)$ and $h_{-\infty} = \lim_{x\to-\infty}h_0(x)$. Here, in order to make sense of Bloch transform framework for the nonlocalized data, we may assume $h_{-\infty} = -h_{+\infty}$. Indeed, for any asymptotic constants $h_{+\infty}$ and $h_{-\infty}$, we have $h_{-\infty} + k = -(h_{+\infty} + k)$ with $k = -(h_{+\infty} + h_{-\infty})/2$.

We are ready to state our main theorem. It says that, assuming (D1)–(D3), $\bar{u}$ is nonlinearly stable in $L^1(\mathbb{R})$ with some modulation $\psi(x,t)$ under small initial perturbations satisfying (16) and (17).

Theorem 2. *Let $\bar{u}(x)$ be a stationary 1-periodic solution of (2) satisfying the spectral stability conditions (D1)–(D3). For a sufficiently small number $E_0 > 0$, we assume initial data $\tilde{u}_0(x)$ and $h_0(x)$ satisfy*

$$\left\|\tilde{u}_0(x - h_0(x)) - \bar{u}(x)\right\|_{L^1(\mathbb{R})\cap H^3(\mathbb{R})} + \left\|\partial_x h_0\right\|_{W^{1,1}(\mathbb{R})\cap H^3(\mathbb{R})} \le E_0, \tag{16}$$

$$\left\|h_0(x) - h_{\pm\infty}\right\|_{L^1(\mathbb{R})} + \left|h_{\pm\infty}\right| \le E_0. \tag{17}$$

Then, for all initial data $\tilde{u}_0$ satisfying (16) and (17), the corresponding solution $\tilde{u}(x,t)$ to (2) satisfies that, for all $t \ge 0$,

$$\left\|\tilde{u}(x - \psi(x,t), t) - \bar{u}(x)\right\|_{L^1(x;\mathbb{R})} \le CE_0, \quad \left\|(\partial_x\psi, \partial_t\psi)(x,t)\right\|_{L^1(x;\mathbb{R})} \le CE_0, \tag{18}$$

for an appropriate modulation $\psi(x,t) \in W^{2,\infty}$ with $\psi(x,0) = h_0(x)$. Here the constant C depends on $M' > 0$ in Lemma 4 and we determine $\psi(x,t)$ in Section 4.

2.1. Discussion and Open Problems. As shown in Theorem 2, $\bar{u}$ is modulationally stable in $L^1(\mathbb{R})$ under nonlocalized initial perturbations. That is, even if we perturb the underlying solution $\bar{u}$ by shifting it slightly out of phase at $\pm\infty$, $\bar{u}$ is still stable in $L^1(\mathbb{R})$ with some modulation determined in Section 4. However, it is just boundedly stable, while the main theorem in [1] implies that the underlying periodic solution $\bar{u}$ is nonlinearly "asymptotically" stable in $L^p(\mathbb{R})$ ($p \ge 2$); that is, $\tilde{u}(x-\psi(x,t),t)$ converges to $\bar{u}$ in $L^p(\mathbb{R})$ as t goes to infinity. It makes sense if we plug $p = 1$ into L^p-estimates in [1]:

$$\left\|\tilde{u}(x - \psi(x,t), t) - \bar{u}(x)\right\|_{L^p(x;\mathbb{R})} \le CE_0(1+t)^{-(1/2)(1-1/p)}, \quad p \ge 2. \tag{19}$$

We emphasize again that we use pointwise estimates of the solution operator $S(t)$ in terms of the localized data $v_0 := \tilde{u}_0(x - h_0(x)) - \bar{u}(x)$ and the nonlocalized data $\bar{u}'h_0$ in order to obtain $L^1(\mathbb{R})$-stability, because we cannot apply the Hausdorff-Young inequality (5) as we mentioned in the Introduction. This is the reason why we need both initial conditions (16) and (17), while (16) is enough to prove $L^p(\mathbb{R})$-stability ($p \geq 2$) in [1].

Pointwise estimates give us more elaborate behaviors of the perturbation v like an exact solution of linear perturbation equation $v_t = Lv$; so pointwise estimate is a common method to get $L^1(\mathbb{R})$-stability. However, we need to make more effort to obtain pointwise bounds. As shown in [3], we need condition (17) in order to obtain pointwise estimate of $S(t)(\bar{u}'h_0)(x)$ for the case $|x| \gg Ct$ with a sufficiently large constant $C > 0$. This issue did not arise in [2] because only localized modulations $\psi(x,t)$ with $h_0(x) = \psi(x,0) \equiv 0$ were considered in [2]. However, the primary difficulty is to obtain pointwise estimates on the Green function $G(x,t;y)$ of the linear operator L for the case $|x-y| \gg Ct$ by using Bloch decomposition. If we solve the difficulty, we might delete condition (17) in Theorem 2, even in the main theorem of [3]. This problem was discussed in more detail in [3]. Pointwise nonlinear stability under nonlocalized perturbations in systems of conservation is an open problem.

The dynamics of modulated periodic traveling waves have been studied in [11] by the WKB approximations. We consider that the wave number of the periodic traveling waves is modulated by the function $\partial_x\psi$. As described in Theorem 2 and [1, 3], $\bar{u}$ is nonlinearly stable under small initial perturbations (16) and (17), with a heat kernel rate of decay in the wave number $\partial_x\psi$. One can clearly see this in the integral representation of $\psi(x)$ in Section 4.

3. Review of Previous Results

This section provides the previous results in [2, 3], particularly how to estimate the solution operator of L in terms of the localized and the nonlocalized data in pointwise sense. For some unknown modulation $\psi(x,t)$, we first define modulated perturbations v of the underlying periodic solution $\bar{u}$ as

$$v(x,t) = \tilde{u}(x-\psi(x,t),t) - \bar{u}(x), \tag{20}$$

for any solution $\tilde{u}$ of (2) near $\bar{u}$. Recalling the linear operator L in (7), we review the nonlinear perturbation equation about v established in [1, 7].

Lemma 3 (nonlinear perturbation equation). *The modulated perturbation v satisfies*

$$(\partial_t - L)(v + \bar{u}'\psi) = \mathcal{N}(x,t), \tag{21}$$

where $\mathcal{N}(x,t) := Q + R_x + (\partial_x^2 + \partial_t)Z + T$ *with*

$$\begin{aligned} Q &:= f(v(x,t)+\bar{u}(x)) - f(\bar{u}(x)) - df(\bar{u}(x))v, \\ R &:= -v\psi_t - v\psi_{xx} + (\bar{u}_x + v_x)\frac{\psi_x^2}{1-\psi_x}, \\ Z &:= v\psi_x, \\ T &:= -(f(v+\bar{u}) - f(\bar{u}))\psi_x. \end{aligned} \tag{22}$$

Applying Duhamel's principle to (21), we obtain

$$\begin{aligned} v(x,t) = {} & -\bar{u}'(x)\psi(x,t) + S(t)(v_0 + \bar{u}'h_0) \\ & + \int_0^t S(t-s)\mathcal{N}(s)\,ds, \end{aligned} \tag{23}$$

where $S(t)$ is the solution operator of L and the formula of $S(t)$ is given by (13).

The goal of this paper is to estimate (23) in $L^1(x;\mathbb{R})$ for an appropriate nonlocalized modulation $\psi(x,t)$. The most important and difficult part is estimating pointwise bounds of $S(t)$ with respect to the localized data v_0 and the nonlocalized data $\bar{u}'h_0$. For the localized data v_0, we use pointwise bounds on Green's function $G(x,t;y)$ of L in [2]

$$G(x,t;y) = \bar{u}'(x)E(x,t;y) + \widetilde{G}(x,t;y), \tag{24}$$

where

$$\begin{aligned} E(x,t;y) &= \frac{1}{\sqrt{4\pi bt}}e^{-|x-y-at|^2/4bt}\tilde{q}(y,0)\chi(t), \\ \left|\widetilde{G}(x,t;y)\right| &\lesssim \left((1+t)^{-1} + t^{-1/2}e^{-\eta t}\right)e^{-|x-y-at|^2/Mt}, \\ \left|\widetilde{G}_y(x,t;y)\right| &\lesssim t^{-1}e^{-|x-y-at|^2/Mt}, \end{aligned} \tag{25}$$

uniformly on $t \geq 0$, for some sufficiently large constants $M > 0$ and $\eta > 0$. Here $\tilde{q}$ is the periodic left eigenfunction of L_0 at $\lambda = 0$ discussed in Section 1.2 and $\chi(t)$ is a smooth cutoff function such that $\chi(t) = 0$ for $0 \leq t \leq 1/2$ and $\chi(t) = 1$ for $t \geq 1$.

For the nonlocalized data $\bar{u}'h_0$, we decompose the solution operator $S(t)$ as follows:

$$S(t) = S_*^p(t) + \widetilde{S}_*(t), \quad S_*^p(t) = \bar{u}'s_*^p(t), \tag{26}$$

with

$$\begin{aligned} s_*^p(t)(\bar{u}'h_0) &:= \int_{-\infty}^{\infty} e^{i\xi x}e^{(-ia\xi - b\xi^2)t}\hat{h}_0(\xi)\,d\xi \\ &= \int_{-\infty}^{\infty} (4\pi bt)^{-1/2}e^{-|x-y-at|^2/4bt}h_0(y)\,dy, \\ \widetilde{S}_*(t)(\bar{u}'h_0) &= (S(t) - S_*^p(t))(\bar{u}'h_0). \end{aligned} \tag{27}$$

Here, $\hat{}$ denotes the Fourier transform. This kind of decomposition was not needed in [2] because we considered only localized modulations $\psi(x,t)$ with $h_0(x) = 0$. Moreover, the decomposition of $S(t)$ here is rather different from the decomposition in [1] because we need pointwise estimates of $\widetilde{S}_*(t)(\bar{u}'h_0)$ obtained in [3] in order to prove $L^1(\mathbb{R})$-stability. To compare, we state the decomposition of the solution operator in [1]

$$S(t) = S^p(t) + \widetilde{S}(t), \quad S^p(t) = \bar{u}'s^p(t), \tag{28}$$

with

$$s^p(t)\left(\overline{u}'h_0\right) := \sum_{j\in\mathbb{Z}}\int_{\pi}^{\pi}\alpha(\xi) \cdot e^{\lambda(\xi)t}e^{i\xi x}\left\langle\widetilde{\phi}(\xi,\cdot)\overline{u}'(\cdot),e^{i2\pi j\cdot}\right\rangle_{L^2([0,1])}\widehat{h}_0(\xi + 2\pi j)\,d\xi,$$
$$\widetilde{S}(t)\left(\overline{u}'h_0\right) = \left(S(t) - S^p(t)\right)\left(\overline{u}'h_0\right). \tag{29}$$

Here, we have the worst term of $s^p(t)$ when $j = 0$. By using normalization $\langle\widetilde{\phi}(0,\cdot),\overline{u}'(\cdot)\rangle_{L^2([0,1])} = 1$ and normalization $|\widetilde{\phi}(0,\cdot) - \widetilde{\phi}(\xi,\cdot)| = O(|\xi|)$, one can actually see that those two decomposition types are not different in the sense that $s^p_*(t)(\overline{u}'h_0) \sim s^p(t)(\overline{u}'h_0)$ and $\widetilde{S}_*(t)(\overline{u}'h_0) \sim \widetilde{S}(t)(\overline{u}'h_0)$.

We now review the pointwise estimates of $S(t)$ on nonlocalized data $\overline{u}'h_0$. Our main purpose is to estimate $S(t)(\overline{u}'h_0)$ in terms of $|\partial_x h_0|$, $|h_{\pm\infty}|$, or $|(h_0 - h_{\pm\infty})(x)|$ which are small data. We first estimate $s^p_*(t)$ which has slower decay than $\widetilde{S}(t)$.

Recalling the Fourier transform of the nth derivative, $\widehat{h_0^{(n)}} = (i\xi)^n\widehat{h_0}(\xi)$, we have

$$\left|\partial_x s^p_*(t)\left(\overline{u}'h_0\right)\right| \leq \int_{-\infty}^{\infty}(4\pi bt)^{-1/2} \cdot e^{-|x-y-at|^2/4bt}\left|\partial_y h_0(y)\right|dy, \tag{30}$$

$$\left|\partial_x^2 s^p_*(t)\left(\overline{u}'h_0\right)\right| \leq \int_{-\infty}^{\infty}(4\pi bt)^{-1/2} \cdot e^{-|x-y-at|^2/4bt}\left|\partial_y^2 h_0(y)\right|dy, \tag{31}$$

$$\left|\partial_t s^p_*(t)\left(\overline{u}'h_0\right)\right| \lesssim \int_{-\infty}^{\infty}(4\pi bt)^{-1/2} \cdot e^{-|x-y-at|^2/4bt}\left(\left|\partial_y h_0(y)\right| + \left|\partial_y^2 h_0(y)\right|\right)dy. \tag{32}$$

We now present pointwise bounds $|\widetilde{S}_*(t)(\overline{u}'h_0)|$ obtained in [3]. See [3] for the proofs.

Lemma 4 (see [3]). *Suppose $|h_{\pm\infty}| \leq E_0$ for sufficiently small $E_0 > 0$. Then, for $|x| \gg Ct$ and sufficiently large $C > 0$,*

$$\left|\left(\widetilde{S}_*(t)\left(\overline{u}'h_0\right)\right)(x)\right| \lesssim \int_{-\infty}^{\infty} t^{-1/2}e^{-|x-y-at|^2/Mt}\left|\left(h_0 - h_{\pm\infty}\right)(y)\right|dy + E_0 e^{-\eta t}e^{-|x-at|^2/M't}. \tag{33}$$

For $|x| \ll Ct$,

$$\left|\left(\widetilde{S}_*(t)\left(\overline{u}'h_0\right)\right)(x)\right| \lesssim \int_{-\infty}^{\infty}\left[\left(1 + |x-y-at| + \sqrt{t}\right)^{-2} + t^{-1/2}e^{-|x-y-at|^2/M't}\right]\left|\partial_y h_0(y)\right|dy, \tag{34}$$

for a sufficiently large number $M'(> M > 0)$ (M denotes the constant in (25)).

Remark 5. We notice that the condition $|h_{\pm\infty}| \leq E_0$ is needed only for the case of $|x| \gg Ct$.

4. Proof of the Main Theorem

In this section, we determine the modulation $\psi(x,t)$ satisfying the nonlocalized initial data $\psi(x,0) = h_0$. We first recall the representation of v obtained from Lemma 3:

$$v(x,t) = -\overline{u}'(x)\psi(x,t) + S(t)\left(v_0 + \overline{u}'h_0\right) + \int_0^t S(t-s)\mathcal{N}(s)\,ds. \tag{35}$$

We now define $\psi(x,t)$ to cancel the bad estimate parts $s^p_*(t)$ from $S(t)$; that is,

$$\psi(x,t) := s^p_*(t)\left(\overline{u}'h_0\right) + \int_{-\infty}^{\infty}E(x,t;y)v_0(y)\,dy + \int_0^t\int_{-\infty}^{\infty}E(x,t-s;y)\mathcal{N}(y,s)\,dy\,ds. \tag{36}$$

Then we can easily check $\psi(x,0) = h_0(x)$ because of a cutoff function in E and the formula of $s^p_*(t)(\overline{u}'h_0)$. If we substitute $\psi(x,t)$ into (23), we rewrite v as

$$v(x,t) = \widetilde{S}_*(t)\left(\overline{u}'h_0\right) + \int_{-\infty}^{\infty}\widetilde{G}(x,t;y)v_0(y)\,dy + \int_0^t\int_{-\infty}^{\infty}\widetilde{G}(x,t-s;y)\mathcal{N}(y,s)\,dy\,ds. \tag{37}$$

We notice that the source term $\mathcal{N}$ consists of $|v|^2$, $|\psi_x|^2$, $|\psi_t|^2$, and $|\psi_{xx}|^2$; so we consider derivatives of $\psi(x,t)$ for $0 \leq t \leq 1$ and $0 \leq m \leq 2$:

$$\partial_t^k\partial_x^m\psi(x,t) = \partial_t^k\partial_x^m s^p_*(t)\left(\overline{u}'h_0\right) + \int_{-\infty}^{\infty}\partial_t^k\partial_x^m E(x,t;y)v_0(y)\,dy + \int_0^t\int_{-\infty}^{\infty}\partial_t^k\partial_x^m E(x,t-s;y)\mathcal{N}(y,s)\,dy\,ds. \tag{38}$$

Remark 6. The main idea of determining $\psi(x,t)$ is to cancel $E(x,t;y)$ and $\overline{u}'s^p_*(t)$ in $S(t)v_0$ and $S(t)(\overline{u}'h_0)$, respectively, because they are too big to handle, especially the source terms. By Lemma 4 and (30)~(32), we intuitively know that $v \sim \psi_x, \psi_t$. Compared with $L^p(\mathbb{R})$-bounds ($p \geq 2$) in [1], we need more elaborate decomposition of the solution operator $S(t)$ in order to estimate $|x| \gg Ct$. This is the reason why the decomposition of $S(t)$ here and in [3] is rather different from the decomposition in [1].

We now prove the main theorem with modulations $\psi(x,t)$ defined in (36).

Proof of Theorem 2. Assume that the initial perturbation $v_0(x) = \bar{u}(x-h_0(x),0)-\bar{u}(x)$ and the initial modulation $h_0(x)$ satisfy (16)~(17). We begin by estimating $\|v(\cdot,t)\|_{L^1(\mathbb{R})}$:

$$\begin{aligned}
&\|v(\cdot,t)\|_{L^1(\mathbb{R})} \\
&\quad \le \int_{-\infty}^{\infty} \left|\widetilde{S}_*(t)\left(\bar{u}' h_0\right)\right| dx \\
&\quad + \int_{-\infty}^{\infty} \left|\int_{-\infty}^{\infty} \widetilde{G}(x,t;y)\, v_0(y)\, dy\right| dx \\
&\quad + \int_{-\infty}^{\infty} \left|\int_0^t \int_{-\infty}^{\infty} \widetilde{G}(x,t-s;y)\, \mathcal{N}(y,s)\, dy\, ds\right| dx \\
&\quad = I + II + III.
\end{aligned} \tag{39}$$

By Lemma 4, for fixed $t>0$,

$$\begin{aligned}
I &= \int_{-\infty}^{\infty} \left|\widetilde{S}_*(t)\left(\bar{u}' h_0\right)\right| dx \lesssim \int_{|x|\gg Ct}\left[\int_{-\infty}^{\infty} t^{-1/2} e^{-|x-y-at|^2/Mt} \left|(h_0-h_{\pm\infty})(y)\right| dy + \left|h_{\pm\infty}\right| e^{-\eta t} e^{-|x-at|^2/M' t}\right] dx \\
&+ \int_{|x|\ll Ct}\left[\int_{-\infty}^{\infty}\left[\left(1+|x-y-at|+\sqrt{t}\right)^{-2} + t^{-1/2} e^{-|x-y-at|^2/Mt}\right]\left|\partial_y h_0(y)\right| dy\right] dx \\
&\lesssim \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\left(1+|x-y-at|+\sqrt{t}\right)^{-2} + t^{-1/2} e^{-|x-y-at|^2/Mt} dx\right] \times \left(\left|(h_0-h_{\pm\infty})(y)\right| + \left|\partial_y h_0(y)\right|\right) dy \\
&+ \int_{-\infty}^{\infty} \left|h_{\pm\infty}\right| e^{-\eta t} e^{-|x-at|^2/M' t} dx.
\end{aligned} \tag{40}$$

Since $\int_{-\infty}^{\infty}(1+|x-y-at|+\sqrt{t})^{-2}dx \lesssim (1+\sqrt{t})^{-1} \le 1$, $\int_{-\infty}^{\infty} t^{-1/2}e^{-|x-y-at|^2/Mt}dx \lesssim \sqrt{M}$, and $\int_{-\infty}^{\infty} e^{-\eta t}e^{-|x-at|^2/M't}dx \lesssim \sqrt{M'}$ for any $t>0$ and any $y\in\mathbb{R}$, we obtain

$$\begin{aligned}
I &\lesssim \int_{-\infty}^{\infty}\left(\left|(h_0-h_{\pm\infty})(y)\right| + \left|\partial_y h_0(y)\right|\right) dy + \left|h_{\pm\infty}\right| \\
&= \left\|(h_0-h_{\pm\infty})(y)\right\|_{L^1(\mathbb{R})} + \left\|\partial_x h_0\right\|_{L^1(\mathbb{R})} + \left|h_{\pm\infty}\right| \\
&\lesssim E_0.
\end{aligned} \tag{41}$$

By recalling pointwise bounds on $\widetilde{G}$ in (25), we have $\|\widetilde{G}(x,t;y)\|_{L^1(y;\mathbb{R})} \lesssim \sqrt{M}((1+t)^{-1}\sqrt{t}+e^{-\eta t}) \lesssim (1+t)^{-1/2} \le 1$ for all $x\in\mathbb{R}$, so that we estimate II as

$$\begin{aligned}
II &\lesssim \int_{-\infty}^{\infty} \left\|\widetilde{G}(x,t;y)\right\|_{L^1(y;\mathbb{R})} \left|v_0(y)\right| dx \lesssim \left\|v_0\right\|_{L^1} \\
&\le E_0.
\end{aligned} \tag{42}$$

In order to estimate III, we first review L^p-nonlinear stability estimate for $p\ge 2$ in [1]: for all $t\ge 0$,

$$\|v(x,t)\|_{H^1(x;\mathbb{R})}, \left\|\nabla_{x,t}\psi(x,t)\right\|_{H^1(x;\mathbb{R})} \lesssim E_0(1+t)^{-1/4}; \tag{43}$$

so

$$\begin{aligned}
\|(Q,R,Z,T)(y,s)\|_{L^1(y;\mathbb{R})} &\le \left\|\left(v,\psi_y,\psi_s\right)\right\|_{H^1(y;\mathbb{R})}^2 \\
&\lesssim E_0(1+s)^{-1/2}.
\end{aligned} \tag{44}$$

Then integration by parts implies that

$$\begin{aligned}
&III \\
&\lesssim \int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\left[(1+t-s)^{-1} + (t-s)^{-1/2} e^{-\eta(t-s)}\right] e^{-|x-y-a(t-s)|^2/M(t-s)} + (t-s)^{-1} e^{-|x-y-a(t-s)|^2/M(t-s)}\right] dx \left|(Q,R,Z,T)(y,s)\right| dy\, ds \\
&\lesssim \int_0^t\int_{-\infty}^{\infty}\left[(1+t-s)^{-1/2} + (t-s)^{-1/2}\right]\left|(Q,R,Z,T)(y,s)\right| dy\, ds \lesssim \int_0^t \left[(1+t-s)^{-1/2} + (t-s)^{-1/2}\right]\left\|(Q,R,Z,T)(y,s)\right\|_{L^1(y;\mathbb{R})} ds \\
&\lesssim E_0 \int_0^t \left[(1+t-s)^{-1/2} + (t-s)^{-1/2}\right](1+s)^{-1/2} ds.
\end{aligned} \tag{45}$$

By separating the last integral into $\int_0^{t/2}$ and $\int_{t/2}^t$, we can easily calculate $III \lesssim E_0$.

Similarly, we estimate derivatives of $\psi(x,t)$ in $L^1(x;\mathbb{R})$. From the definition of ψ in (36),

$$
\begin{aligned}
\partial_x \psi(x,t) &= \partial_x s_*^p(t)\left(\overline{u}' h_0\right) \\
&\quad + \int_{-\infty}^{\infty} E_x(x,t;y)\, v_0(y)\, dy \\
&\quad + \int_0^t \int_{-\infty}^{\infty} E_x(x,t-s;y)\, N(y,s)\, dy\, ds, \\
\partial_t \psi(x,t) &= \partial_t s_*^p(t)\left(\overline{u}' h_0\right) \\
&\quad + \int_{-\infty}^{\infty} E_t(x,t;y)\, v_0(y)\, dy \\
&\quad + \int_0^t \int_{-\infty}^{\infty} E_t(x,t-s;y)\, N(y,s)\, dy\, ds.
\end{aligned} \tag{46}
$$

Here, we used $E(x,0;y) = 0$ because there is a cutoff function in $E(x,t;y)$. Recalling (30) and (32) and applying $\int_{-\infty}^{\infty} t^{-1/2} e^{-|x-y-at|^2/Mt} dx \lesssim \sqrt{M}$ for any $t > 0$ and any $y \in \mathbb{R}$, we obtain

$$
\begin{aligned}
&\left\| \partial_x s_*^p(t)\left(\overline{u}' h_0\right) \right\|_{L^1(x;\mathbb{R})}, \left\| \partial_t s_*^p(t)\left(\overline{u}' h_0\right) \right\|_{L^1(x;\mathbb{R})} \\
&\quad \lesssim \left\| \partial_x h_0 \right\|_{W^{1,1}(\mathbb{R})} \le E_0.
\end{aligned} \tag{47}
$$

Again, from the cutoff function $\chi(t)$ in $E(x,t;y)$, we can easily obtain

$$
\left| E_x(x,t;y) \right|, \left| E_t(x,t;y) \right| \lesssim (1+t)^{-1/2} e^{-|x-y-at|^2/Mt}; \tag{48}
$$

so, for any $y \in \mathbb{R}$,

$$
\left\| E_x(x,t;y) \right\|_{L^1(x;\mathbb{R})}, \left\| E_t(x,t;y) \right\|_{L^1(x;\mathbb{R})} \lesssim \sqrt{M}. \tag{49}
$$

Thus, we have

$$
\begin{aligned}
&\left\| \int_{-\infty}^{\infty} E_x(x,t;y)\, v_0(y)\, dy \right\|_{L^1(x;\mathbb{R})}, \\
&\quad \left\| \int_{-\infty}^{\infty} E_t(x,t;y)\, v_0(y)\, dy \right\|_{L^1(x;\mathbb{R})} \lesssim \left\| v_0 \right\|_{L^1(\mathbb{R})} \le E_0.
\end{aligned} \tag{50}
$$

Estimates of the last integrals of (46) follow similarly as in (45); so, by (48),

$$
\begin{aligned}
&\left\| \int_0^t \int_{-\infty}^{\infty} E_x(x,t-s;y)\, N(y,s)\, dy\, ds \right\|_{L^1(x;\mathbb{R})}, \\
&\quad \left\| \int_0^t \int_{-\infty}^{\infty} E_t(x,t-s;y)\, N(y,s)\, dy\, ds \right\|_{L^1(x;\mathbb{R})} \\
&\quad \lesssim \int_0^t (1+t-s)^{-1/2} \left\| (Q,R,S,T)(y,s) \right\|_{L^1(y;\mathbb{R})} ds \\
&\quad \lesssim E_0 \int_0^t (1+t-s)^{-1/2} (1+s)^{-1/2} ds \lesssim E_0.
\end{aligned} \tag{51}
$$

This completes the proof. □

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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