The Factorization for a Class of Hom-Coalgebras

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1. Introduction

The motivation to introduce Hom-type algebras comes for examples related to q-deformations of Witt and Virasoro algebras, which play an important role in physics, mainly in conformal field theory. Hom-structures (Lie algebras, algebras, coalgebras, Hopf algebras) have been intensively investigated recently. Makhlouf and Silvestrov ([1, 2]) generalized the associativity to twisted associativity and naturally proposed the notion of Hom-associative algebras and were first to introduce Hom-coalgebras, Hom-bialgebras, Hom-Hopf algebras, and related objects.

The crossed products of algebras were independently introduced in [3, 4]. In [3], Blattner et al. showed the equivalence of crossed products and cleft extensions. In [5], Blattner and Montgomery gave several characterizations of crossed products. Lu and Wang [6] generalized the result in [3] to the case of Hom-Hopf algebras. The concept of crossed coproduct appeared as a dual version of the usual crossed product for Hopf algebras and it was studied in several papers; see [7–9]. In [8] the authors studied cleft coextension, a dual notion for that of cleft extension, and it was proved that a cleft coextension is isomorphic to a crossed coproduct. Naturally, what are the structure and the relation of crossed coproduct and cleft coextensions in the sense of Hom-structure?

In [10], Radford showed that a bialgebra with a projection had a factorization. In [11], Caenepeel et al. generalized Radford’s result. It is natural to ask under what conditions a Hom-coalgebra can be factorized into Hom-crossed coproduct. These two problems motivate us writing this paper.

This paper is organized as follows.

In Section 2, we recall some basic definitions and results, such as Hom-bialgebras, Hom-Hopf algebra, Hom-(co)module, Hom-Hopf module, and Hom-module coalgebras. In Section 3, let (H, α) be a Hom-Hopf algebra and (C, β) a Hom-coalgebra. We give the definition of Hom-Hopf algebra H coacting weakly on Hom-coalgebra C from the left, introduce the notion of Hom-crossed coproduct, and then discuss the necessary and sufficient conditions for C ⋊ H to be Hom-crossed coproduct (see Theorem 4). In Section 4, we introduce the definition of the cleft coextension and discuss the equivalence between the Hom-crossed coproducts and cleft coextensions (see Theorem 10). Furthermore, we discuss the relation between cleft coextension and Hom-module coalgebra with the Hom-Hopf module structure and obtain a Hom-coalgebra factorization for Hom-module coalgebra with the Hom-Hopf module structure (see Theorem 15).

2. Preliminaries

In this paper, all the vector spaces, tensor products, and homomorphisms are over a fixed field k. For a coalgebra C, we write Δ(c) = c1 ⊗ c2 for any c ∈ C (summation omitted).
We now recall from [2,12–14] some definitions and results about the Hom-Hopf algebras, Hom-(co)modules, and so on.

2.1. Hom-Hopf Algebra. A Hom-algebra is a quadruple \((A, \mu, 1_A, \alpha)\) (abbr. \((A, \alpha)\)), where \(A\) is a linear space, \(\mu : A \otimes A \to A\) is a linear map, with notation \(\mu(a \otimes a') = aa'\), \(1_A \in A\), and \(\alpha \in \text{Aut}_A(A)\), such that, for any \(a, a', a'' \in A\),

\[
\begin{align*}
\alpha(aa') &= \alpha(a)\alpha(a'), \\
1_A = 1_A a &= \alpha(a), \\
\alpha(a)(a'a'') &= (aa')\alpha(a'''), \\
\alpha(1_A) &= 1_A.
\end{align*}
\]

A linear map \(f : (A, \mu_A, 1_A, \alpha_A) \to (B, \mu_B, 1_B, \alpha_B)\) is called a morphism of Hom-algebra if \(\alpha_B f = f \alpha_A\) and \(f(1_A) = 1_B\), and \(f \mu_A = \mu_B f \otimes f\).

A Hom-coalgebra is a quadruple \((C, \Delta, \epsilon, \beta)\) (abbr. \((C, \beta)\)), where \(C\) is a linear space, \(\Delta : C \to \text{Hom}_C \otimes C, \epsilon : C \to k\) are linear maps, and \(\beta \in \text{Aut}_B(C)\), such that, for any \(c \in C\),

\[
\begin{align*}
\beta(c_1) \otimes \beta(c_2) &= \beta(c_1) \otimes \beta(c_2), \\
\epsilon(c_1) &= c_1 \epsilon(c_2) = \beta(c), \\
\beta(c_1) \otimes c_2 \otimes c_2 &= c_1 \otimes c_2 \otimes \beta(c_2), \\
\epsilon &\beta = \epsilon.
\end{align*}
\]

A linear map \(f : (C, \Delta_C, \epsilon_C, \beta_C) \to (D, \Delta_D, \epsilon_D, \beta_D)\) is called a morphism of Hom-coalgebra if \(\beta_D f = f \beta_C, \epsilon_D f = \epsilon_C\), and \(\Delta_C f = (f \otimes f)\Delta_C\).

A Hom-bialgebra is a sextuple \((H, \mu, 1_H, \Delta, \epsilon, \psi)\) (abbr. \((H, \psi)\)), where \((H, \mu, 1_H, \psi)\) is a Hom-algebra and \((H, \Delta, \epsilon, \psi)\) is a Hom-coalgebra, such that \(\Delta, \epsilon\) are morphisms of Hom-algebra; that is,

\[
\begin{align*}
\Delta(hh') &= \Delta(h) \Delta(h'), \\
\Delta(1_H) &= 1_H \otimes 1_H, \\
\epsilon(hh') &= \epsilon(h) \epsilon(h'), \\
\epsilon(1_H) &= 1.
\end{align*}
\]

Furthermore, if there exists a linear map \(S : H \to H\) such that

\[
\begin{align*}
S(h_1)h_2 &= h_1S(h_2) = \epsilon(h)1_H, \\
S(\gamma(h)) &= \gamma(S(h)),
\end{align*}
\]

then we call \((H, \mu, 1_H, \Delta, \epsilon, S, \gamma)\) (abbr. \((H, S, \gamma)\)) a Hom-Hopf algebra.

Let \((H, S, \gamma)\) be a Hom-Hopf algebra; for \(S\) we have the following properties for any \(h, g \in H\):

\[
\begin{align*}
S(h_1)S(h_2) &= S(h_2)S(h_1), \\
S(h_2) &= S(g)S(h), \\
\epsilon S &= \epsilon.
\end{align*}
\]

2.2. Hom-Hopf Module. Let \((A, \beta)\) be a Hom-algebra; a right \((A, \beta)\)-Hom-module is a triple \((M, \cdot, \alpha)\), where \(M\) is a linear space, \(\cdot : M \otimes A \to M\) is a linear map, and \(\alpha\) is an automorphism of \(M\), such that, for any \(a, a', a'' \in A\) and \(m, m' \in M\),

\[
\begin{align*}
\alpha(m \cdot (aa')) &= (m \cdot a) \cdot \beta(a'), \\
m \cdot 1_A &= \alpha(m), \\
\alpha(m \cdot a) &= \alpha(m) \cdot \beta(a).
\end{align*}
\]

Let \((M, \cdot, \alpha_M)\) and \((N, \cdot, \alpha_N)\) be two right \((A, \beta)\)-Hom-modules. Then a linear morphism \(f : M \to N\) is called a morphism of right \((A, \beta)\)-Hom-modules if \(f(m) \cdot \alpha_M(a) = f(m) \cdot \alpha_N(a)\), and \(\alpha\) is an automorphism of \(M\), such that, for any \(m \in M\),

\[
\alpha(m(0)) \otimes m(1) = m(0) \otimes m(1) \otimes \beta(m(1)),
\]

\[
m(0) \epsilon(m(1)) = \alpha(m),
\]

\[
\alpha(m(0)) \otimes \alpha(m(1)) = \alpha(m(0)) \otimes \beta(m(1)).
\]

Let \((M, \cdot, \rho_M, \alpha_M)\) and \((N, \cdot, \rho_N, \alpha_N)\) be two right \((C, \beta)\)-Hom-comodules. Then a linear morphism \(f : M \to N\) is called a morphism of right \((C, \beta)\)-Hom-comodules if \(f(m(0)) \otimes f(m(1)) = f(m(0)) \otimes f(m(1)) \otimes \beta(m(1))\), and \(\alpha\) is an automorphism of \(M\), such that, for all \(m \in M, h \in H\),

\[
\rho (m \cdot h) = m(0) \cdot h_1 \otimes m(1)h_2.
\]

2.3. The Fundamental Theorem of Hom-Hopf Module. Let \((H, \alpha)\) be a Hom-Hopf algebra and \((B, \rho, \beta)\) a right \(H\)-Hom-Hopf module, and set \(A = B^{\text{colit}} = \{b \in B \mid \rho(b) = \beta(b) \otimes 1\}\), then

\[
\varphi : A \otimes H \to B, \\
\varphi : h \otimes h \mapsto b \cdot h
\]

is an isomorphism of right \(H\)-Hopf module.

2.4. Hom-Module Coalgebra. Recall from [15], let \((H, \beta)\) be a Hom-Hopf algebra and \((C, \alpha)\) a Hom-coalgebra, and if \((C, \cdot, \alpha)\) is a left \((H, \beta)\)-Hom-module, for all \(c \in C, h \in H\), the following conditions hold:

\[
(h \cdot c)_1 \otimes (h \cdot c)_2 = h_1 \cdot c_1 \otimes h_2 \cdot c_2, \\
\epsilon(h \cdot c) = \epsilon(h) \epsilon(c),
\]

then \((C, \cdot, \alpha)\) is called an \((H, \beta)\)-Hom-module coalgebra.

3. Hom-Crossed Coproducts

Let \((H, \alpha)\) be a Hom-Hopf algebra and \((C, \beta)\) a Hom-coalgebra. In this section, we give the definition of \(H\) coating
weakly on $C$ from the left and introduce Hom-crossed coproduct. Then we discuss the necessary and sufficient conditions for $C \rtimes H$ to be Hom-crossed coproduct and get some properties about it.

**Remark 3.** If the cocycle $\lambda$ is convolution invertible, we will denote its convolution inverse by $\lambda^{-1}(c) = \lambda_1(c) \otimes \lambda_2^{-1}(c)$.

**Theorem 4.** $(C \rtimes H, \beta \otimes \alpha)$ is a Hom-crossed coproduct if and only if the following conditions hold.

**(CU) Normal Cocycle Condition**

$$(\text{id} \otimes \varepsilon) \lambda = (\varepsilon \otimes \text{id}) \lambda = 1_H \varepsilon. \quad (12)$$

**(C) Cocycle Condition**

$$\lambda_1(c_1) \alpha^{-1}(\lambda_1(c_2)_1) \otimes \lambda_2(c_1) \alpha^{-1}(\lambda_1(c_2)_2) \otimes \alpha(\lambda_2(c_2))$$

$$\lambda = \alpha^{-1}(c_{1-1}) \lambda_1(c_2) \otimes \alpha^{-1}(\lambda_1(c_2)_1) \alpha_{-1}(\lambda_2(c_2)_2). \quad (13)$$

**(TC) Twisted Comodule Condition**

$$\lambda_1(c_1) \alpha^{-2}(c_{2-1}) \otimes \lambda_2(c_1) \alpha^{-2}(c_{2-2}) \otimes c_0 = \alpha^{-1}(c_{1-1}) \lambda_1(c_2) \otimes \alpha^{-2}(c_{10-1}) \lambda_2(c_2) \otimes \beta^{-1}(c_{100}). \quad (14)$$

**Proof.** Directly computing, we can get that $\varepsilon \otimes \varepsilon$ is the counit of $(C \rtimes H, \beta \otimes \alpha)$ if and only if (CU) holds.

Now, we prove that if $(C \rtimes H, \beta \otimes \alpha)$ is a Hom-crossed coproduct then the conditions (C) and (TC) are satisfied. Because of coassociativity of $(C \rtimes H, \beta \otimes \alpha)$, we can get

$$\Delta_A (c \otimes h) = c_1 \otimes (\beta \otimes \alpha) \Delta_A (c \otimes h) = c_1$$

$$\otimes \left(\alpha^{-4}(c_{211-1}) \alpha^{-3}(\lambda_1(c_{22}))\right)$$

$$\cdot \left(\left(\alpha^{-5}(c_{21-1}) \alpha^{-4}(\lambda_1(c_{22}))\right) \alpha^{-2}(h_{11})\right)$$

$$\otimes \beta^{-2}(c_{210}) \otimes \alpha^{-2}(\lambda_2(c_{22}))$$

$$\cdot \left(\left(\alpha^{-5}(c_{21-2}) \alpha^{-4}(\lambda_1(c_{22}))\right) \alpha^{-2}(h_{22})\right).$$

We say that $(C \rtimes H, \beta \otimes \alpha)$ is a Hom-crossed coproduct if $\Delta_A$ is coassociative and $\varepsilon \otimes \varepsilon$ is the counit for all $c \in C$ and $h \in H$.

**Remark 4.** If the cocycle $\lambda$ is a convolution invertible, we will denote its convolution inverse by $\lambda_{-1}(c) = \lambda_1(c) \otimes \lambda_2^{-1}(c)$.
If coproduct is reduced to Hom-smash coproduct.

Example 5

\[ \lambda(\epsilon) = \epsilon(\epsilon) 1 \]

For condition (C), the left hand side is

\[ \lambda_1(\epsilon_1) \alpha^-2(\epsilon_2, \epsilon_1) \otimes \lambda_2(\epsilon_1) \alpha^-2(\epsilon_2, \epsilon_1) \]

\[ = \epsilon(\epsilon) 1 \]

and the right hand side is

\[ \alpha^-1(\epsilon_1, \epsilon_2) \alpha^-2(\epsilon_2, \epsilon_1) \alpha^-1(\epsilon_2, \epsilon_1) \]

\[ = \epsilon(\epsilon) 1 \]

This completes the proof.

This completes the proof.

(1) Consider the case when \( \lambda \) is trivial, that is, \( \lambda(\epsilon) = \epsilon(\epsilon) 1 \) for all \( \epsilon \in C \). Then the Hom-crossed coproduct is reduced to Hom-smash coproduct.

Proof. If \( \lambda(\epsilon) = \epsilon(\epsilon) 1 \), then \( (\text{id} \otimes \epsilon) \lambda(\epsilon) = \epsilon(\epsilon) (\text{id} \otimes \epsilon)(1 \otimes 1) = 1 \text{id} \epsilon(\epsilon) \). Similarly we can get \( (\epsilon \otimes \text{id}) \lambda = 1 \text{id} \epsilon(\epsilon) \), so (CU) is satisfied.

For condition (C), the left hand side is

\[ \lambda_1(\epsilon) \alpha^- (\lambda_1(\epsilon) \lambda_2(\epsilon)) \alpha^- (\lambda_1(\epsilon) \lambda_2(\epsilon)) \]

\[ = \epsilon(\epsilon) 1 \]

and the right hand side is

\[ \alpha^- (\lambda_1(\epsilon) \lambda_2(\epsilon)) \alpha^- (\lambda_1(\epsilon) \lambda_2(\epsilon)) \]

\[ = \epsilon(\epsilon) 1 \]

so (C) is satisfied.

(2) Let \( H_2 \) be 2-dimension Hopf group algebra with a basis \{1, g\}. Then \( (H_2, \text{id}) \) forms a Hom-Hopf algebra. Let \( H_4 \) be a vector space with a basis \{e, c, x, y\}. Define the Hom-coalgebra structure on \( H_4 \) as follows.

The automorphism \( \beta : H_4 \to H_4 \) is given by

\[ \beta(e) = e, \]
\[ \beta(c) = c, \]
\[ \beta(x) = -x, \]
\[ \beta(y) = -y; \]

the comultiplication and counit are given by

\[ \Delta(e) = e \otimes e, \]
\[ \Delta(c) = c \otimes c, \]
\[ \Delta(x) = -x \otimes c - e \otimes x, \]
\[ \Delta(y) = -y \otimes e - c \otimes y, \]
\[ \epsilon(e) = 1, \]
\[ \epsilon(c) = 1, \]
\[ \epsilon(x) = 0, \]
\[ \epsilon(y) = 0. \]

It is easy to see that \( H_4 \) is a Hom-coalgebra.

Now consider the coaction \( \beta^{H_4} : H_4 \to H_4 \otimes H_4 \) defined by

\[ e_{-1} \otimes e_0 = 1 \otimes e, \]
\[ c_{-1} \otimes c_0 = 1 \otimes c, \]
\[ x_{-1} \otimes x_0 = -g \otimes x, \]
\[ y_{-1} \otimes y_0 = -g \otimes y. \]

Then after a direct computation, we get that \( H_4 \) coacts weakly on \( H_4 \) from the left. Further, recall from (1) that if we define \( \lambda : H_4 \to H_4 \otimes H_4 \) by \( \lambda(h) = \epsilon(h) 1 \otimes 1 \), then \( (H_4 \times H_2, \beta \otimes \text{id}) \) is a Hom-crossed coproduct.
Proof. We can prove that the condition (CU) and the condition (C) hold for any \( c \in H_4 \), the same as Example 5(1). Then we only prove that the condition (TC) holds. By the proof of Example 5(1), for \( x \in H_4 \), we can get that the left hand side of the condition (TC) is

\[
\lambda_1(x_1)x_{2-11} \otimes \lambda_2(x_1)x_{2-12} \otimes x_{20} = x_{1-11} \otimes x_{1-12} \otimes \beta(x_0) = g \otimes x \otimes x,
\]

(24)

and the right hand side of the condition (TC) is

\[
x_{1-1} \lambda_1(x_2) \otimes x_{10-1} \lambda_2(x_2) \otimes \beta^{-1}(x_{100}) = x_{-1} \otimes x_{0-1} \otimes \beta^{-1}(x_{00}) = g \otimes x \otimes x,
\]

(25)

so the condition (TC) holds for \( x \in H_4 \). Similarly, we can get the condition (TC) holds for \( e, c, y \in H_4 \).

(3) Let \((H, \Delta, \mu)\) be a Hopf algebra and \((C, \Delta)\) a coalgebra, and \( H \) weakly coacts on \( C \) from the left. Assume that \( \alpha \) is a Hopf automorphism of \( H \) and \( \beta \) is a coalgebra isomorphism of \( C \). Then we have Hom-Hopf algebra \((H_\alpha, \Delta_\alpha, \mu_\alpha, \alpha)\) and Hom-coalgebra \((C_\beta, \Delta_\beta, \beta)\) (see [2]). Furthermore assume that \( \rho(\beta(c)) = (\alpha \otimes \beta)(c) \), and define the coaction \( \rho(c) = (\alpha \otimes \beta)(c) \), then \( H_\alpha \) weakly coacts on \( C_\beta \) from the left. If \( C \rtimes_\lambda H \) is a crossed coproduct and \( \Delta_\beta = (\alpha \otimes \alpha)\lambda \), then \( C_\beta \rtimes_\lambda H_\alpha \) is a Hom-crossed coproduct.

Proof. If \( C \rtimes_\lambda H \) is a crossed coproduct, then we can get

\[
(id \otimes \epsilon) \lambda = (\epsilon \otimes id) \lambda = 1_{H_\alpha};
\]

\[
\lambda_1(c_1) \otimes \lambda_2(c_1) \lambda_1(c_2) \lambda_2(c_2)
\]

\[
= c_{1-1} \lambda_1(c_2) \otimes \lambda_1(c_1) \lambda_2(c_2)\]

\[
\otimes \lambda_2(c_1) \lambda_2(c_2); \quad (27)
\]

\[
\lambda_1(c_1) c_{2-11} \otimes \lambda_2(c_1) c_{2-12} \otimes c_{20} = c_{1-1} \lambda_1(c_2) \otimes c_{0-1} \lambda_2(c_2) \otimes c_{100},
\]

(28)

From [2], we know that the multiplication and comultiplication of Hom-Hopf algebra \((H_\alpha, \Delta_\alpha, \mu_\alpha, \alpha)\) are, respectively, given by

\[
\mu_\alpha(h \otimes g) = \alpha(h) \alpha(g),
\]

\[
\Delta_\alpha(h) = \alpha(h_1) \otimes \alpha(h_2),
\]

(29)

\( \forall h, g \in H_\alpha \),

and the comultiplication of Hom-coalgebra \((C_\beta, \Delta_\beta, \beta)\) is given by

\[
\Delta_\beta(c) = \beta(c_1) \otimes \beta(c_2), \quad \forall c \in C_\beta.
\]

(30)

First, we prove that \( H_\alpha \) weakly coacts on \( C_\beta \) from the left. We only prove that (WI) of Definition 1 holds; the other two are easy to get. For any \( c \in C_\beta \),

\[
\alpha^2(c_1) \otimes \alpha_2(c_2) \quad (in \ H_\alpha \otimes C_\beta \otimes C_\beta)
\]

\[
= \alpha^3(c_1) \otimes \beta^2(c_1) \otimes \beta^2(c_2) \quad (in \ H \otimes C \otimes C)
\]

\[
= \alpha^3(c_1) \otimes \beta^2(c_1) \otimes \beta^2(c_2) \quad (in \ H \otimes C \otimes C)
\]

\[
= \alpha(c_1) \otimes \alpha_2(c_2), \quad (in \ H_\alpha \otimes C_\beta \otimes C_\beta),
\]

so \( H_\alpha \) weakly coacts on \( C_\beta \) from the left.

Then we prove that \( C_\beta \rtimes_\lambda H_\alpha \) is a Hom-crossed coproduct. It is easy to see that the condition (CU) holds. For the condition (C),

\[
\lambda_1(c_1) \alpha^{-1}(\lambda_1(c_2)_1) \otimes \lambda_2(c_1) \alpha^{-1}(\lambda_1(c_2)_2)
\]

\[
\otimes \alpha(\lambda_2(c_2)) \quad (in \ H_\alpha \otimes H_\alpha \otimes H_\alpha)
\]

\[
= \alpha^2(\lambda_1(c_1) \lambda_1(c_2)) \otimes \alpha^2(\lambda_2(c_1) \lambda_2(c_2))
\]

\[
\otimes \alpha(\lambda_2(c_2)) \quad (in \ H \otimes H \otimes H)
\]

\[
= \alpha^2(\lambda_1(c_1) \lambda_1(c_2)) \otimes \alpha^2(\lambda_1(c_1) \lambda_1(c_2))
\]

\[
\otimes \alpha(\lambda_2(c_2)) \quad (in \ H \otimes H \otimes H)
\]

\[
\alpha^{-1}(\lambda_1(c_1) \lambda_1(c_2)) \otimes \alpha^{-1}(\lambda_1(c_1) \lambda_1(c_2))
\]

\[
\otimes \alpha^{-1}(\lambda_2(c_2)) \quad (in \ H_\alpha \otimes H_\alpha \otimes H_\alpha)
\]

(31)

so the condition (CU) holds. Similarly we can prove that the condition (TC) holds by (28).

For a Hom-crossed coproduct, we can get the following properties, which are useful for the latter conclusions.

**Lemma 6.** Let \( C \rtimes_\lambda H \) be a Hom-crossed coproduct with invertible cocycle \( \lambda \). Then the following equalities hold for any \( c \in C \):

(i)

\[
c_{-1} \otimes \lambda_1(c_0) \otimes \lambda_2(c_0)
\]

\[
= (\alpha^{-3}(\lambda_1(c_{11})) \alpha^{-4}(\lambda_1(c_{12})) \alpha^{-1}(\lambda_1(c_1)))
\]

\[
\otimes (\alpha^{-3}(\lambda_2(c_{11})) \alpha^{-4}(\lambda_1(c_{12})) \alpha^{-2}(\lambda_2(c_2)))
\]

\[
\otimes \alpha^{-2}(\lambda_2(c_{12})) \alpha^{-2}(\lambda_2(c_2));
\]

(33)

(ii)

\[
c_{-1} \otimes \lambda_1(c_0) \otimes \lambda_2^{-1}(c_0)
\]

\[
= \alpha^{-1}(\lambda_1(c_1)) (\alpha^{-4}(\lambda_1^{-1}(c_{11})) \alpha^{-3}(\lambda_1^{-1}(c_{12})))
\]

\[
\otimes \alpha^{-2}(\lambda_2(c_{11})) (\alpha^{-4}(\lambda_1^{-1}(c_{12})) \alpha^{-3}(\lambda_1^{-1}(c_{22})))
\]

\[
\otimes \alpha^{-2}(\lambda_2(c_{12})) \alpha^{-2}(\lambda_2^{-1}(c_{21})).
\]

(34)
Proof. Applying \(\alpha^{-3} \otimes \alpha^{-3} \otimes \alpha^{-3}\) to both sides of (C) and then multiplying it (by convolution) to the right by the following map, we can get (i):

\[
c \mapsto \alpha^{-1} \lambda_1^{-1} (c) \otimes \alpha^{-2} \left( \lambda_2^{-1} (c_1) \right) \otimes \alpha^{-2} \left( \lambda_2^{-1} (c_2) \right).
\] (35)

Let us denote by \(F_1, F_2, F_3 : C \to H \otimes H \otimes H\), where the map is defined by

\[
F_1 (c) = c_1 \otimes \lambda_1 (c_0) \otimes \lambda_2 (c_0).
\] (36)

We get from (i) that a right convolution inverse for \(F_1\) is given by

\[
F_2 (c)
= \alpha^{-1} \left( \lambda_1 (c_1) \right) \left( \alpha^{-4} \left( \lambda_1^{-1} (c_2) \right) \alpha^{-3} \left( \lambda_1^{-1} (c_22) \right) \right)
\]
\[
\otimes \alpha^{-2} \left( \lambda_2 (c_1) \right) \left( \alpha^{-4} \left( \lambda_2^{-1} (c_2) \right) \alpha^{-3} \left( \lambda_2^{-1} (c_22) \right) \right)
\]
\[
\otimes \alpha^{-3} \left( \lambda_2 (c_1) \right) \alpha^{-2} \left( \lambda_2^{-1} (c_1) \right).
\] (37)

Let \(F_3\) be given by \(F_3 (c) = c_1 \otimes \lambda_1^{-1} (c_0) \otimes \lambda_2^{-1} (c_0)\). Since \((F_3 \ast F_1) (c) = \varepsilon (c) 1 \otimes 1 \otimes 1\). We get that \(F_3\) is a left convolution inverse of \(F_1\), so \(F_3 = F_2\). Therefore, we prove that (ii) holds.

Remark 7. Note that if \((C \times_1 H, \beta \otimes \alpha)\) is a Hom-crossed product, then the map \(\pi : C \times_1 H \to C\), defined by \(\pi (c \otimes h) = \varepsilon (h) \beta^{-1} (c)\), is a Hom-coalgebra map, and we will also define the map \(\theta : C \times_1 H \to H, \theta (c \otimes h) = \varepsilon (c) \alpha^{-1} (h)\).

The following result is the generalization of Proposition 2.1 in [8].

**Proposition 8.** Let \(B = C \times_1 H\) be a Hom-crossed coproduct. Then \(\theta\) is convolution invertible in \(Hom (B, H)\) if and only if \(\lambda\) is convolution invertible in \(Hom (C, H \otimes H)\).

Proof. Assume first that \(\lambda\) is convolution invertible; for any \(c \times h \in B\), define \(\mu : B \to H\), by

\[
\mu (c \times h) = \varepsilon (h) \lambda^{-3} \left( \lambda_1^{-1} (c) \right) \alpha^{-3} (h) \alpha^{-2} \left( \lambda_2^{-1} (c) \right).
\] (38)

Now we show that \(\mu\) is a convolution inverse of \(\theta\).

\[
(\mu \ast \theta) (c \times h) = \varepsilon (c) \lambda^{-3} (c_1) \alpha^{-4} (c_121) \alpha^{-3} (\lambda_1 (c_22))
\]
\[
\cdot \alpha^{-1} (h_1) \varepsilon (c_210) \lambda^{-3} (\lambda_1 (c_22)) \alpha^{-1} (h_2)
\]
\[
= \varepsilon (c) \lambda^{-3} (\lambda_1 (c_1)) \alpha^{-1} (h_1)
\]
\[
\cdot \varepsilon^{-1} \left( \alpha^{-4} (c_210) \alpha^{-3} (\lambda_1 (c_22)) \alpha^{-1} (h_1) \right)
\]
\[
\cdot \alpha^{-2} \left( \lambda_2^{-1} (c_1) \right) \varepsilon (c_210) \left( \alpha^{-3} \left( \lambda_2 (c_2) \right) \alpha^{-2} (h_2) \right)
\]
\[
= \varepsilon (c) \lambda^{-3} (\lambda_1 (c_1)) \alpha^{-3} (h_1)
\]
\[
\cdot \alpha^{-3} \left( \lambda_2^{-1} (c_1) \lambda_2 (c_2) \right) \alpha^{-2} (h_2) = \varepsilon (c) \varepsilon (h).
\] (39)

So \(\mu\) is a left inverse for \(\theta\); in order to show that \(\mu\) is a right inverse we compute as follows:

\[
(\theta \ast \mu) (c \times h) = \theta (c_1) \alpha^{-4} (c_121) \alpha^{-3} (\lambda_1 (c_22))
\]
\[
\cdot \alpha^{-1} (h_1) \beta^{-1} (c_210) \alpha^{-2} (\lambda_2 (c_22)) \alpha^{-1} (h_2)
\]
\[
= \varepsilon (c_1) \alpha^{-5} (c_121) \alpha^{-3} (\lambda_1 (c_22)) \alpha^{-2} (h_1)
\]
\[
\cdot \beta^{-1} (c_210) \alpha^{-5} (\lambda_1 (c_22)) \alpha^{-2} (h_2)
\]
\[
\cdot \alpha^{-4} (\lambda_2^{-1} (c_22)) = \alpha^{-4} (c_121) \alpha^{-3} (\lambda_1 (c_22))
\]
\[
\cdot \alpha^{-2} (h_1) \beta^{-1} (c_210) \alpha^{-5} (\lambda_1 (c_22)) \alpha^{-2} (\lambda_2 (c_22))
\]
\[
\cdot \alpha^{-3} (\lambda_2^{-1} (c_22)) \alpha^{-3} (\lambda_2^{-1} (c_210)) = (\alpha^{-3} (c_121))
\]
\[
\cdot \alpha^{-2} (h_1) \alpha^{-3} (\lambda_2 (c_22))
\]
\[
\cdot \beta^{-1} (c_210) \alpha^{-5} (\lambda_1 (c_22)) \alpha^{-2} (\lambda_2 (c_22))
\]
\[
\cdot \alpha^{-4} (\lambda_2^{-1} (c_210)) = (\alpha^{-3} (c_121))
\]
\[
\cdot \alpha^{-2} (h_1) \beta^{-1} (c_210) \alpha^{-5} (\lambda_1 (c_22)) \alpha^{-2} (\lambda_2 (c_22))
\]
\[
\cdot \alpha^{-3} (\lambda_2^{-1} (c_210)) \alpha^{-3} (\lambda_2^{-1} (c_210)) = \varepsilon (c_1) \varepsilon (h).
\] (40)

So \(\lambda\) is convolution invertible.

Next we prove that if \(\theta\) is convolution invertible, then \(\lambda\) is also convolution invertible. Recalling

\[
\pi : B \to C,
\]
\[
c \otimes h \mapsto \varepsilon (h) \beta^{-1} (c),
\]
we prove that \(\lambda \pi\) is convolution invertible.

Firstly, we prove that

\[
(\lambda \pi) (b) = \alpha^{-3} (\theta (h_1)) \alpha^{-3} (\theta^{-1} (b_1))
\]
\[
\otimes \alpha^{-3} (\theta (h_2)) \alpha^{-3} (\theta^{-1} (b_2)).
\] (41)
By applying $\Delta_{J_H}$ we get

\[
\begin{align*}
(\alpha^{-4}(c_{1-1})\alpha^{-3}(\lambda_1(c_2)))\alpha^{-2}(h_1),
\end{align*}
\]

and we prove that $\omega$ is the convolution inverse of $\lambda \pi$. On the other hand,

\[
\begin{align*}
(\lambda \pi \ast \omega)(b) &= (\lambda \pi)(b) \omega(b) = (\alpha^{-3}(\theta(b_1)))
\end{align*}
\]

So $\omega$ is a left convolution inverse for $\lambda \pi$. On the other hand,

\[
\begin{align*}
\cdot \theta^{-1}(\beta^{-1}(c_{10}) \ominus \alpha^{-1}(\lambda_2(c_2)) \alpha^{-1}(h_2))_2 = \epsilon(c)
\end{align*}
\]

At the same time, we have

\[
\begin{align*}
\cdot \epsilon(h) 1 \ominus 1.
\end{align*}
\]
\[
\begin{align*}
\cdot (\alpha^{-4}(\theta(b_{211})) \alpha^{-4}(\theta(b_{212})) \alpha^{-4}(\theta(b_{221}))) \\
= \alpha^{-1}(\theta(b_{11})) \alpha^{-2}(\theta(b_{221})) \otimes \alpha^{-1}(\theta(b_{12})) \\
\cdot \alpha^{-3}(\theta(b_{221})) (\epsilon(b_{21})) &= \alpha^{-1}(\theta(b_{11})) \\
\cdot \alpha^{-1}(\theta(b_{11})) \otimes \alpha^{-1}(\theta(b_{12})) &= 1 \otimes 1 \epsilon(d).
\end{align*}
\]

Thirdly, let \( j : C \to B, j(c) = c \otimes 1 \), then we have
\[
(\pi \otimes \text{id}) \Delta(j(c)) = (\pi \otimes \text{id}) \Delta(\beta(c) \otimes 1)
\]
\[
= \pi(\beta(c_1) \otimes \alpha^{-2}(c_{21-1}) \alpha^{-1}(\lambda_1(c_{22}))) \\
\otimes \beta^{-1}(c_{210}) \otimes \lambda_2(c_{22}).
\]
\[
= c_1 \epsilon(c_{21-1}) \epsilon(\lambda_1(c_{22})) \otimes \beta^{-1}(c_{210}) \otimes \lambda_2(c_{22})
\]
\[
= c_1 \otimes c_{21} \otimes \epsilon(c_{22}) = c_1 \otimes \beta(c_2) \otimes 1
\]
\[
= c_1 \otimes j(c_2).
\]

Since \((\lambda \pi) \ast \omega = 1\), we obtain that
\[
(\lambda \pi \ast \omega)(j(c)) = (\lambda \pi)(j(c)) \omega(j(c)_2)
\]
\[
= \lambda(c_1) \omega(j(c)_2) = \epsilon(c) \cdot 1.
\]

So \( \lambda \) has a right convolution inverse \( \omega j \) in Hom\((C, H \otimes H)\). On the other hand,
\[
(\lambda \pi \ast \omega j) \pi(b) = \lambda(\pi(b_1)) \omega(j(\pi(b)_2))
\]
\[
= \lambda(\pi(b_1)) \omega(j(\pi(b)_2)) = \epsilon(\pi(b)) \cdot 1.
\]

It shows that \( \omega \pi j \) is a right convolution inverse for \( \lambda \pi \) in Hom\((B, H \otimes H)\); thus \( \omega \pi j = \omega j \).

Finally, we denote that \( \lambda' = \omega j \), because we have that \((\lambda \pi) \ast (\lambda' \pi) = (\lambda' \pi) \ast (\lambda \pi) = 1\) in Hom\((B, H \otimes H)\) and \( \pi \) is a surjective Hom-coalgebra map, and we obtain \( \lambda \ast \lambda' = \lambda' \ast \lambda = 1 \) in Hom\((C, H \otimes H)\). So \( \lambda' \) is the convolution inverse of \( \lambda \).

\[\square\]

4. Two Equivalent Characterizations of Cleft Coextensions

In this section, we introduce the notion of the cleft coextension and then discuss two equivalent characterizations of cleft coextensions.

Let \((H, \alpha)\) be a Hom-Hopf algebra and \((B, \beta)\) a right \(H\)-Hom-module coalgebra. Let \(H^+\) denote the augmentation Hom-ideal ker \(e\) which is a Hom-Hopf ideal, then \(BH^+\) is a Hom-coideal of \(B\), and \(B/BH^+\) is a Hom-coalgebra with a trivial right \(H\)-Hom-module structure.

Definition 9. Let \((H, \alpha)\) be a Hom-Hopf algebra and \((B, \beta)\) a right \(H\)-Hom-module coalgebra. Denote by \(C = B/BH^+\);

if there exists a right \(H\)-Hom-module map \(\phi : B \to H\) which is convolution invertible, then \(B/C\) is said to be a cleft coextension and write \((B, \phi, \beta)\).

By Proposition 8, we get the main result of this section, which is the generalization of Theorem 2.10 in [8].

Theorem 10. The coextension \(B/C\) is cleft if and only if \(B\) is isomorphic to a Hom-crossed coproduct \(C \ltimes H\) with invertible cocycle \(\lambda\).

Proof. If \(B\) is isomorphic to the Hom-crossed coproduct \(C \ltimes H\), we prove that the coextension \(B/C\) is cleft. First, we show that \(\phi : C \ltimes H \to H\) is a right \(H\)-Hom-module map. For any \(h, g \in H\), we get
\[
\phi(c \otimes h) \cdot g = \phi(c \cdot h) g = \epsilon(c) \alpha^{-1}(h) g.
\]

On the other hand,
\[
\phi((c \otimes h) \cdot g) = \phi(c \cdot g_1 \otimes hg_2) = \epsilon(c \cdot g_1) \alpha^{-1}(hg_2)
\]
\[
= \epsilon(c) \alpha^{-1}(h) g,
\]

and by Proposition 8, \(\lambda\) is an invertible cocycle, so \(\phi\) is convolution invertible. Hence \(C \ltimes H/C\) is cleft, so \(B/C\) is cleft too.

Conversely, if the coextension \(B/C\) is cleft, let us denote by \(\tilde{\epsilon}\) the image in \(C\) of the element \(c \in B\). Define a weak coaction of \(H\) on \(C\) by
\[
\rho : C \to H \otimes C,
\]
\[
\rho(\tilde{\epsilon}) = \alpha^{-2}(\phi(c_{11})) \alpha^{-1}(\phi^{-1}(c_{21})) \otimes \beta^{-1}(\tilde{\epsilon}_{12}).
\]

The definition is well defined. The conditions (W1), (W2), and (W3) hold which can be directly verified.

Define \(\lambda : C \to H \otimes H\), which is given by
\[
\lambda(\tilde{\epsilon}) = \alpha^{-3}(\phi(c_{11})) \alpha^{-3}(\phi^{-1}(c_{21}))
\]
\[
\otimes \alpha^{-3}(\phi(c_{12})) \alpha^{-3}(\phi^{-1}(c_{22})),
\]

and \(\omega : C \to H \otimes H\) is given by
\[
\omega(\tilde{\epsilon}) = \alpha^{-4}(\phi(c_{11})) \alpha^{-2}(\phi^{-1}(c_{21}))
\]
\[
\otimes \alpha^{-4}(\phi(c_{12})) \alpha^{-3}(\phi^{-1}(c_{22})).
\]

Similar to the proof of Proposition 8, we can obtain that \(\lambda \ast \omega = \omega \ast \lambda = 1\), that is, \(\omega\) is the convolution inverse of \(\lambda\). With convolution inverse \(\lambda\) and weak coaction of \(H\) on \(C\) as the above, we can define a comultiplication \(\Delta\) and a counit \(\epsilon\) as the Definition 2. The comultiplication \(\Delta\) is given by
\[
\Delta(\tilde{\epsilon} \otimes \epsilon) = \tilde{\epsilon}_1 \otimes (\alpha^{-4}(\tilde{\epsilon}_{21-1}) \alpha^{-3}(\lambda_1(\tilde{\epsilon}_{22}))) \alpha^{-1}(h_1)
\]
\[
\otimes \beta^{-2}(\tilde{\epsilon}_{210}) \otimes \alpha^{-2}(\lambda_2(\tilde{\epsilon}_{22})) \alpha^{-1}(h_2) = \tilde{\epsilon}_1
\]
\[
\otimes ((\alpha^{-6}(\phi(c_{211})) \alpha^{-5}(\phi^{-1}(c_{212}))))}.
Lemma II. Let \((H,\alpha)\) be a Hom-Hopf algebra and \((B,\cdot,\rho,\beta)\) a right \(H\)-Hom-Hopf module; set \(A = B^{\text{co}H} = \{b \in B \mid \rho(b) = \beta(b) \otimes 1\}\) and \(C = B/H^*\). Then

1. \(C = A\) as vector spaces, and the isomorphism is \(\psi : C \to A\), \(\psi(b) = b_{(0)} \cdot S(b_{(1)})\).

2. If \((B,\cdot)\) is a right \(H\)-Hom-module coalgebra, then there exists a Hom-coalgebra structure on \(A\) such that \(C = A\) as Hom-coalgebras.

Proof. (1) It is easy to see that \(\psi\) is an isomorphism of vector spaces.

(2) In order to prove that \(C = A\) as Hom-coalgebras, we first give the Hom-coalgebra structure on \(A\). The comultiplication on \(A\) is given by

\[
\Delta(a) = \beta^{-3}(a_{(0)(0)}) \cdot S\left(\alpha^{-3}(a_{(1)(1)})\right)
\]

\[
\otimes \beta^{-3}(a_{(2)(0)}(0)) \cdot S\left(\alpha^{-3}(a_{(2)(1)(0)})\right),
\]

for any \(a \in A\). With the above defined comultiplication, we can prove that \(A\) is a Hom-coalgebra. Furthermore we can get that \(\psi\), which is as similar as defined in (1), is also a Hom-coalgebra morphism. So \(C = A\) as Hom-coalgebras.

We then discuss the relation between cleft coextension and Hom-module coalgebra with the Hom-Hopf module structure.

Proposition 12. Let \((B,\cdot,\phi,\beta)\) be a cleft coextension. One assumes that \(H\) coacts on \(B\) to the right by

\[
\rho(b) = \beta^{-2}(b_{11}) \cdot \alpha^{-3}(\phi^{-1}(b_{12}) \phi(b_{21}))
\]

\[
\otimes \alpha^{-1}(\phi(b_{22})),
\]

then

1. \((B,\rho,\beta)\) is a right \(H\)-Hom-comodule;

2. \((B,\cdot,\rho,\beta)\) is a right \(H\)-Hom-Hopf module.

Proof. Note that \(\phi\) is convolution invertible right \(H\)-Hom-module map, where the convolution inverse of \(\phi\) is denoted by \(\phi^{-1}\). We can get

\[h_1 \alpha^{-1} \phi^{-1}(b \cdot h_2) = \epsilon(h) \alpha\left(\phi^{-1}(b)\right).\]

In fact

\[
\epsilon(h)\phi^{-1}(\beta^3(b)) = \phi^{-1}\left(\beta^3(b_{11})\right) \epsilon(b_2) \epsilon(h)
\]

\[= \phi^{-1}\left(\beta^3(b_{11})\right)\left(\phi \circ \phi^{-1}(b_2) \cdot h_1\right)
\]

\[= \phi^{-1}\left(\beta^3(b_{11})\right)\left(\phi(b_{21}) \cdot h_1\right) \phi^{-1}\left(b_2 \cdot h_2\right)
\]

\[= \left(\phi^{-1}\left(\beta^3(b_{11})\right)\left(\phi(b_{21})\cdot h_1\right)\right) \alpha\left(h_1\right) \alpha\left(\phi^{-1}\left(b_2 \cdot h_2\right)\right)
\]

\[= \left(\phi^{-1}\left(b_{11}\right)\right)\alpha\left(h_1\right) \alpha\left(\phi^{-1}\left(\beta^{-1}(b_2) \cdot h_2\right)\right)
\]
Assume that $(\beta \otimes \epsilon) \rho(\cdot) = \delta_0$; thus $(B, \rho, \beta)$ is a right $H$-Hom-Hopf module.
Conversely, it is sufficient to show that \( \phi \) is a right \( H \)-Hom-module map. Using the fact that \((B, \cdot, \rho, \beta)\) is a right \( H \)-Hom-Hopf module and \( \phi(b) = \varepsilon(b)(0) \beta^{-1}(b(1)) \), for any \( b \in B \) and \( h \in H \), we get

\[
\phi(b \cdot h) = \varepsilon((b \cdot h)(0)) \beta^{-1}((b \cdot h)(1)) = \varepsilon(b)(0) \beta^{-1}(b(1)) \alpha^{-1}(h) = \phi(b) \cdot h.
\]

(70)

The proof is complete. \( \square \)

5. The Factorization of Hom-Module Coalgebra with Hopf Module Structure

In this section, we always assume that \((H, \alpha)\) is a Hom-Hopf algebra and \((B, \cdot, \rho, \beta)\) is a cleft coextension. Based on this assumption, \( B \) can not only form a right \( H \)-Hom-comodule, but also form a right \( H \)-Hom-Hopf module. Furthermore, we obtain the factorization of Hom-coalgebra \( B \).

Now, we can obtain a Hom-coalgebra factorization for Hom-module coalgebra with the Hom-Hopf module structure.

**Theorem 14.** Let \((H, \alpha)\) be a Hom-Hopf algebra and \((B, \cdot, \rho, \beta)\) a right \( H \)-Hom-Hopf module. One assumes that \((B, \cdot, \beta)\) is a right \( H \)-Hom-module coalgebra and set \( A = B^{\text{cot}} = \{ a \in B \mid \rho(a) = \beta(a) \otimes 1 \} \). The comultiplication on \( A \otimes H \) can be defined

\[
\Delta(a \otimes h) = \beta^{-3}(a(0)(0)(0)) \cdot S(\alpha^{-3}(a(0)(1))) \otimes \alpha^{-2}(a(1)) \alpha^{-1}(h) \otimes \beta^{-3}(a(2)(0)(0)) \tag{71}
\]

and the counit can be defined

\[
\varepsilon(a \otimes h) = \varepsilon(a) \varepsilon(h), \tag{72}
\]

for any \( a \in A, h \in H \), then \((A \otimes H, \alpha \otimes \beta)\) is a Hom-coalgebra, and one writes it as \( A \square H \). Moreover, \( B = A \square H \) as Hom-coalgebras.

**Proof.** Since \((B, \cdot, \rho)\) is a right \( H \)-Hom-Hopf module, \( \Delta \) is well defined and the map

\[
\varphi : A \otimes H \rightarrow B,
\]

\[
a \otimes h \mapsto a \cdot h \tag{73}
\]

is an isomorphism of right \( H \)-Hom-Hopf module.

We first verify that \((A \otimes H, \Delta, \varepsilon, \beta \otimes \alpha)\) is a Hom-coalgebra. It is easy to see that \((\varepsilon \otimes (\beta \otimes \alpha)) \Delta = ((\beta \otimes \alpha) \otimes \varepsilon) \Delta\). It is sufficient to show the coassociativity. For any \( a \in A, h \in H \), we calculate as follows:

\[
(\Delta \otimes (\beta \otimes \alpha)) \Delta(a \otimes h) = (\beta^{-6}(a(0)(0)(0)(0)) \cdot S(\alpha^{-6}(a(0)(1)(1))) \otimes (\alpha^{-5}(a(0)(0)(1))) S(\alpha^{-5}(a(0)(1)(1))) \notag \]
\[
\cdot S(\alpha^{-6}(a(0)(0)(1))) S(\alpha^{-6}(a(0)(1)(1))) \notag \]
\[
= (\beta^{-6}(a(0)(0)(0)(0)) \cdot S(\alpha^{-6}(a(0)(1))) S(\alpha^{-6}(a(0)(1)(1))) \notag \]
\[
\cdot S(\alpha^{-6}(a(0)(0)(1))) S(\alpha^{-6}(a(0)(1)(1))) \notag \]
\[
\cdot S(\alpha^{-6}(a(0)(0)(1))) S(\alpha^{-6}(a(0)(1)(1))) \notag \]
\[
= (\beta^{-6}(a(0)(0)(0)(0)) \cdot S(\alpha^{-6}(a(0)(1))) S(\alpha^{-6}(a(0)(1)(1))) \notag \]
\[
\cdot S(\alpha^{-6}(a(0)(0)(1))) S(\alpha^{-6}(a(0)(1)(1))) \notag \]
\[
\cdot S(\alpha^{-6}(a(0)(0)(1))) S(\alpha^{-6}(a(0)(1)(1))) \notag \]
\[
= (\beta^{-6}(a(0)(0)(0)(0)) \cdot S(\alpha^{-6}(a(0)(1))) S(\alpha^{-6}(a(0)(1)(1))) \notag \]
\[
\cdot S(\alpha^{-6}(a(0)(0)(1))) S(\alpha^{-6}(a(0)(1)(1))) \notag \]
\[
\cdot S(\alpha^{-6}(a(0)(0)(1))) S(\alpha^{-6}(a(0)(1)(1))) \notag \]
\[
= (\beta^{-6}(a(0)(0)(0)(0)) \cdot S(\alpha^{-6}(a(0)(1))) S(\alpha^{-6}(a(0)(1)(1))) \notag \]
\[
\cdot S(\alpha^{-6}(a(0)(0)(1))) S(\alpha^{-6}(a(0)(1)(1))) \notag \]
\[
\cdot S(\alpha^{-6}(a(0)(0)(1))) S(\alpha^{-6}(a(0)(1)(1))) \notag \]
\[
= (\beta^{-6}(a(0)(0)(0)(0)) \cdot S(\alpha^{-6}(a(0)(1))) S(\alpha^{-6}(a(0)(1)(1))) \notag \]
\[
\cdot S(\alpha^{-6}(a(0)(0)(1))) S(\alpha^{-6}(a(0)(1)(1))) \notag \]
\[
\cdot S(\alpha^{-6}(a(0)(0)(1))) S(\alpha^{-6}(a(0)(1)(1))) \notag \]
\[
= (\beta^{-6}(a(0)(0)(0)(0)) \cdot S(\alpha^{-6}(a(0)(1))) S(\alpha^{-6}(a(0)(1)(1))) \notag \]
\[
\cdot S(\alpha^{-6}(a(0)(0)(1))) S(\alpha^{-6}(a(0)(1)(1))) \notag \]
\[
\cdot S(\alpha^{-6}(a(0)(0)(1))) S(\alpha^{-6}(a(0)(1)(1))) \notag \]
In a similar way
\[
\begin{align*}
\Delta &\cdot (\Delta \otimes \alpha) \\
&= \beta^{-3}(a_{1(0)1(0)}) \cdot S (\alpha^{-3}(a_{1(0)1(0)})) \\
&\otimes \alpha^{-3}(a_{1(0)1(1)}) \\
&\otimes (S (\alpha^{-5}(a_{1(1)21})) \alpha^{-2}(a_{1(1)21})) \alpha^{-2}(h_{11}) \\
&\otimes \beta^{-3}(a_{0(2)0(0)}) \cdot S (\alpha^{-2}(a_{1(0)2(0)})) \\
&\otimes \alpha^{-3}(a_{1(0)2(1)}) \\
&\cdot ((S (\alpha^{-4}(a_{1(1)1})) \alpha^{-4}(a_{1(1)1})) \alpha^{-2}(h_{12})) \\
&\otimes \beta^{-2}(a_{0(2)0(0)}) \cdot S (\alpha^{-2}(a_{1(0)2(0)})) \otimes \alpha^{-1}(a_{2(3)}) h_{2} \\
&= \beta^{-4}(a_{1(0)1(0)}) \cdot S (\alpha^{-4}(a_{1(0)1(0)})) \\
&\otimes \alpha^{-2}(a_{0(2)0(0)}) \cdot S (\alpha^{-2}(a_{1(0)2(0)})) \otimes \alpha^{-1}(a_{2(3)}) h_{2} \\
&= \beta^{-3}(a_{1(0)1(0)}) \alpha^{-1}(h_{11}) \otimes \beta^{-4}(a_{1(0)2(0)}) \\
&\cdot S (\alpha^{-4}(a_{1(1)1})) \alpha^{-2}(a_{1(1)21}) \alpha^{-2}(h_{11}) \\
&\otimes \beta^{-2}(a_{0(2)0(0)}) \cdot S (\alpha^{-2}(a_{1(0)2(0)})) \otimes \alpha^{-1}(a_{2(3)}) h_{2} \\
&= \beta^{-4}(a_{1(0)1(0)}) \cdot S (\alpha^{-4}(a_{1(0)1(0)})) \\
&\otimes \alpha^{-2}(a_{0(2)0(0)}) \cdot S (\alpha^{-2}(a_{1(0)2(0)})) \otimes \alpha^{-1}(a_{2(3)}) h_{2} \\
&\otimes \alpha^{-3}(a_{1(0)1(1)}) \alpha^{-1}(h_{11}) \otimes \beta^{-4}(a_{1(0)2(0)}) \\
&\cdot S (\alpha^{-4}(a_{1(1)1})) \alpha^{-2}(a_{1(1)21}) \alpha^{-2}(h_{11}) \\
&\otimes \beta^{-2}(a_{0(2)0(0)}) \cdot S (\alpha^{-2}(a_{1(0)2(0)})) \otimes \alpha^{-1}(a_{2(3)}) h_{2} \\
&= \beta^{-3}(a_{1(0)1(1)}) \alpha^{-1}(h_{11}) \otimes \beta^{-4}(a_{1(0)2(0)}) \\
&\cdot S (\alpha^{-4}(a_{1(1)1})) \alpha^{-2}(a_{1(1)21}) \alpha^{-2}(h_{11}) \\
&\otimes \beta^{-2}(a_{0(2)0(0)}) \cdot S (\alpha^{-2}(a_{1(0)2(0)})) \otimes \alpha^{-1}(a_{2(3)}) h_{2}.
\end{align*}
\]  

Thus, \((A \otimes H, \Delta, \epsilon, \beta \otimes \alpha)\) is a Hom-coalgebra.

Next we show that \(\varphi\) is a Hom-coalgebra morphism.

\[
\varphi ((a \otimes h) \otimes (a \otimes h)) = \beta^{-3}(a_{1(0)0}) \cdot S (\alpha^{-3}(a_{1(0)0})) \\
\otimes \alpha^{-2}(a_{2(1)}) \cdot S (\alpha^{-1}(h_{21}) \otimes \beta^{-3}(a_{22(0)}) \\
\cdot S (\alpha^{-3}(a_{22(0)})) \otimes \alpha^{-2}(a_{22(1)}) \alpha^{-1}(h_{22}).
\]  

(75)

By the above results, we obtain the following result.

**Theorem 15.** Let \((H, \alpha)\) be a Hom-Hopf algebra and \((B, \cdot, \phi, \beta)\) a cleft coextension, then

\[
B = C \rtimes \alpha H = A \square H
\]  

(77)

as Hom-coalgebras, where \(C = B/BH^+\), and

\[
A = \{ \lambda \in B \mid \beta^{-2}(h_{11}) \cdot \alpha^{-3}(\phi^{-1}(h_{12}) \phi(b_{1})) \\}
\]

(78)

\[\otimes \alpha^{-1}(\phi(b_{2})) = \beta(b) \otimes 1\].

**Proof.** By Theorems 10 and 14, we have \(B = C \rtimes \alpha H\) and \(B = A \square H\) as Hom-coalgebras, respectively. Since \((B, \cdot, \phi, \beta)\) is a cleft coextension, we get \((B, \cdot, \rho, \beta)\) is a right \(H\)-Hom-Hopf module by Proposition 12, where the comodule structure is defined by

\[
\rho(b) = \beta^{-2}(h_{11}) \cdot \alpha^{-3}(\phi^{-1}(h_{12}) \phi(b_{1})) \\
\otimes \alpha^{-1}(\phi(b_{2})).
\]

(79)

We know that

\[
\psi : C \rightarrow B^{coH},
\]

(80)

is an isomorphism of Hom-coalgebras by Lemma 11. Set

\[
\overline{\psi} = \psi \otimes id,
\]

(81)

\[\overline{\psi} : C \rtimes \alpha H \rightarrow A \square H.\]
We only need to verify that $\psi$ is a Hom-coalgebra morphism. First we prove that
\[
\psi (\overline{b}) = b_1 \cdot \phi^{-1} (b_2). \quad (82)
\]
In fact
\[
\psi (\overline{b}) = b_{(0)} \cdot S (b_{(1)}) = \beta^{-3} (b_{11}) \cdot \alpha^{-3} (\phi^{-1} (b_{12}) \phi (b_2)) \cdot S (\alpha^{-1} (\phi (b_2))) = \beta^{-1} (b_{11}) \cdot \alpha^{-1} (\phi^{-1} (b_{12})) \epsilon (\phi (b_2)) = b_1 \cdot \phi^{-1} (b_2).
\]
On one hand, we can calculate
\[
\begin{align*}
\Delta \psi (\overline{b} \otimes h) & \overset{\text{83}}{=} \Delta (b_1 \cdot \phi^{-1} (b_2) \otimes h) \\
& = \beta^{-3} \left( (b_1 \cdot \phi^{-1} (b_2))_{12(0)(0)} \right) \\
& \quad \cdot S (\alpha^{-3} \left( (b_1 \cdot \phi^{-1} (b_2))_{10(0)(1)} \right)) \\
& \quad \otimes \alpha^{-2} \left( (b_1 \cdot \phi^{-1} (b_2))_{01(1)} \right) \alpha^{-1} (h_1) \\
& \quad \otimes \beta^{-2} \left( (b_1 \cdot \phi^{-1} (b_2))_{02(0)(0)} \right) \\
& \quad \cdot S (\alpha^{-3} \left( (b_1 \cdot \phi^{-1} (b_2))_{00(1)(1)} \right)) \\
& \quad \otimes \alpha^{-2} \left( (b_1 \cdot \phi^{-1} (b_2))_{20(1)(1)} \right) \alpha^{-1} (h_2) \\
& = \beta^{-3} (b_{1100(0)} \cdot \phi^{-1} (b_{211})) \\
& \quad \cdot S (\alpha^{-3} (b_{1110(0)} \cdot \phi^{-1} (b_{212})) \\
& \quad \otimes \alpha^{-2} (b_{1111(0)} \cdot \phi^{-1} (b_{212})) \alpha^{-1} (h_1) \\
& \quad \otimes \beta^{-2} (b_{2000(0)} \cdot \phi^{-1} (b_{221})) \\
& \quad \cdot S (\alpha^{-3} (b_{2010(0)} \cdot \phi^{-1} (b_{222})) \\
& \quad \otimes \alpha^{-2} (b_{2211(0)} \cdot \phi^{-1} (b_{222})) \alpha^{-1} (h_2) \\
& = \beta^{-3} (b_{1100(0)} \cdot \phi^{-1} (b_{222})) \alpha^{-1} (h_2) = \beta^{-3} (b_{1110(0)}) \\
& \quad \cdot (\alpha^{-4} (\phi^{-1} (b_{221})) S (\alpha^{-4} (\phi^{-1} (b_{222})))) \\
& \quad \cdot S (\alpha^{-3} (b_{1100(0)})) \otimes \alpha^{-2} (b_{1101(0)} \cdot \phi^{-1} (b_{222})) \alpha^{-1} (h_1) \\
& \quad \otimes \beta^{-2} (b_{1200(0)} \\
& \quad \cdot (\alpha^{-4} (\phi^{-1} (b_{222}))) S (\alpha^{-4} (\phi^{-1} (b_{222}))).
\end{align*}
\]
For all $b, h \in H$, we define $\rho' : C \rightarrow H \otimes C$ by
\[
\rho' (\overline{b}) = \overline{b}_{-1} \otimes \overline{b}_0 = \phi (b_{11}) \alpha (\phi^{-1} (b_{22})) \otimes \beta^{-1} (\overline{b}_{12}),
\]
and $\lambda : C \rightarrow H \otimes H$ is defined by
\[
\lambda (\overline{b}) = \lambda_1 (\overline{b}) \otimes \lambda_2 (\overline{b}) = \phi (b_{11}) \alpha^{-2} (\phi^{-1} (b_{22})).
\]
On the other hand, we have
\[
(\psi \otimes \overline{\psi}) \Delta_h (\overline{b} \otimes h) = (\psi \otimes \overline{\psi}) (\overline{b} \otimes \alpha^{-4} (\overline{b}_{21-}) \\
\cdot \alpha^{-3} (\lambda_1 (\overline{b}_{22}) \otimes \beta^{-2} (\overline{b}_{210})) \\
\cdot \alpha^{-2} (\alpha_2 (\overline{b}_{22}) \otimes \alpha^{-2} (\phi^{-1} (b_{22})_2)).
\]
\( \alpha^{-4} (\phi(b_{211} \cdot \alpha^{-1}(b_{212})) \alpha^{-3} (\phi(b_{212})) \cdot \alpha^{-2} (\phi^{-1}(b_{2222} \cdot \alpha^{-1}(h_{1}))) \cdot (\phi^{-1}(b_{2222})) \alpha^{-1} (h_{1}) \otimes (\alpha^{-3}(b_{211})) \</p>

\( \alpha^{-2} (\phi^{-1}(b_{2222})) \alpha^{-2} (\phi^{-1}(b_{2222})) \alpha^{-1} (h_{2}))) \</p>

\( = (\psi \otimes \psi)(b_{1} \otimes (\alpha^{-3}(b_{211}))) \)</p>

\( \cdot ((\alpha^{-4}(\phi^{-1}(b_{212}))) \cdot ((\phi^{-1}(b_{2222})))) \alpha^{-1} (h_{1}) \otimes (\beta^{-3}(b_{2222})) \cdot (\alpha^{-2}(\phi(b_{2222}))) \alpha^{-1} (h_{2})) \</p>

\( = (\psi \otimes \psi)(b_{1} \otimes (\alpha^{-2}(\phi(b_{2222}))) \epsilon(b_{2222})) \)</p>

\( \cdot \alpha^{-4} (\phi^{-1}(b_{2222})) \alpha^{-1} (h_{1}) \otimes (\beta^{-2}(b_{2222})) \)</p>

\( \otimes \alpha^{-2} (\phi^{-1}(b_{2222})) \alpha^{-2} (\phi^{-1}(b_{2222}))) \alpha^{-1} (h_{2})) \)</p>

\( = (\psi \otimes \psi)(b_{1} \otimes (\alpha^{-2}(\phi(b_{2222}))) \alpha^{-3} (\phi(b_{2222}))) \)</p>

\( \cdot \alpha^{-1} (h_{1}) \otimes (\beta^{-2}(b_{2222})) \otimes \alpha^{-2} (\phi(b_{2222})) \)</p>

\( \cdot \alpha^{-1} (\phi^{-1}(b_{2222})) \alpha^{-1} (h_{2}) \</p>

\( \otimes (\alpha^{-2}(\phi(b_{2222}))) \alpha^{-3} (\phi(b_{2222}))) \alpha^{-1} (h_{1}) \</p>

\( \otimes (\beta^{-2}(b_{2222} \cdot (\phi^{-1}(b_{2222}))) \alpha^{-1} (h_{1}) \}</p>

\( \cdot \alpha^{-1} (\phi^{-1}(b_{2222})) \alpha^{-1} (h_{2}) = (83) \</p>

\( = b_{11} \cdot (\phi^{-1}(b_{12})) \)</p>

\( \otimes \alpha^{-2}(\phi(b_{2222}))) \alpha^{-3} (\phi(b_{2222})) \alpha^{-1} (h_{1}) \)</p>

\( \otimes \beta^{-2}(b_{2222} \cdot (\phi^{-1}(b_{2222}))) \alpha^{-1} (h_{1}) \)</p>

\( \cdot \alpha^{-1} (\phi^{-1}(b_{2222})) \alpha^{-1} (h_{2}) = \beta^{-2}(b_{1221}) \)</p>

\( \cdot \alpha^{-2}(\phi(b_{12})) \alpha^{-1} (h_{1}) \otimes (\alpha^{-2}(\phi(b_{12}))) \)</p>

\( \cdot \alpha^{-1} (\phi^{-1}(b_{12})) \alpha^{-1} (h_{2}) \otimes (\beta^{-2}(b_{1221})) \cdot \alpha^{-2}(\phi(b_{12})) \)</p>

\( \cdot \alpha^{-1} (\phi^{-1}(b_{12})) \alpha^{-1} (h_{2}) \)</p>

\( \cdot \alpha^{-1} (\phi^{-1}(b_{211})) \alpha^{-1} (h_{2}) \</p>

\( \) (87)

Therefore, \( B = C \times_{\lambda} H = A \Box H \) as Hom-coalgebras. \( \Box \)

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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