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Research Article

Contrast Expansion Method for Elastic Incompressible Fibrous Composites

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Contrast parameter expansion of the elastic fields for 2D composites is developed by Schwarz's method and by the method of functional equations for the case of circular inclusions. A computationally efficient algorithm is described and implemented in symbolic form to compute the local fields in 2D elastic composites and the effective shear modulus for macroscopically isotropic composites. The obtained new analytical formula contains high-order terms in the contrast parameter and explicitly demonstrates dependence on the location of inclusions. As a numerical example, the hexagonal array is considered.

1. Introduction

The general potential theory of mathematical physics yields methods of integral equation to numerically solve various boundary value problems. Integral equations for plane elastic problems were constructed by Muskhelishvili [1], first, extended to doubly periodic problems in [2], and developed in [3–5] and papers cited therein. The obtained results were applied to computations of the effective properties of the elastic media. Integral equations are efficient for the numerical investigation of nondilute composites when interactions of inclusions have to be taken into account. Another type of integral equation based on the generalized alternating method of Schwarz was proposed by Mikhlin [6] and developed in [7].

Consider a problem on the plane $\mathbb{R}^2$ isomorphic to the complex plane $\mathbb{C}$. Let $D := \mathbb{C} \cup \{\infty\} \setminus (\bigcup_{k=1}^{n} D_k \cup \partial D_k)$ where the boundary ∂D_k of D_k is a simple closed Lyapunov curve oriented in clockwise sense. Schwarz's method for a multiply-connected domain $D = \widehat{\mathbb{C}} \setminus \bigcup_{k=1}^{n}(D_k \cup \partial D_k)$ is based on the separate solutions of the simple boundary value problems for simply connected domains $\widehat{\mathbb{C}} \setminus (D_k \cup \partial D_k)$ $(k = 1, 2, \ldots, n)$. It was demonstrated in [7] that Schwarz's method can be realized as the iterative schemes constructed on contrast and on concentration parameters considered as small perturbation parameters and precisely described below. Convergence was proved for Laplace's equation for the contrast expansion.

The main advantage of Schwarz's method consists in analytical solution to the problems discussed when the physical parameters are presented in symbolic form in the final exact or approximate formulae. Such formulae were recently obtained in [8] for biharmonic functions which describes elastic materials for a circular multiply-connected domain. The results of [8] are based on the concentration expansion.

In the present paper, we apply Schwarz's method based on the contrast parameter expansion for a circular multiply-connected domain. Schwarz's method is used in the form of the functional equations method [7–9]. Elastic isotropic materials are described through two independent moduli. In order to make the presentation clear, we simplify the problem by consideration of incompressible materials when Poisson's ratio ν is equal to 0.5. Then, we introduce only one contrast parameter:

$$\rho = \frac{\mu_1/\mu - 1}{\mu_1/\mu + 1}, \tag{1}$$

where μ_1 and μ denote the shear modulus of inclusions D_k and matrix D, respectively.

In the present paper, we deduce a computationally efficient algorithm implemented in symbolic form to compute the local fields in 2D elastic composites and the effective shear modulus μ_e for macroscopically isotropic composites. The obtained new analytical formula is valid up to $O(\rho^3)$ and explicitly demonstrates dependence on the location of inclusions. Such an approach has advantages over pure numerical methods when dependencies of the effective constants on the mechanical properties of constitutes and on geometrical structure are required.

The numerical examples from Section 6 give sufficiently accurate values of μ_e for all admissible ρ, that is, for $|\rho| \leq 1$ and for f not exceeding 0.4.

2. Contrast Expansion Method

Consider a finite number n of inclusions as mutually disjoint simply connected domains D_k $(k = 1, 2, \ldots, n)$ in the complex plane $\mathbb{C} \equiv \mathbb{R}^2$. It is worth noting that the number n is given in a symbolic form with an implicit purpose to pass to the limit $n \to \infty$ later.

The components of the stress tensor can be determined by the Kolosov-Muskhelishvili formulae [1]:

$$\sigma_{xx} + \sigma_{yy} = \begin{cases} 4\,\mathrm{Re}\,\varphi_k'(z), & z \in D_k, \\ 4\,\mathrm{Re}\,\varphi_0'(z), & z \in D, \end{cases}$$
$$\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} = \begin{cases} -2\left[z\overline{\varphi_k''(z)} + \overline{\psi_k'(z)}\right], & z \in D_k, \\ -2\left[z\overline{\varphi_0''(z)} + \overline{\psi_0'(z)}\right], & z \in D. \end{cases} \tag{2}$$

The strain tensor components of incompressible materials $\epsilon_{xx} = -\epsilon_{yy}$, ϵ_{xy} are determined by the formulae [1]:

$$\epsilon_{xx} + i\epsilon_{xy} = \begin{cases} -\dfrac{1}{2\mu_1}\left[z\overline{\varphi_k''(z)} + \overline{\psi_k'(z)}\right], & z \in D_k, \\ -\dfrac{1}{2\mu}\left[z\overline{\varphi_0''(z)} + \overline{\psi_0'(z)}\right], & z \in D. \end{cases} \tag{3}$$

Let the stress tensor be applied at infinity:

$$\boldsymbol{\sigma}^\infty = \begin{pmatrix} \sigma_{xx}^\infty & \sigma_{xy}^\infty \\ \sigma_{yx}^\infty & \sigma_{yy}^\infty \end{pmatrix}. \tag{4}$$

The uniform shear stress will be considered in the paper when

$$\sigma_{xx}^\infty = \sigma_{yy}^\infty = 0, \qquad \sigma_{xy}^\infty = \sigma_{yx}^\infty = 1. \tag{5}$$

Then,

$$\psi_0(z) = iz + \psi(z), \tag{6}$$

where the functions $\varphi_0(z)$ and $\psi(z)$ are analytical in D and bounded at infinity. The functions $\varphi_k(z)$ and $\psi_k(z)$ are analytical in D_k and twice differentiable in the closures of the considered domains.

It follows from (1) that

$$\frac{\mu_1}{\mu} = \frac{1+\rho}{1-\rho}. \tag{7}$$

The perfect bonding at the matrix-inclusion interface can be expressed by two equations [1]:

$$\varphi_k(t) + t\overline{\varphi_k'(t)} + \overline{\psi_k(t)} = \varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)}, \quad t \in \partial D_k,\ k = 1, 2, \ldots, n, \tag{8}$$

$$\varphi_k(t) - t\overline{\varphi_k'(t)} - \overline{\psi_k(t)} = \frac{1+\rho}{1-\rho}\left(\varphi_0(t) - t\overline{\varphi_0'(t)} - \overline{\psi_0(t)}\right), \quad t \in \partial D_k,\ k = 1, 2, \ldots, n. \tag{9}$$

The problem (8)-(9) is the classic boundary value problem of the plane elasticity having the unique solution up to additive constants corresponding to rigid shifts of medium. It follows from [1] that solutions of (8)-(9) analytically depend on ρ for sufficiently small $|\rho|$. One can see also Chapter 2, Section 3 in [3], where the problem (8)-(9) is reduced to the Fredholm integral equation shortly written as

$$\Phi = \left(\frac{\mu_1}{\mu} - 1\right) K\Phi + g, \tag{10}$$

where K is a compact integral operator in the space of the Hölder continuous functions.

We are looking for the complex potentials in the contrast expansion form. For instance, $\varphi_k(z)$ for sufficiently small $|\rho|$ has the following form:

$$\varphi_k(z) = \sum_{l=0}^{\infty} \varphi_k^{(l)}(z)\,\rho^l. \tag{11}$$

Then, the problem (8)-(9) is reduced to the following cascade of boundary value problems. Equation (8) becomes

$$\varphi_k^{(l)}(t) + t\overline{\varphi_k^{(l)\prime}(t)} + \overline{\psi_k^{(l)}(t)} = \varphi_0^{(l)}(t) + t\overline{\varphi_0^{(l)\prime}(t)} + \overline{\psi_0^{(l)}(t)}, \quad t \in \partial D_k,\ k = 1, 2, \ldots, n,\ l = 0, 1, \ldots. \tag{12}$$

Equation (9) yields the cascade

$$\varphi_k^{(0)}(t) - t\overline{\varphi_k^{(0)\prime}(t)} - \overline{\psi_k^{(0)}(t)} = \varphi_0^{(0)}(t) - t\overline{\varphi_0^{(0)\prime}(t)} - \overline{\psi_0^{(0)}(t)},$$
$$\varphi_k^{(l)}(t) - t\overline{\varphi_k^{(l)\prime}(t)} - \overline{\psi_k^{(l)}(t)} = F_{l-1}(t), \quad t \in \partial D_k,\ k = 1, 2, \ldots, n,\ l = 1, 2, \ldots, \tag{13}$$

where

$$F_{l-1}(t) = 2\left[\varphi_0^{(l-1)}(t) - t\overline{\varphi_0^{(l-1)\prime}(t)} - \overline{\psi_0^{(l-1)}(t)}\right]. \quad (14)$$

Addition and subtraction of the boundary condition (12) for $l = 0$ and the first condition (13) yield

$$\begin{aligned} \varphi_k^{(0)}(t) &= \varphi_0^{(0)}(t), \\ \psi_k^{(0)}(t) &= \psi_0^{(0)}(t), \\ & t \in \partial D_k,\ k = 1, 2, \ldots, n. \end{aligned} \quad (15)$$

Application of principle of analytic continuation to (15) implies analytic continuation of the functions $\varphi_k^{(0)}(z)$ and $\varphi_0^{(0)}(z)$ into all the domains D_k. Then, (6) gives the exact formulae for the zero approximation:

$$\begin{aligned} \varphi_0^{(0)}(z) &= 0, \\ \psi_0^{(0)}(z) &= iz \end{aligned} \quad (16)$$

and $F_0(t) = 2i\bar{t}$.

Ultimately, we arrive at the following cascade. First, we solve n boundary value problems separately for every domain D_k:

$$\varphi_k^{(1)}(t) - t\overline{\varphi_k^{(1)\prime}(t)} - \overline{\psi_k^{(1)}(t)} = 2i\bar{t}, \quad t \in \partial D_k. \quad (17)$$

Next, we solve the problem following from (13):

$$\begin{aligned} &\varphi_0^{(1)}(t) + t\overline{\varphi_0^{(1)\prime}(t)} + \overline{\psi_0^{(1)}(t)} \\ &\quad = \varphi_k^{(1)}(t) + t\overline{\varphi_k^{(1)\prime}(t)} + \overline{\psi_k^{(1)}(t)}, \\ &\qquad t \in \partial D_k,\ k = 1, 2, \ldots, n. \end{aligned} \quad (18)$$

This is a boundary value problem for the multiply-connected domain D on the functions $\varphi_0^{(1)}(z)$, $\psi_0^{(1)}(z)$ analytic in D and twice continuously differentiable in the closure of D including infinity. The described above step is the first step of the iterative scheme when we pass from $\varphi_0^{(0)}(z)$ and $\psi_0^{(0)}(z)$ to $\varphi_0^{(1)}(z)$ and $\psi_0^{(1)}(z)$. The step $(l+1)$ consists in the solution to the problems for every domain D_k:

$$\begin{aligned} &\varphi_k^{(l+1)}(t) - t\overline{\varphi_k^{(l+1)\prime}(t)} - \overline{\psi_k^{(l+1)}(t)} = F_l(t), \\ &\qquad t \in \partial D_k,\ k = 1, 2, \ldots, n, \end{aligned} \quad (19)$$

and, further, the problem for the domain D

$$\begin{aligned} &\varphi_0^{(l+1)}(t) + t\overline{\varphi_0^{(l+1)\prime}(t)} + \overline{\psi_0^{(l+1)}(t)} \\ &\quad = \varphi_k^{(l+1)}(t) + t\overline{\varphi_k^{(l+1)\prime}(t)} + \overline{\psi_k^{(l+1)}(t)}, \\ &\qquad t \in \partial D_k,\ k = 1, 2, \ldots, n. \end{aligned} \quad (20)$$

Therefore, the conjugation problem (8)-(9) is reduced to the sequence of the problems separately for the domains D_k $(k = 1, 2, \ldots, n)$ and D. The described iterative scheme is computationally effective if the inclusions D_k have simple shape. The next section is devoted to its explicit realization for circular inclusions.

3. Method of Functional Equations

Consider the circular inclusions $\mathbb{D}_k := \{z \in \mathbb{C} : |z - a_k| < r\}$ $(k = 1, 2, \ldots, n)$ and $\mathbb{D} := \mathbb{C} \cup \{\infty\} \setminus (\bigcup_{k=1}^{n} \mathbb{D}_k \cup \partial\mathbb{D}_k)$, where $\partial\mathbb{D}_k := \{t \in \mathbb{C} : |t - a_k| = r\}$. Introduce the new unknown functions

$$\begin{aligned} \Phi_k(z) &= \left(\frac{r^2}{z - a_k} + \overline{a_k}\right)\varphi_k'(z) + \psi_k(z), \\ &\qquad |z - a_k| \le r, \end{aligned} \quad (21)$$

analytic in $\mathbb{D}_k$ except at the point a_k, where its principal part has the form $r^2(z - a_k)^{-1}\varphi_k'(a_k)$.

Let $z_{(k)}^* = r^2(\overline{z - a_k})^{-1} + a_k$ denote the inversion with respect to the circle $\partial\mathbb{D}_k$. If a function $f(z)$ is analytic in $|z - a_k| < r$, then $\overline{f(z_{(k)}^*)}$ is analytic in $|z - a_k| > r$. The problem (8), (9) was reduced in [9] (see equations (5.6.11) and (5.6.16) in Chapter 5), [10] to the system of functional equations:

$$\begin{aligned} &\varphi_k(z) \\ &\quad = \rho \sum_{m \ne k} \left[\overline{\Phi_m\left(z_{(m)}^*\right)} - (z - a_m)\overline{\varphi_m'(a_m)}\right] \\ &\qquad - \rho\overline{\varphi_k'(a_k)}(z - a_k) + p_0, \end{aligned} \quad (22)$$

$$\begin{aligned} &\Phi_k(z) \\ &\quad = \rho \sum_{m \ne k} \overline{\varphi_m\left(z_{(m)}^*\right)} \\ &\qquad + \rho \sum_{m \ne k} \left(\overline{z_{(k)}^*} - \overline{z_{(m)}^*}\right)\left[\left(\overline{\Phi_m\left(z_{(m)}^*\right)}\right)' - \overline{\varphi_m'(a_m)}\right] \\ &\qquad + (1 + \rho) iz - \rho \sum_{k=1}^{n} \frac{r^2\overline{\varphi_k'(a_k)}}{z - a_k} + q_0, \\ &\qquad\qquad |z - a_k| \le r,\ k = 1, 2, \ldots, n, \end{aligned} \quad (23)$$

where p_0 and q_0 are constants. The unknown functions $\varphi_k(z)$ and $\Phi_k(z)$ $(k = 1, 2, \ldots, n)$ are related by $2n$ equations (22)-(23). One can see that the functional equations do not contain integral operators but contain compositions of $\varphi_k(z)$ and $\Phi_k(z)$ with inversions.

Following [11], we first introduce the Hardy-Sobolev space $\mathscr{H}^{(2,2)}(\mathbb{D}_k)$ separately for each $k = 1, \ldots, N$ as the space of functions analytic in D_k satisfying the conditions:

$$\sup_{0<r<r_k} \int_0^{2\pi} \left|f^{(q)}\left(re^{i\theta} + a_k\right)\right|^2 d\theta < \infty, \quad q = 0, 1, 2, \quad (24)$$

where $f^{(q)}$ denotes the derivative $d^q f/dz^q$. The norm is introduced as follows:

$$\|f\|_{\mathscr{H}^{(2,2)}(\mathbb{D}_k)}^2 = \|f\|_{\mathscr{H}^2(\mathbb{D}_k)}^2 + \|f'\|_{\mathscr{H}^2(\mathbb{D}_k)}^2 + \|f''\|_{\mathscr{H}^2(\mathbb{D}_k)}^2, \quad (25)$$

where the classic Hardy norm

$$\|f\|_{\mathscr{H}^2(\mathbb{D}_k)}^2 = \sup_{0<r<r_k} \int_0^{2\pi} \left|f\left(re^{i\theta} + a_k\right)\right|^2 d\theta \quad (26)$$

is used.

Using the above designations (24)–(26) we introduce the space $\mathscr{H}^{(2,2)}(\bigcup_{k=1}^{N}\mathbb{D}_k)$ shortly denoted by $\mathscr{H}^{(2,2)}$ of functions f analytic in $\bigcup_{j=1}^{N}\mathbb{D}_k$ endowed with the norm:

$$\|f\|^2_{\mathscr{H}^{(2,2)}} := \sum_{k=1}^{N}\|f_k\|^2_{\mathscr{H}^{(2,2)}(\mathbb{D}_k)}, \tag{27}$$

where $f(z) = f_k(z)$ for $z \in \mathbb{D}_k$ $(k = 1,\dots,N)$.

The functional-differential equations (22)-(23) include the meromorphic functions $\Phi_k(z)$ not belonging to $\mathscr{H}^{(2,2)}$. They were written as

$$\phi = \mathscr{A}\phi + \eta \tag{28}$$

on the vector-function $\phi(z) = (\varphi_k(z), \psi_k(z))^T$ introduced in all D_k by substitution (21) in the space $\mathscr{H}^{(2,2)} \times \mathscr{H}^{(2,2)}$. The operator $\mathscr{A}$ and the given vector-function η are determined by (22)-(23). Equation (28) was explicitly written in [11] as the system of functional-differential equations

$$\begin{aligned}\varphi_k(z) = \rho\sum_{m\neq k}\Big[&z\overline{\varphi'_m\left(z^*_{(m)}\right)} + \overline{\psi_m\left(z^*_{(m)}\right)}\\ &-(z-a_m)\overline{\varphi'_m(a_m)}\Big] - \rho\overline{\varphi'_k(a_k)}(z-a_k) + p_0,\\ &|z-a_k| \le r_k,\ k = 1,2,\dots,N,\end{aligned} \tag{29}$$

$$\begin{aligned}\psi_k(z) = \rho\sum_{m\neq k}\overline{\varphi_m\left(z^*_{(m)}\right)} - \rho\sum_{m\neq k}\overline{z^*_{(m)}}\Big[&\overline{\varphi'_m\left(z^*_{(m)}\right)}\\ &+ z\left(\overline{\varphi'_m\left(z^*_{(m)}\right)}\right)' + \left(\overline{\psi_m\left(z^*_{(m)}\right)}\right)' - \overline{\varphi'_m(a_m)}\Big]\\ &- \rho\sum_{m\neq k}\frac{r_m^2}{z-a_m}\overline{\psi'_m(a_m)} + (1+\rho)iz + \rho\overline{a_k}\overline{\varphi'_k(a_k)}\\ &+ q_0, \quad |z-a_k| \le r_k,\ k = 1,2,\dots,N.\end{aligned} \tag{30}$$

It was proved in [11] that the operator $\mathscr{A}$ is compact in $\mathscr{H}^{(2,2)} \times \mathscr{H}^{(2,2)}$. One can see that the contrast parameter ρ plays the role of the spectral parameter; hence, ϕ can be written in the form of power series in ρ. This implies that the method of successive approximations applied to (29) converges in $\mathscr{H}^{(2,2)} \times \mathscr{H}^{(2,2)}$ for sufficiently small ρ. Let $\varphi^*_k(z)$, $\psi^*_k(z)$ $(k = 1,2,\dots,n)$ be a solution of (29). This unique solution belongs to $\mathscr{H}^{(2,2)} \times \mathscr{H}^{(2,2)}$. The given vector-function η is twice differentiable in the closures of D_k. Hence, the pumping principle [9, page 22] can be applied as follows. The shifts in composition operators of the right part of (29) are the shift strictly into domains. Hence, if we substitute $\varphi^*_k(z)$, $\psi^*_k(z)$ into (29), we obtain that $\varphi^*_k(z)$, $\psi^*_k(z)$ are twice differentiable in the closures of D_k. The equivalent method of successive approximations can be applied to (22)-(23) more conveniently in computations.

The functions $\varphi(z)$ and $\psi(z)$ are determined through $\varphi_k(z)$ and $\Phi_k(z)$ up to additive constants. For instance,

$$\varphi(z) = \frac{\rho}{1+\rho}\sum_{m=1}^{n}\left[\overline{\Phi_m\left(z^*_{(m)}\right)} - (z-a_m)\overline{\varphi'_m(a_m)}\right], \quad z \in \mathbb{D}. \tag{31}$$

4. Method of Successive Approximations

Application of successive approximations to functional equations is equivalent to Schwarz's method described at the end of Section 2. It follows, for instance, from the uniqueness of the analytic expansion in ρ near zero. Moreover, each iteration for the functional equations corresponds to an iteration step in Schwarz's method since the coefficients in the series in ρ are also uniquely determined. It can be also established directly form formulae written below.

Using the series (11) for $\varphi_k(z)$ and analogous series for other functions and applying successive approximations to the functional equations we obtain the following iteration scheme. The zeroth approximation is

$$\begin{aligned}\varphi^{(0)}_k(z) &= 0,\\ \Phi^{(0)}_k(z) &= iz.\end{aligned} \tag{32}$$

The next approximations for $s = 1,2,\dots$ are

$$\begin{aligned}\varphi^{(s)}_k(z) &= \sum_{m\neq k}\left(\overline{\Phi^{(s-1)}_m\left(z^*_m\right)} - (z-a_m)\overline{\left(\varphi^{(s-1)}_m\right)'(a_m)}\right)\\ &\quad - (z-a_k)\overline{\left(\varphi^{(s-1)}_k\right)'(a_k)},\end{aligned} \tag{33}$$

$$\begin{aligned}\Phi^{(s)}_k(z) &= \sum_{m\neq k}\overline{\varphi^{(s-1)}_m\left(z^*_m\right)} - \sum_{m=1}^{n}\frac{r^2}{z-a_m}\overline{\left(\varphi^{(s-1)}_m\right)'(a_m)}\\ &\quad + \sum_{m\neq k}\left(\overline{z^*_k} - \overline{z^*_m}\right)\\ &\quad \cdot\left(\left(\overline{\Phi^{(s-1)}_m\left(z^*_m\right)}\right)' - \overline{\left(\varphi^{(s-1)}_m\right)'(a_m)}\right) + (\delta_{0s} + \delta_{1s})\\ &\quad \cdot iz,\end{aligned} \tag{34}$$

where $\delta_{j,k}$ denotes the Kronecker symbol.

Introduce the functions

$$F^{(s)}(z) = \sum_{m=1}^{n}\left[\overline{\Phi^{(s)}_m\left(z^*_{(m)}\right)} - (z-a_m)\overline{\varphi^{(s)\prime}_m(a_m)}\right]. \tag{35}$$

Then,

$$\sum_{m=1}^{n}\left[\overline{\Phi_m\left(z^*_{(m)}\right)} - (z-a_m)\overline{\varphi'_m(a_m)}\right] = \sum_{s=0}^{\infty}F^{(s)}(z)\rho^s. \tag{36}$$

It follows from (31) that $\varphi(z)$ can be written in the following form:

$$\varphi(z) = \sum_{s=0}^{\infty}\left(\sum_{j=0}^{s-1}(-1)^{s-j-1}F^{(j)}(z)\right)\rho^{s}, \tag{37}$$

where the following expansion is used:

$$\frac{\rho}{1+\rho} = \sum_{l=0}^{\infty}(-1)^{l}\rho^{l+1}. \tag{38}$$

The sth approximation for $\varphi(z)$ becomes

$$\varphi^{(s)}(z) = \sum_{j=0}^{s-1}(-1)^{s-j-1}F^{(j)}(z), \quad s = 1, 2, \ldots \tag{39}$$

and $\varphi^{(0)}(z) = 0$.

5. Shear Modulus

5.1. General Iterative Scheme. Introduce the average value over a sufficiently large rectangle Q_n containing all the inclusions $\mathbb{D}_k$:

$$\langle w\rangle_n = \frac{1}{|Q_n|}\iint_{Q_n} w\,dx_1dx_2. \tag{40}$$

The averaged shear modulus $\mu_e^{(n)}$ of the considered finite composite is introduced as the ratio

$$\mu_e^{(n)} = \frac{\langle\sigma_{12}\rangle_n}{2\langle\epsilon_{12}\rangle_n}. \tag{41}$$

It is related to the effective shear modulus μ_e for macroscopically isotropic composites by the limit $\mu_e = \lim_{n\to\infty}\mu_e^{(n)}$. It was demonstrated in [8] that (41) can be transformed into

$$\frac{\mu_e}{\mu} = \frac{1+\operatorname{Re}A}{1-\operatorname{Re}A}, \tag{42}$$

where

$$A = \lim_{n\to\infty}\frac{1}{2|Q_n|}\sum_{k=1}^{n}\int_{\partial\mathbb{D}_k}\varphi(t)\,dt. \tag{43}$$

Here, $\varphi(z)$ is a solution to the problem with n inclusions.

In order to determine A, we first calculate the integral $\int_{\partial\mathbb{D}_k}\varphi(z)dz$. Equation (37) implies that

$$\int_{\partial\mathbb{D}_k}\varphi(t)\,dt = \sum_{s=0}^{\infty}\left(\sum_{j=0}^{s-1}(-1)^{s-j-1}\int_{\partial\mathbb{D}_k}F^{(j)}(t)\,dt\right)\rho^{s}. \tag{44}$$

Application of (35) and Cauchy's integral theorem yield

$$\int_{\partial\mathbb{D}_k}F^{(s)}(z)\,dz = \sum_{m=1}^{n}\int_{\partial\mathbb{D}_k}\overline{\Phi_m^{(s)}\left(z_{(m)}^{*}\right)}dz. \tag{45}$$

The presented iterative scheme can be easily realized numerically. But we are interested in analytical formulae which can be obtained by symbolic computations. In the next sections, we perform symbolic computations to determine $\varphi(z)$ and the effective shear modulus in the third-order approximation.

5.2. Second-Order Approximation. We are looking for $\varphi(z)$ in the form

$$\begin{aligned}\varphi(z) &= \varphi^{(1)}(z)\rho + \varphi^{(2)}(z)\rho^2 + \varphi^{(3)}(z)\rho^3\\ &\quad + O\left(|\rho|^4\right).\end{aligned} \tag{46}$$

It follows from (37) that for the third-order approximation we need the integrals (45) for $j = 0, 1, 2$. Using the second equation (32), we have

$$\begin{aligned}\int_{\partial\mathbb{D}_k}F^{(0)}(z)\,dz &= \sum_{m=1}^{n}\int_{\partial\mathbb{D}_k}\overline{\Phi_m^{(0)}\left(z_{(m)}^{*}\right)}dz\\ &= -i\sum_{m=1}^{n}\int_{\partial\mathbb{D}_k}\overline{z_{(m)}^{*}}dz.\end{aligned} \tag{47}$$

The functions $\overline{z_{(m)}^{*}} = r^2/(z-a_m) + \overline{a_m}$ are analytic in $\mathbb{D}_k$ for $m \neq k$. Therefore, the integrals with $m \neq k$ vanish in (47) by Cauchy's integral theorem and (47) can be calculated by residues

$$\int_{\partial\mathbb{D}_k}F^{(0)}(z)\,dz = -i\int_{\partial\mathbb{D}_k}\overline{z_{(k)}^{*}}dz = 2\pi r^2. \tag{48}$$

In order to use (37) for $s = 1$ we find from (34)

$$\begin{aligned}\Phi_k^{(1)}(z) = i\Bigg(&z\\ &+\sum_{m\neq k}\left(\left(\overline{z_k^{*}}-\overline{a_k}\right)-\left(\overline{z_m^{*}}-\overline{a_m}\right)+\left(\overline{a_k}-\overline{a_m}\right)\right)\\ &\cdot\frac{r^2}{(z-a_m)^2}\Bigg).\end{aligned} \tag{49}$$

Using the expansion for $m \neq k$,

$$\begin{aligned}&(z-a_m)^{-l}\\ &\quad = \sum_{j=0}^{\infty}(-1)^{j}\binom{l+j-1}{j}(a_k-a_m)^{-(l+j)}(z-a_k)^{j},\end{aligned} \tag{50}$$

we have

$$\begin{aligned}\left(\overline{z_k^{*}}-\overline{a_k}\right)\frac{r^2}{(z-a_m)^2} &= \frac{r^2}{z-a_k}\frac{r^2}{(z-a_m)^2}\\ &= r^4(a_k-a_m)^{-2}(z-a_k)^{-1}\\ &\quad - r^4\sum_{j=0}^{\infty}(-1)^{j}(j+2)(a_k-a_m)^{-(j+3)}(z-a_k)^{j},\end{aligned} \tag{51}$$

$$\begin{aligned}\left(\overline{z_m^{*}}-\overline{a_m}\right)\frac{r^2}{(z-a_m)^2} &= \frac{r^4}{2}\\ &\quad\cdot\sum_{j=0}^{\infty}(-1)^{j}(j+1)(j+2)(a_k-a_m)^{-(j+3)}(z-a_k)^{j}.\end{aligned} \tag{52}$$

Along similar lines

$$\left(\overline{a_k}-\overline{a_m}\right)\frac{r^2}{\left(z-a_m\right)^2}=r^2\sum_{j=0}^{\infty}(-1)^j\binom{j+1}{j}\left(\overline{a_k}-\overline{a_m}\right)\cdot\left(a_k-a_m\right)^{-(j+2)}\left(z-a_k\right)^j. \tag{53}$$

Subtracting (51) and (52) we obtain

$$\left(\overline{z_k^*}-\overline{a_k}\right)\frac{r^2}{\left(z-a_m\right)^2}-\left(\overline{z_m^*}-\overline{a_m}\right)\frac{r^2}{\left(z-a_m\right)^2}$$

$$=r^4\left(a_k-a_m\right)^{-2}\left(z-a_k\right)^{-1}-\frac{r^4}{2}\cdot\sum_{j=0}^{\infty}(-1)^j(j+2)(j+3)\left(a_k-a_m\right)^{-(j+3)}\left(z-a_k\right)^j. \tag{54}$$

Using the above formulae we obtain from (49) the exact formula:

$$\Phi_m^{(1)}(z)=i\left(z+\sum_{m_1\neq m}\left(r^4\left(a_m-a_{m_1}\right)^{-2}\left(z-a_m\right)^{-1}-\frac{r^4}{2}\sum_{j=0}^{\infty}(-1)^j(j+2)(j+3)\left(a_m-a_{m_1}\right)^{-(j+3)}\left(z-a_m\right)^j+r^2\sum_{j=0}^{\infty}(-1)^j(j+1)\left(\overline{a_m}-\overline{a_{m_1}}\right)\left(a_m-a_{m_1}\right)^{-(j+2)}\left(z-a_m\right)^j\right)\right). \tag{55}$$

We are now ready to calculate the integral:

$$\int_{\partial\mathbb{D}_k}F^{(1)}(z)\,dz=\int_{\partial\mathbb{D}_k}\sum_{m=1}^{n}\overline{\Phi_m^{(1)}\left(z_{(m)}^*\right)}dz=-i\int_{\partial\mathbb{D}_k}\sum_{m=1}^{n}\left(\overline{z_{(m)}^*}+\sum_{m_1\neq m}\left(r^4\left(\overline{a_m-a_{m_1}}\right)^{-2}\left(\overline{z_{(m)}^*-a_m}\right)^{-1}-\frac{r^4}{2}\sum_{j=0}^{\infty}(-1)^j(j+2)(j+3)\left(\overline{a_m-a_{m_1}}\right)^{-(j+3)}\left(\overline{z_{(m)}^*-a_m}\right)^j+r^2\sum_{j=0}^{\infty}(-1)^j(j+1)\left(a_m-a_{m_1}\right)\left(\overline{a_m-a_{m_1}}\right)^{-(j+2)}\left(\overline{z_{(m)}^*-a_m}\right)^j\right)\right)dz. \tag{56}$$

Using Cauchy's integral theorem and residues we obtain

$$\int_{\partial\mathbb{D}_k}\sum_{m=1}^{n}\left(\overline{z_{(m)}^*-a_m}\right)^{-1}dz=\int_{\partial\mathbb{D}_k}\sum_{m=1}^{n}\frac{z-a_m}{r^2}dz=0,$$

$$\int_{\partial\mathbb{D}_k}\sum_{m=1}^{n}\left(\overline{z_{(m)}^*-a_m}\right)^{j}dz=\int_{\partial\mathbb{D}_k}\left(\frac{r^2}{z-a_k}\right)^j dz=\begin{cases}2\pi i r^2 & \text{for } j=1\\ 0 & \text{otherwise.}\end{cases} \tag{57}$$

This yields

$$\int_{\partial\mathbb{D}_k}F^{(1)}(z)\,dz=2\pi r^2+12\pi r^6\sum_{m\neq k}\left(\overline{a_k-a_m}\right)^{-4}-4\pi r^4\sum_{m\neq k}\left(a_k-a_m\right)\left(\overline{a_k-a_m}\right)^{-3}. \tag{58}$$

5.3. Third-Order Approximation. The third-order approximation requires advanced and long computations presented below. In order to calculate $\varphi^{(3)}$, we need the following function:

$$\Phi_m^{(2)}(z)=-i\left[\sum_{m_1\neq m}\sum_{m_2\neq m_1}\overline{\left(z_{m_1}^*\right)_{m_2}^*}-\sum_{m_1=1}^{n}\frac{r^2}{z-a_{m_1}}\sum_{m_2\neq m_1}\frac{r^2}{\left(\overline{a_{m_1}}-\overline{a_{m_2}}\right)^2}+\sum_{m_1\neq m}\left(\left(\overline{z_m^*}-\overline{a_m}\right)-\left(\overline{z_{m_1}^*}-\overline{a_{m_1}}\right)+\left(\overline{a_m}-\overline{a_{m_1}}\right)\right)\left(\frac{r^2}{\left(z-a_{m_1}\right)^2}-\sum_{m_2\neq m_1}\left(\frac{r^2}{\left(r^2/\left(z-a_{m_1}\right)+\overline{a_{m_1}}-\overline{a_{m_2}}\right)^2}+\frac{2r^4}{\left(r^2/\left(z-a_{m_1}\right)+\overline{a_{m_1}}-\overline{a_{m_2}}\right)^3\left(z-a_{m_1}\right)}-\frac{3r^6}{\left(r^2/\left(z-a_{m_1}\right)+\overline{a_{m_1}}-\overline{a_{m_2}}\right)^4\left(z-a_{m_1}\right)^2}+\frac{2r^4\left(a_{m_1}-a_{m_2}\right)}{\left(r^2/\left(z-a_{m_1}\right)+\overline{a_{m_1}}-\overline{a_{m_2}}\right)^3\left(z-a_{m_1}\right)^2}\right)\right)-\sum_{m_2\neq m_1}\frac{r^2}{\left(\overline{a_{m_1}}-\overline{a_{m_2}}\right)^2}\right]. \tag{59}$$

It is obtained from (34) for $s=2$ by substitution $\Phi_k^{(1)}$ given by (49) and $\varphi_k^{(1)}$ calculated by (33) with $s=1$.

Below, we describe general recurrent formulae for an arbitrary sth approximation and explicitly write the

third-order terms when $s = 3$. The following double series in ρ and $(z - a_k)$ are used:

$$\varphi_k(z) = \sum_{s=0}^{\infty}\left(\sum_{j=0}^{\infty}\varphi_{k,j}^{(s)}(z-a_k)^j\right)\rho^s, \tag{60}$$

$$\Phi_k(z) = \frac{r^2}{z-a_k}\varphi_{m,1} + \sum_{j=0}^{\infty}\gamma_{m,j}(z-a_k)^j. \tag{61}$$

The coefficients in the power series (61) are presented as series in ρ:

$$\begin{aligned}\varphi_{m,l} &= \sum_{s=0}^{\infty}\varphi_{m,l}^{(s)}\rho^s,\\ \gamma_{m,l} &= \sum_{s=0}^{\infty}\gamma_{m,l}^{(s)}\rho^s.\end{aligned} \tag{62}$$

Substitution of (60)–(62) into (22) and selection of coefficients in the same powers of ρ and $(z - a_k)$ yield

$$\begin{aligned}&\varphi_{k,j}^{(s)}\\ &= -\delta_{j,1}\overline{\varphi_{k,1}^{(s-1)}}\\ &\quad + (-1)^j\sum_{m\neq k}\sum_{l=1}^{\infty}\overline{\gamma_{m,l}^{(s-1)}}r^{2l}\binom{l+j-1}{j}(a_k-a_m)^{-(l+j)}\end{aligned} \tag{63}$$

with the zeroth approximation $\varphi_{k,j}^{(0)} = 0$. Along similar lines, (23) yields

$$\begin{aligned}\gamma_{k,j}^{(s)} &= i\delta_{j,1}(\delta_{0,s}+\delta_{1,s}) + \sum_{m\neq k}\Bigg(-(-1)^j\\ &\quad\cdot r^2\overline{\varphi_{m,1}^{(s-1)}}(a_k-a_m)^{-(j+1)} + (-1)^j\\ &\quad\cdot\sum_{l=1}^{\infty}\overline{\varphi_{m,l}^{(s-1)}}r^{2l}\binom{l+j-1}{j}(a_k-a_m)^{-(l+j)} + (-1)^j\\ &\quad\cdot\sum_{l=1}^{\infty}l\overline{\gamma_{m,l}^{(s-1)}}r^{2(l+1)}\binom{l+j+2}{j+1}(a_k-a_m)^{-(l+j+2)}\\ &\quad - (-1)^j\left(\overline{a_k-a_m}\right)\\ &\quad\cdot\sum_{l=1}^{\infty}l\overline{\gamma_{m,l}^{(s-1)}}r^{2l}\binom{l+j}{j}(a_k-a_m)^{-(l+j+1)}\Bigg)\end{aligned} \tag{64}$$

with the zeroth approximation $\gamma_{k,j}^{(0)} = i\delta_{1,j}$.

The expanded form of $\varphi(z)$ follows from (31):

$$\varphi(z) = \frac{\rho}{1+\rho}\sum_{k=1}^{n}\sum_{s=0}^{\infty}\sum_{j=0}^{\infty}\overline{\gamma_{k,j}^{(s)}}r^{2j}(z-a_k)^{-j}\rho^s. \tag{65}$$

Equation (65) is in accordance with (37) where

$$F^{(j)}(z) = \sum_{m=1}^{n}\sum_{l=0}^{\infty}\overline{\gamma_{m,l}^{(j)}}r^2(z-a_m)^{-l}. \tag{66}$$

Using (66) we calculate the integral

$$\begin{aligned}\int_{\partial\mathbb{D}_k}F^{(j)}(t)\,dt &= \int_{\partial\mathbb{D}_k}\sum_{m=1}^{n}\sum_{l=0}^{\infty}\overline{\gamma_{m,l}^{(j)}}r^2(t-a_m)^{-l}\,dt\\ &= \int_{\partial\mathbb{D}_k}\sum_{l=0}^{\infty}\overline{\gamma_{k,l}^{(j)}}r^2(t-a_k)^{-l}\,dt\\ &= 2\pi i\overline{\gamma_{k,1}^{(j)}}r^2.\end{aligned} \tag{67}$$

The coefficients $\gamma_{m,l}^{(j)}$ for $j = 0, 1, 2$ can be explicitly written by the iterations (64):

$$\begin{aligned}\gamma_{k,1}^{(0)} &= i,\\ \gamma_{k,1}^{(1)} &= i - ir^2\left(2\sum_{m\neq k}\frac{\overline{a_k-a_m}}{(a_k-a_m)^3} - 6r^2\sum_{m\neq k}\frac{1}{(a_k-a_m)^4}\right),\end{aligned} \tag{68}$$

$$\begin{aligned}\gamma_{k,1}^{(2)} &= i\Bigg(-r^4\sum_{k_1\neq k}\sum_{k_2\neq k_1}\frac{1}{(a_k-a_{k_1})^2}\frac{1}{(\overline{a_{k_1}}-\overline{a_{k_2}})^2}\\ &\quad + 6r^4\sum_{k_1\neq k}\frac{1}{(a_k-a_{k_1})^4} - 2r^2\sum_{k_1\neq k}\frac{\overline{a_k}-\overline{a_{k_1}}}{(a_k-a_{k_1})^3}\\ &\quad + \sum_{l=1}^{\infty}(-1)^l\\ &\quad\cdot\Bigg(-r^{2+2l}l\sum_{k_1\neq k}\sum_{k_2\neq k_1}\frac{1}{(a_k-a_{k_1})^{l+1}}\frac{1}{(\overline{a_{k_1}}-\overline{a_{k_2}})^{l+1}}\\ &\quad - \frac{1}{4}r^{2(l+3)}l((l+2)(l+3))^2\sum_{k_1\neq k}\frac{1}{(\overline{a_k}-\overline{a_{k_1}})^{l+3}}\\ &\quad\cdot\sum_{k_1\neq k}\frac{1}{(a_k-a_{k_1})^{l+3}} + \frac{1}{2}r^{2(l+2)}l(l+1)(l+2)(l+3)\\ &\quad\cdot\sum_{k_1\neq k}\frac{a_k-a_{k_1}}{(\overline{a_k}-\overline{a_{k_1}})^{l+2}}\sum_{k_1\neq k}\frac{1}{(a_k-a_{k_1})^{l+3}} + \frac{1}{2}\\ &\quad\cdot r^{2(l+2)}l(l+1)(l+2)(l+3)\sum_{k_1\neq k}\frac{1}{(\overline{a_k}-\overline{a_{k_1}})^{l+3}}\\ &\quad\cdot\sum_{k_1\neq k}\frac{\overline{a_k}-\overline{a_{k_1}}}{(a_k-a_{k_1})^{l+2}} - r^{2(l+1)}l(l+1)^2\\ &\quad\cdot\sum_{k_1\neq k}\frac{a_k-a_{k_1}}{(\overline{a_k}-\overline{a_{k_1}})^{l+2}}\sum_{k_1\neq k}\frac{\overline{a_k}-\overline{a_{k_1}}}{(a_k-a_{k_1})^{l+2}}\Bigg)\Bigg).\end{aligned} \tag{69}$$

The limit (43) has to be found. Using (44) and (67) we express A through $\gamma_{k,1}^{(j)}$:

$$A = \sum_{s=0}^{\infty}\left(\sum_{j=0}^{s-1}(-1)^{s-j-1}\pi i r^2\lim_{n\to\infty}\frac{1}{|Q_n|}\sum_{k=1}^{n}\overline{\gamma_{k,1}^{(j)}}\right)\rho^s. \tag{70}$$

Taking into account terms up to $O(\rho^4)$ we rest $\gamma_{k,1}^{(j)}$ for $j = 0, 1, 2$; hence,

$$\begin{aligned} A = {} & \pi i r^2 \left(\lim_{n\to\infty} \frac{1}{|Q_n|} \sum_{k=1}^{n} \overline{\gamma_{k,1}^{(0)}} \right) \rho \\ & + \pi i r^2 \lim_{n\to\infty} \frac{1}{|Q_n|} \sum_{k=1}^{n} \left(\overline{\gamma_{k,1}^{(1)}} - \overline{\gamma_{k,1}^{(0)}} \right) \rho^2 \\ & + \pi i r^2 \lim_{n\to\infty} \frac{1}{|Q_n|} \sum_{k=1}^{n} \left(\overline{\gamma_{k,1}^{(2)}} - \overline{\gamma_{k,1}^{(1)}} + \overline{\gamma_{k,1}^{(0)}} \right) \rho^3 \\ & + O\left(\rho^4\right). \end{aligned} \tag{71}$$

The above limits are calculated by use of the formalism developed in [8, 12]. It was supposed that N equal nonoverlapping disks belong to a parallelogram (fundamental) cell periodically extended to the complex plane by two linearly independent translation vectors. As an example, we shortly present the transformation of the term of $\gamma_{k,1}^{(1)}$ given by (68) multiplied by $\pi i r^2$ and complexly conjugated in accordance with (71):

$$T_n = 12\pi r^6 \sum_{m\neq k} \frac{1}{\left(\overline{a_k - a_m}\right)^4}. \tag{72}$$

We are interested in the limit associated with the Eisenstein summation [8, 12]:

$$T = \lim_{n\to\infty} \frac{T_n}{2|Q_n|}. \tag{73}$$

The normalized e-sums were introduced in [8, 12] by means of the Eisenstein summation:

$$e_p = \lim_{n\to\infty} \frac{1}{N^{1+p/2} |Q_n|^{p/2}} \sum_{m\neq k} \frac{1}{(a'_k - a'_m)^p}, \tag{74}$$

$$\begin{aligned} & e_{pp} \\ & \quad = \lim_{n\to\infty} \frac{1}{N^{1+p} |Q_n|^{p}} \sum_{m\neq k} \sum_{k\neq k_1} \frac{1}{(a'_k - a'_m)^p \left(\overline{a'_k - a'_{k_1}}\right)^p}, \end{aligned} \tag{75}$$

$$e_p^{(1)} = \lim_{n\to\infty} \frac{1}{N^{1+(p-1)/2} |Q_n|^{(p-1)/2}} \sum_{m\neq k} \frac{\overline{a'_k - a'_m}}{(a'_k - a'_m)^p}, \tag{76}$$

where $a'_k = |Q_n|^{-1/2} a_k$. This means that the rectangle Q_n is normalized by the $\mathbb{C}$-linear transformation $z \to z' = |Q_n|^{-1/2} z$ to Q'_n having the unit area. Therefore, $T = 6(f^3/\pi^2)\overline{e_4}$. Other terms are transformed by the same method. As a result we arrive at the following formula:

$$\begin{aligned} A = {} & f\rho + 2f \left(3\frac{f^2}{\pi^2}\overline{e_4} - \frac{f}{\pi}\overline{e_3^{(1)}} \right) \rho^2 + f \left(-\frac{f^2}{\pi^2} e_{2,2} \right. \\ & + \sum_{l=1}^{\infty} (-1)^l \left(-\left(\frac{f}{\pi}\right)^{(l+1)} l e_{l+1,l+1} \right. \\ & - \frac{1}{4} \left(\frac{f}{\pi}\right)^{(l+3)} l\left((l+2)(l+3)\right)^2 \left|e_{l+3}\right|^2 \\ & + \frac{1}{2} \left(\frac{f}{\pi}\right)^{(l+2)} l(l+1)(l+2)(l+3)\, \overline{e_{l+2}^{(1)}} e_{l+3} \\ & + \frac{1}{2} \left(\frac{f}{\pi}\right)^{(l+2)} l(l+1)(l+2)(l+3)\, \overline{e_{l+3}} e_{l+2}^{(1)} \\ & \left. \left. - \left(\frac{f}{\pi}\right)^{(l+1)} l(l+1)^2 \left|e_{l+2}^{(1)}\right|^2 \right) \right) \rho^3 + O\left(\rho^4\right), \end{aligned} \tag{77}$$

where

$$f = \lim_{n\to\infty} \frac{N\pi r^2}{|Q_n|} \tag{78}$$

denotes the concentration. The absolute convergence of series in R_3 follows from the standard root test (Cauchy's criterion).

The effective shear modulus is calculated by (42):

$$\frac{\mu_e}{\mu} = \frac{1 + f\rho + R_2\rho^2 + R_3\rho^3}{1 - f\rho - R_2\rho^2 - R_3\rho^3}, \tag{79}$$

where

$$R_2 = 2\left(3\frac{\overline{e_4}}{\pi^2} f^3 - \frac{\overline{e_3^1}}{\pi} f^2 \right), \tag{80}$$

$$\begin{aligned} R_3 = {} & f \left[\frac{f^2}{\pi^2} e_{2,2} + \sum_{l=1}^{\infty} (-1)^l \left(-\left(\frac{f}{\pi}\right)^{(l+1)} l e_{l+1,l+1} \right. \right. \\ & - \frac{1}{4} \left(\frac{f}{\pi}\right)^{(l+3)} l\left((l+2)(l+3)\right)^2 \left|e_{l+3}\right|^2 \\ & + \frac{1}{2} \left(\frac{f}{\pi}\right)^{(l+2)} l(l+1)(l+2)(l+3)\, \overline{e_{l+2}^{(1)}} e_{l+3} \\ & + \frac{1}{2} \left(\frac{f}{\pi}\right)^{(l+2)} l(l+1)(l+2)(l+3)\, \overline{e_{l+3}} e_{l+2}^{(1)} \\ & \left. \left. - \left(\frac{f}{\pi}\right)^{(l+1)} l(l+1)^2 \left|e_{l+2}^{(1)}\right|^2 \right) \right]. \end{aligned} \tag{81}$$

Expansion of (79) in ρ yields

$$\begin{aligned} \frac{\mu_e}{\mu} = {} & 1 + 2f\rho + 2\left(f^2 + R_2\right)\rho^2 \\ & + 2\left(f^3 + 2fR_2 + R_3\right)\rho^3 + O\left(\rho^4\right). \end{aligned} \tag{82}$$

6. Numerical Simulations

The asymptotic formula (82) is a new theoretical formula obtained in the present paper. The computationally effective formulae and algorithms for the absolutely convergent sums e_p and e_{pp} ($p > 2$) were developed in [13, 14]. However, the numerical implementation of the conditionally convergent series (74)-(75) for $p = 2$ and (76) for $p = 3$ requires further investigations.

Formula (42) is similar to the famous Clausius-Mossotti approximation (Maxwell's formula) [15, Section 10.4] applied for calculation of the effective conductivity of dilute composites. It is surprising that the Eisenstein summation and Maxwell's self-consistent formalism are based on the different summation definitions of the conditionally convergent series [16, 17]. Using an analogy with the conductivity problem we now justify the proper limit value of e_3^1, that is, (76) for $p = 3$. The Hashin-Shtrikman bounds [15, Ch. 23] for 2D incompressible elastic media become

$$h^+ = \frac{1 + f\rho}{1 - f\rho},$$
$$h^- = \frac{(\rho + 1)((f - 1)\rho + 1)}{(\rho - 1)((f - 1)\rho - 1)} \quad \text{for } \rho < 0. \tag{83}$$

In the case $\rho > 0$, the bounds h^+ and h^- are replaced with each other. One can check that the upper and lower bounds (83) coincide up to $O(\rho^3)$:

$$h = 1 + 2f\rho + 2f^2\rho^2 + O\left(\rho^3\right). \tag{84}$$

This implies that the coefficients in the term $f^2\rho^2$ of the expressions (82) and (84) must be equal. It follows from (80) that e_3^1 must vanish in the framework of the considered Maxwell's formalism. The definition of $e_{2,2}$ is not essential in the final formulae since the terms with $e_{2,2}$ are cancelled in (81). The above demonstration is based on the formal pure mathematical arguments. Its physical interpretation has been not clear yet.

Consider a numerical example, the regular hexagonal lattice, when the e-sums become the Eisenstein-Rayleigh lattice sums $e_k^{(1)} = S_k^{(1)}$ and $e_k = S_k$ calculated by algorithm presented in [8, 18]. Only the nonzero values of S_k and $S_k^{(1)}$ are presented in Table 1.

The effective shear modulus for the regular hexagonal lattice becomes

$$\frac{\mu_e}{\mu} = \frac{1 + f\rho + H_3\rho^3}{1 - f\rho - H_3\rho^3} + O\left(\rho^4\right), \tag{85}$$

Table 1: The nonzero values of the lattice sums for the hexagonal array.

k	S_{k+1}	$S_k^{(1)}$
5	3.80815	4.2426
11	2.53532	2.93754
17	1.64402	1.89818
23	1.06787	1.23308
29	0.693603	0.800903
35	0.450508	0.520202
41	0.292614	0.337881
47	0.190058	0.21946
53	0.123446	0.142544
59	0.0801808	0.0925848

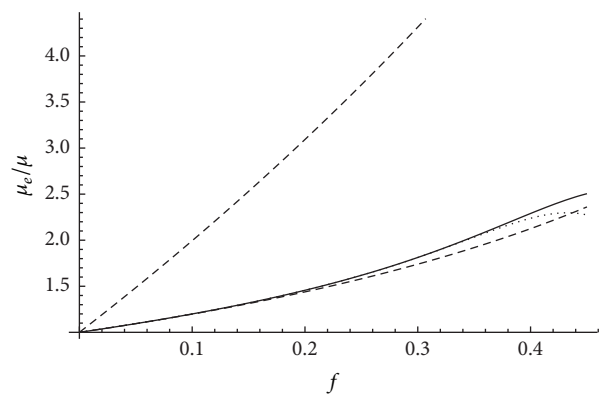

Figure 1: Dependence of the effective shear modulus for the hexagonal array on the concentration f for $\rho = 0.9$. The data are computed by (85) (solid line) and its polynomial expansion up to $O(\rho^4)$ (dotted line). The Hashin-Shtrikman bounds (83) are shown by dashed lines.

where

$$H_3 = f\left[f^2 + \sum_{l=1}^{\infty}(-1)^l\left(-\left(\frac{f}{\pi}\right)^{(l+1)} lS_{l+1}^2 - \frac{1}{4}\left(\frac{f}{\pi}\right)^{(l+3)} l\left((l+2)(l+3)\right)^2 S_{l+3}^2 + \left(\frac{f}{\pi}\right)^{(l+2)} l(l+1)(l+2)(l+3)S_{l+2}^{(1)}S_{l+3} - \left(\frac{f}{\pi}\right)^{(l+1)} l(l+1)^2\left(S_{l+2}^{(1)}\right)^2\right)\right]. \tag{86}$$

The typical dependence of the effective modulus (85) on f is displayed in Figure 1 for ρ and in Figure 2 for negative ρ. A dependence on ρ is given in Figure 3. The presented and other graphs demonstrate sufficiently good precision for $f < 0.4$ and for arbitrary $|\rho|$.

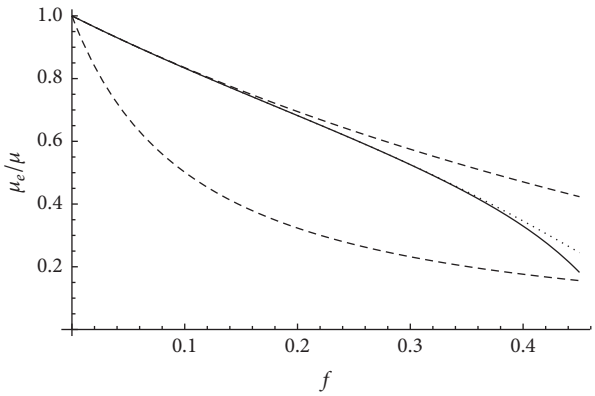

Figure 2: The same dependencies as in Figure 1 but for $\rho = -0.9$.

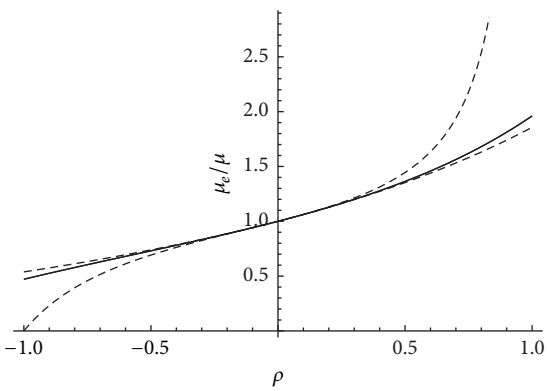

Figure 3: Dependence of the effective shear modulus for the hexagonal array on ρ for $f = 0.3$. The designations of lines are the same as in Figure 1. Equation (85) and its polynomial expansion coincide (solid line).

7. Conclusion

Schwarz's alternating method is applied to 2D elastic problem for dispersed composites. It is realized for circular inclusions in symbolic form. Exact and approximate formulae for the local fields and for the effective shear modulus are established. In general, Schwarz's method can be realized as expansion on the concentration f and on the contrast parameter ρ. The concentration expansion was recently realized in [8]. In the present paper, we use the contrast parameter expansion. These two expansions in f and ρ yield two different computational schemes [7]. The effective modulus formulae are the same in the second-order approximations and begin to differ in the third-order terms in f and ρ, respectively. Schwarz's method is used in the form of the functional equations method for circular inclusions [7–9].

We develop a computationally efficient algorithm implemented in symbolic form to compute the local fields in 2D elastic incompressible composites and the effective shear modulus for macroscopically isotropic composites. The new analytical formula (79) contains the third-order term in ρ and explicitly demonstrates dependence on the location of inclusions. The theory is supplemented by a numerical example on the hexagonal array of inclusions. Figures 1–3 illustrate the dependence of the effective shear modulus on f and ρ. These numerical examples give sufficiently accurate values of μ_e for all admissible ρ, that is, for $|\rho| \leq 1$ and for $f < 0.4$.

One can expect that the precision of μ_e will increase by using of the next approximation terms ρ^m $(m > 3)$. As it is noted in Introduction, Schwarz's method can be based on two different expansion, in ρ used here and in f used in [12]. The both expansions contain the locations of inclusions in symbolic form. The expansions in f were held for smaller f but for an arbitrary ν [12]. At the present time, we are inclined to use the contrast parameter expansions. However, we suppose that the choice between different variants of Schwarz's method will depend on the further implementation of high-order symbolic-numerical codes.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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