Research Article

Rota-Baxter Operators on 3-Dimensional Lie Algebras and the Classical $R$-Matrices

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Our aim is to classify the Rota-Baxter operators of weight 0 on the 3-dimensional Lie algebra whose derived algebra's dimension is 2. We explicitly determine all Rota-Baxter operators (of weight zero) on the 3-dimensional Lie algebras $g$. Furthermore, we give the corresponding solutions of the classical Yang-Baxter equation in the 6-dimensional Lie algebras $g \ltimes \text{ad}^* g^*$ and the induced left-symmetry algebra structures on $g$.

1. Introduction

In physics, the Yang-Baxter equation is a consistency equation which was first introduced in the field of statistical mechanics. It depends on the idea that, in some scattering situations, particles may preserve their momentum while changing their quantum internal states. Rota-Baxter algebra started with the probability study and has since found applications in many areas of mathematics and physics, such as quasi-symmetric functions, number theory, dendriform algebras, and Yang-Baxter equations.

A Rota-Baxter operator (of weight zero) on an associative algebra $A$ is defined to be a linear map $P : g \to g$ satisfying

$$P(x)P(y) = P(P(x)y + xP(y)), \quad \forall x, y \in A. \quad (1)$$

Rota-Baxter operators (on associative algebras) were introduced by Baxter to solve an analytic formula in probability [1–4]. It has been related to other areas in mathematics and mathematical physics [5–9]. A Rota-Baxter operator (of weight zero) on a Lie algebra $(g, [\cdot, \cdot])$ is a linear operator $P : g \to g$ such that

$$[P(x), P(y)] = P([P(x), y] + [x, P(y)]), \quad \forall x, y \in g. \quad (2)$$

In fact, a Rota-Baxter operator is also called the operator form of the classical Yang-Baxter equation [10–13]. Let $g$ be a Lie algebra and $r = \sum_i a_i \otimes b_i \in g \otimes g$. $r$ is called a classical $R$-matrix if it is a solution of the classical Yang-Baxter equation (CYBE) in $g$: that is,

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \quad (3)$$

in $U(g)$, where $U(g)$ is the universal enveloping algebra of $g$ and

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1, \quad r_{13} = \sum_i a_i \otimes 1 \otimes b_i, \quad r_{23} = \sum_i 1 \otimes a_i \otimes b_i. \quad (4)$$

Set $r^{21} = \sum_i b_i \otimes a_i$. It is easy to obtain that $r$ is skew-symmetric if and only if $r = -r^{21}$. Semenov-Tian-Shansky proved in [14] that $r$ is skew-symmetric and there is a nondegenerate symmetric invariant bilinear form on Lie algebra $g$; relation (2) is equivalent to relation (3) when the weight is zero. Furthermore, Rota-Baxter operators of weights 0 and 1 on a Lie algebra $g$ give rise to solutions of CYBE on the double Lie algebra $g \ltimes \text{ad}^* g^*$ over the direct sum $g \oplus g^*$ of the Lie algebra $g$ and its dual space $g^*$ [12, 15, 16]. Moreover, we can get some solutions of CYBE in $g \ltimes \text{ad}^* g^*$ Lie algebras through Rota-Baxter operators of any weight on $g$. 

In [12], the authors gave all Rota-Baxter operators (of weight zero) on 3-dimensional simple Lie algebra \(\mathfrak{sl}(2, \mathbb{C})\). The aim of this paper is to determine the Rota-Baxter operators (of weight zero) on the 3-dimensional Lie algebra which is not simple, and the dimension of its derived algebra is 2. We will determine the Rota-Baxter operators on the Lie algebra \(g\) and give a family of solutions of CYBE in \(g \ltimes \text{ad}^* g^*\). This paper is organized as follows. In Section 2, we give the classification theorem of Rota-Baxter operators (of weight zero) on \(g\). In Section 3, we give the corresponding solutions of CYBE in \(g \ltimes \text{ad}^* g^*\). In Section 4, we give the corresponding left-symmetry structure on \(g\).

2. The Rota-Baxter Operators on \(g\) (of Weight Zero)

2.1. Notations and the Classification Theorem. Let \(g\) be a 3-dimensional linear Lie algebra whose standard (Cartan-Weyl) basis consists of \(e_1, e_2, e_3\) over the field of complex numbers \(\mathbb{C}\) with the following Lie brackets:

\[
\begin{align*}
[e_1, e_2] &= e_1, \\
[e_1, e_3] &= 0, \\
[e_2, e_3] &= e_1 + e_3.
\end{align*}
\] (5)

Thus, a linear operator \(P : g \to g\) is determined by

\[
\begin{pmatrix}
P(e_1) \\
P(e_2) \\
P(e_3)
\end{pmatrix} =
\begin{pmatrix}
b_{11} & b_{12} & b_{13} \\
b_{21} & b_{22} & b_{23} \\
b_{31} & b_{32} & b_{33}
\end{pmatrix}
\begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix},
\] (6)

where \(b_{ij} \in \mathbb{C}, 1 \leq i, j \leq 3\). \(P\) is a Rota-Baxter operator on \(g\) if the above matrix \((b_{ij})_{3 \times 3}\) satisfies (2). Here is our main theorem.

**Theorem 1.** All Rota-Baxter operators of weight zero on \(g\) are listed in their matrices form with respect to the Cartan-Weyl basis below, where \(a, b,\) and \(c\) are nonzero complex numbers.

\[
P_1 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
a & 1 & 0
\end{pmatrix},
\]

\[
P_2 = \begin{pmatrix}
0 & 0 & 0 \\
0 & a & -a^2 \\
0 & 1 & -a
\end{pmatrix},
\]

\[
P_3 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix},
\]

\[
P_4 = \begin{pmatrix}
0 & 0 & 0 \\
\frac{a}{b} & a & -a^2 \\
\frac{1}{b} & 1 & -a
\end{pmatrix},
\]

\[
P_5 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
P_6 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
P_7 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
P_8 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
P_9 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
P_{10} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
P_{11} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
P_{12} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
P_{13} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
P_{14} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
P_{15} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
P_{16} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\[ P_{17} = \begin{pmatrix} 0 & 0 & 1 \\ a & 0 & b \\ 0 & 0 & -1 \end{pmatrix}, \]
\[ P_{18} = \begin{pmatrix} a & 0 & 1 \\ 0 & 0 & b \\ a^2 - a & 0 & a - 1 \end{pmatrix} \quad (a \neq \frac{1}{2}), \]
\[ P_{19} = \begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ a^2 - a & 0 & a - 1 \end{pmatrix} \quad (a \neq \frac{1}{2}), \]
\[ P_{20} = \begin{pmatrix} 0 & 0 & 1 \\ a & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \]
\[ P_{21} = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \]
\[ P_{22} = \begin{pmatrix} a & 0 & 1 \\ 0 & 0 & 0 \\ a^2 - a & 0 & a - 1 \end{pmatrix} \quad (a \neq \frac{1}{2}), \]
\[ P_{23} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & a \\ -\frac{1}{4} & 0 & -\frac{1}{2} \end{pmatrix}, \]
\[ P_{24} = \begin{pmatrix} a & 0 & 1 \\ b & 0 & c \\ a^2 - a & 0 & a - 1 \end{pmatrix} \quad (a \neq \frac{1}{2}), \]
\[ P_{25} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & a \\ -\frac{1}{4} & 0 & -\frac{1}{2} \end{pmatrix}, \]
\[ P_{26} = \begin{pmatrix} 1 & 0 & 1 \\ a & 0 & 0 \\ -\frac{1}{4} & 0 & -\frac{1}{2} \end{pmatrix}, \]
\[ P_{27} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -\frac{1}{4} & 0 & -\frac{1}{2} \end{pmatrix}, \]
\[ P_{28} = \begin{pmatrix} 1 & 0 & 1 \\ a & 0 & b \\ -\frac{1}{4} & 0 & -\frac{1}{2} \end{pmatrix}, \]
\[ P_{29} = \begin{pmatrix} 0 & 1 & 2a \\ 0 & a & 2a^2 \\ 0 & -\frac{1}{2} & -a \end{pmatrix}, \]
\[ P_{30} = \begin{pmatrix} a & 1 & 2a \\ 0 & 0 & 0 \\ -\frac{a}{2} & -\frac{1}{2} & -a \end{pmatrix}, \]
\[ P_{31} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}, \]
\[ P_{32} = \begin{pmatrix} a & 1 & 2(a + b) \\ ab & b & -2b^2 - 2ab \\ -\frac{a}{2} & -\frac{1}{2} & -(a + b) \end{pmatrix}. \]

2.2. Reduction to Quadratic Equations. In order to show that \( P \) is a Rota-Baxter operator, we only check the following:

\[ [P(e_1), P(e_2)] = P([P(e_1), e_2] + e_1, P(e_2)), \]
\[ [P(e_2), P(e_3)] = P([P(e_2), e_3] + e_2, P(e_3)), \]
\[ [P(e_1), P(e_3)] = P([P(e_1), e_3] + e_1, P(e_3)). \]

It follows from (5) and (6) that

\[ [P(e_2), P(e_3)] = (b_21b_32 - b_22b_31 + b_23b_33 - b_32b_31) e_1 \]
\[ + (b_22b_33 - b_23b_32) e_3, \]

while

\[ P([P(e_2), e_3] + e_2, P(e_3))] \]
\[ = P((b_22 - b_31 + b_33) e_1) - P((b_22 + b_33) e_2) \]
\[ = (b_22b_{11} - b_31b_{11} + b_33b_{11} + b_22b_{31} + b_33b_{31}) e_1 \]
\[ + (b_22b_{12} - b_31b_{12} + b_33b_{12} + b_22b_{32} + b_33b_{32}) e_2 \]
\[ + (b_22b_{13} - b_31b_{13} + b_33b_{13} + b_22b_{33} + b_33b_{33}) e_3. \]

Comparing the coefficients in (9) and (10), we have

\[ b_{21}b_{32} - 2b_2b_31 + b_2b_33 - b_32b_{32} - b_22b_{11} + b_31b_{11} \]
\[ - b_{33}b_{11} - b_{33}b_{31} = 0, \]
\[ b_{22}b_{12} - b_31b_{12} + b_33b_{12} + b_22b_{32} + b_33b_{32} = 0, \]
\[ b_{22}b_{13} + b_22b_{33} - b_31b_{13} + b_33b_{13} + b_{33}b_{33} = 0. \]

Similarly, from

\[ [P(e_1), P(e_2)] = P([P(e_1), e_2] + e_1, P(e_2)), \]
\[ [P(e_1), P(e_3)] = P([P(e_1), e_3] + e_1, P(e_3)), \]
we obtain the following six equations:

\[ b_1 b_{11} + b_2 b_{22} - b_3 b_{11} - b_3 b_{22} - b_1 b_{22} + b_3 b_{11} = 0, \]  
\[ b_1 b_{11} + b_2 b_{22} - b_3 b_{11} - b_3 b_{22} = 0, \]  
\[ b_1 b_{13} + 2b_2 b_{13} - b_3 b_{13} - b_3 b_{33} - b_2 b_{23} = 0, \]  
\[ b_1 b_{33} - 2b_2 b_{33} = b_1 b_{33} - b_3 b_{33} - b_2 b_{13} = 0, \]  
\[ b_2 b_{12} + b_2 b_{12} = 0, \]  
\[ b_1 b_{13} + 2b_2 b_{13} = 0. \]  

2.3. Solving the Quadratic Equations. Equation (19) implies 
\[ b_2 (b_1^2 + 2b_2^2) = 0. \]  
To solve the quadratic equations (11), (12), (15), (16), (17), (18), (19), and (20), we distinguish the following cases depending on whether \( b_1 = 0 \) or not.

Case 1. \( b_2 = 0, b_2 + 2b_3 \neq 0. \) That is, \( b_3 = 0, b_3 \neq 0, \) taking \( b_3 = 1. \) Equation (16) implies \( b_1 = 0. \) Equation (15) implies \( b_1 = 0. \) Equation (11) implies \( b_2 = b_2^2 b_1 + b_2 + b_3. \) Equation (12) implies \( b_3 = -b_2. \) Equation (13) implies \( b_3 = b_3 + b_3, \) that is, \( b_3 + b_3, (b_3 - b_3) = 0. \)

Subcase 2.1. If \( b_3 = 0, \) then (13) implies \( b_3 = 0. \) Equation (15) implies \( b_1 = 0. \) Equation (11) implies \( b_2 = b_2 b_3 = 0. \)

Subcase 2.1.1. If \( b_2 = 0, b_3 = 0, \) we obtain

\[ P = \begin{pmatrix} 0 & 0 & 0 \\ b_{21} & 0 & b_{23} \\ 0 & 0 & 0 \end{pmatrix}. \]  

Taking \( b_2 = a, b_3 = b, \) we obtain \( P_5. \) Taking \( b_2 = a, b_3 = b, \) we obtain \( P_5. \) Taking \( b_2 = 0, b_3 = 0, \) we obtain \( P_5. \) Taking \( b_2 = 0, b_3 = 0, \) we obtain \( P_5. \)

Subcase 2.1.2. If \( b_2 = 0, b_3 \neq 0, \) taking \( b_3 = 0, \) we obtain

\[ P = \begin{pmatrix} 0 & 0 & 0 \\ b_{21} & 0 & b_{23} \\ 1 & 0 & 0 \end{pmatrix}. \]  

Taking \( b_2 = a, b_3 = b, \) we obtain \( P_5. \) Taking \( b_2 = a, b_3 = b, \) we obtain \( P_5. \) Taking \( b_2 = 0, b_3 = 0, \) we obtain \( P_5. \) Taking \( b_2 = 0, b_3 = 0, \) we obtain \( P_5. \)

Subcase 2.2. If \( b_2 \neq 0, b_3 \neq 0, \) taking \( b_2 = 1, \) we obtain

\[ P = \begin{pmatrix} 0 & 0 & 0 \\ b_{21} & 1 & b_{23} \\ 0 & 0 & 0 \end{pmatrix}. \]  

Taking \( b_2 = a, b_3 = b, \) we obtain \( P_3. \) Taking \( b_2 = a, b_3 = a, \) we obtain \( P_3. \) Taking \( b_2 = 0, b_3 = 0, \) we obtain \( P_3. \)

Subcase 2.2.1. If \( b_1 + b_3 \neq 0, b_3 + b_3 - 1 = 0, (b_1, 1/2), \) and then \( b_2 = 0, b_3 = b_1 - b_1, \) we obtain

\[ P = \begin{pmatrix} b_{11} & 0 & 1 \\ b_{21} & 0 & b_{23} \\ b_{31} - b_{31} & 0 & 1 \end{pmatrix}. \]  

Taking \( b_1 = 0, b_3 = a, b_3 = b, \) we obtain \( P_{17}. \) Taking \( b_1 = 0, b_3 = b (a \neq 1/2), \) we obtain \( P_{18}. \) Taking \( b_1 = 0, b_3 = b (a \neq 1/2), \) we obtain \( P_{19}. \) Taking \( b_1 = 0, b_3 = b (a \neq 1/2), \) we obtain \( P_{20}. \) Taking \( b_1 = 0, b_2 = a, b_3 = b, \) we obtain \( P_{21}. \) Taking \( b_1 = 0, b_2 = b, b_3 = 0 (a \neq 1/2), \) we obtain \( P_{22}. \) Taking \( b_1 = 0, b_2 = 0, b_3 = 0, \) we obtain \( P_{23}. \) Taking \( b_1 = a, b_2 = b, b_3 = c (a \neq 1/2), \) we obtain \( P_{24}. \)

Subcase 2.2.2. If \( b_1 + b_3 = 0, b_2 = -b_2 - b_2 - 1 \neq 0, (b_2, 1/2), \) and then \( b_2 = (-2b_2 - 1)/2, b_3 = (2b_2 - 4b_1 + 1)/2, (3.4) \) implies \( 8b_1 - 12b_2 + 6b_1 - 1 = 0. \) Then we have \( 8(b_1 - 1/2)^2 = 0, \) \( b_1 = 1/2, \) giving a contradiction.

Subcase 2.2.3. If \( b_1 + b_3 = 0, b_1 - b_3 - 1 = 0, \) that is, \( b_1 = 1/2, b_3 = -1/2, \) and then \( b_2 = 0, b_3 = -1/2, \) we obtain

\[ P = \begin{pmatrix} 1/2 & 0 & 1 \\ b_{21} & 0 & b_3 \\ -1 & 0 & 1/2 \end{pmatrix}. \]  

Taking \( b_2 = 0, b_3 = a, \) we obtain \( P_{25}. \) Taking \( b_2 = a, b_3 = 0, \) we obtain \( P_{26}. \) Taking \( b_2 = 0, b_3 = 0, \) we obtain \( P_{27}. \) Taking \( b_2 = a, b_3 = b, \) we obtain \( P_{28}. \)

Case 3. Assume \( b_2 \neq 0, (b_2 + 2b_3) = 0. \) Taking \( b_2 = 1, \) then \( b_2 = -1/2. \) Equation (16) implies \( b_3 = 2b_1 + 2b_2. \) Equations (12) and (18) imply \( b_1 = (b_2 + b_3)/2. \) Equation (17) implies \( 0 = b_2 = -b_1 b_2 - b_1 b_2 - b_1 b_3 - b_2 b_3 - b_1^2 b_2 - b_1 b_2. \) Equations (11), (15), and (17) imply \( b_1^2 + b_2^2 + b_3^2 + 2b_1 b_2 + 2b_1 b_3 + 2b_2 b_3 = 0. \)
Then we have \((b_1 + b_2 + b_3)^2 = 0\). So \(b_3 = -(b_1 + b_2)\), \(b_{21} = b_1 b_{22}, b_{33} = -2b_{22} - 2b_1 b_{22}, b_3 = -b_1 / 2\). We obtain

\[
P = \begin{pmatrix}
1 & 2b_1 + 2b_2 \\
2b_1 b_2 & -2b_2 - 2b_1 b_{22} \\
-1/2 & -1/2
\end{pmatrix}.
\]

Taking \(b_1 = 0, b_2 = a\), we obtain \(P_{30}\). Taking \(b_1 = a, b_2 = 0\),
we obtain \(P_{31}\). Taking \(b_1 = a, b_2 = b\), we obtain \(P_{32}\).

### 3. Solutions of the CYBE in \(g \ltimes_{ad^*} g^*\)

In this section, we will give some solutions of CYBE in \(g \ltimes_{ad^*} g^*\). Let \((g, [\cdot, \cdot])\) be a Lie algebra and \(\beta : g \to \text{gl}(V)\) a representation of \(g\). On the vector space \(g \otimes V\), there is natural Lie algebra structure (denoted by \(g \ltimes \beta\)) given by

\[
[x_1 + v_1, x_2 + v_2] = [x_1, x_2] + \beta(x_1) v_2 - \beta(x_2) v_1,
\]

\(x_1, x_2, v_1, v_2 \in V\).

Let \(\beta^* : g \to \text{gl}(V^*)\) be the dual representation of \(\beta\). A linear map \(\bar{P} : V \to g\) can be identified as an element \(\bar{P}\) in \(g \otimes V^* \subset (g \ltimes \beta) V^* \otimes (g \ltimes \beta^*) V^*\) as follows. Let \(\{v_1, v_2, \ldots, v_m\}\) be a basis of \(V\) and \(\{v_1^*, v_2^*, \ldots, v_m^*\}\) the dual basis in \(V^*\): that is, \(v_i^*(v_j) = \delta_{ij}\). Let \(\{e_1, e_2, \ldots, e_n\}\) be a basis of \(g\). Set \(\bar{P}(v_i) = \sum_{j=1}^n a_{ij} e_j\), \(1 \leq i \leq n\). Since, as a vector space, \(\text{Hom}(V, g) \equiv g \otimes V^*\), then

\[
\bar{P} = \sum_{i=1}^n a_{ij} e_j \otimes v_i^*,
\]

\[
\subseteq (g \ltimes \beta^*) V^* \otimes (g \ltimes \beta^*) V^*.
\]

**Lemma 2** (see [15]). Let \(g\) be a Lie algebra; let \((V, \beta)\) be a \(g\)-module. A linear map \(\bar{P} : g \to g\) is a Rota-Baxter operator if and only if \(r = P - P^{21}\) is a skew-symmetric solution of CYBE in \(g \ltimes_{ad^*} g^*\).

Now consider the adjoint representation of \(g\), \((g, \text{ad})\) which is a \(g\)-module. Let \(e_1, e_2, e_3\) be the Cartan-Weyl basis. Using Lemma 2 and relation (29), we can obtain a family of solutions of CYBE in \(g \ltimes_{ad^*} g^*\) through the Rota-Baxter operators on \(g\) given in Theorem 1.

**Theorem 3.** The following tensors are solutions of the classical Yang-Baxter equation in \(g \ltimes_{ad^*} g^*\), where \(a, b, \) and \(c\) are nonzero complex numbers

\[
r_1 = (ae_1 + e_2) \otimes e_3 - e_3 \otimes (ae_1 + e_2),
\]

\[
r_2 = (ae_2 + a^2 e_3) \otimes e_3 + (e_2 + ae_3) \otimes e_3 - e_3 \otimes (ae_2 + a^2 e_3),
\]

\[
r_3 = e_2 \otimes e_3 - e_3 \otimes e_2.
\]

\[
r_4 = (abe_1 + ae_2 - a^2 e_3) \otimes e_3 + (be_1 + e_2 - ae_3) \otimes e_3
\]

\[
- e_2 \otimes (abe_1 + ae_2 - a^2 e_3) - e_3
\]

\[
\otimes (be_1 + e_2 - ae_3),
\]

\[
r_5 = (ae_1 + e_3) \otimes e_3 - e_3 \otimes (ae_1 + e_3),
\]

\[
r_6 = e_1 \otimes e_2 - e_3 \otimes e_3 + e_3 \otimes e_1,
\]

\[
r_7 = e_3 \otimes e_2 - e_2 \otimes e_3,
\]

\[
r_8 = 0,
\]

\[
r_9 = (ae_1 + be_3) \otimes e_3 + e_1 \otimes e_3 - e_2 \otimes (ae_1 + be_3)
\]

\[
- e_3 \otimes e_1,
\]

\[
r_{10} = ae_1 \otimes e_3 + e_1 \otimes e_3 - e_3 \otimes e_1 - e_3 \otimes e_1,
\]

\[
r_{11} = ae_3 \otimes e_3 + e_1 \otimes e_3 - e_2 \otimes ae_3 - e_3 \otimes e_1,
\]

\[
r_{12} = e_1 \otimes e_3 - e_3 \otimes e_1,
\]

\[
r_{13} = (ae_1 + e_2 + be_3) \otimes e_3 - e_3 \otimes (ae_1 + e_2 + be_3),
\]

\[
r_{14} = (ae_1 + e_2) \otimes e_3 - e_3 \otimes (ae_1 + e_2),
\]

\[
r_{15} = (e_2 + ae_3) \otimes e_3 - e_3 \otimes (e_2 + ae_3),
\]

\[
r_{16} = e_2 \otimes e_3 - e_2 \otimes e_3,
\]

\[
r_{17} = e_3 \otimes e_1 + (ae_1 + be_3) \otimes e_3 - e_3 \otimes e_3
\]

\[
- e_3 \otimes e_3 - e_3 \otimes (ae_1 + be_3) + e_3 \otimes e_3,
\]

\[
r_{18} = (ae_1 + e_3) \otimes e_3 + be_1 \otimes e_2
\]

\[
+ ((a^2 - a)e_1) + (a - 1)e_3 \otimes e_3 - e_3 \otimes (ae_1 + e_3)
\]

\[
- e_2 \otimes be_3 - e_3 \otimes ((a^2 - a)e_1 + (a - 1)e_3),
\]

\[
(a \neq \frac{1}{2}),
\]

\[
r_{19} = (ae_1 + e_3) \otimes e_3 + be_1 \otimes e_2
\]

\[
+ ((a^2 - a)e_1) + (a - 1)e_3 \otimes e_3 - e_3 \otimes (ae_1 + e_3)
\]

\[
- e_2 \otimes be_3 - e_3 \otimes ((a^2 - a)e_1 + (a - 1)e_3),
\]

\[
(a \neq \frac{1}{2}),
\]

\[
r_{20} = e_3 \otimes e_1 + ae_3 \otimes e_2 - e_3 \otimes e_3 - e_1 \otimes e_3 - e_3 \otimes e_3
\]

\[
\otimes ae_3 + e_3 \otimes e_3,
\]

\[
r_{21} = e_4 \otimes e_1 + ae_1 \otimes e_2 - e_3 \otimes e_3 - e_1 \otimes e_3 - e_3 \otimes e_3
\]

\[
\otimes ae_1 + e_3 \otimes e_3,
\]

\[
r_{22} = e_1 \otimes e_3 + ae_3 \otimes e_3 - e_3 \otimes e_3 - e_3 \otimes e_3
\]

\[
\otimes ae_1 + e_3 \otimes e_3,
\]
\[\begin{align*}
\text{Lemma 4 (see [13]). Let } g & \text{ be a Lie algebra; } P \text{ is called a solution of the classical Yang-Baxter equation. Define a new operation on } g \text{ by } \\
x \ast y & = [P(x), y], \quad \forall x, y \in g. \\
\text{Then } (g, \ast) & \text{ is a left-symmetric algebra.}
\end{align*}\]

According to Theorem 3 and Lemma 4, we can get some left-symmetric algebras of \( g \).

\textbf{Theorem 5.} Some left-symmetric algebras of \( g \) (of weight zero) are determined:

1. \( e_5 \ast e_1 = -e_1, e_5 \ast e_2 = ae_1, e_5 \ast e_3 = e_1 + e_5; \)
2. \( e_2 \ast e_1 = -ae_1, e_2 \ast e_2 = a^2 (e_1 + e_3), e_2 \ast e_3 = a(e_1 + e_3), e_3 \ast e_1 = -e_1, e_3 \ast e_2 = a(e_1 + e_3), e_3 \ast e_3 = e_1 + e_5; \)
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References

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