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Research Article

Traveling Wave Solutions of Space-Time Fractional Generalized Fifth-Order KdV Equation

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The Korteweg-de Vries (KdV) equation, especially the fractional higher order one, provides a relatively accurate description of motions of long waves in shallow water under gravity and wave propagation in one-dimensional nonlinear lattice. In this article, the generalized $\exp(-\Phi(\xi))$-expansion method is proposed to construct exact solutions of space-time fractional generalized fifth-order KdV equation with Jumarie's modified Riemann-Liouville derivatives. At the end, three types of exact traveling wave solutions are obtained which indicate that the method is very practical and suitable for solving nonlinear fractional partial differential equations.

1. Introduction

Nonlinear fractional differential equations (FDEs) as a special category of nonlinear partial differential equations (PDEs) have its variety of applications in physics, biology, chemistry, fluid flow, electrical networks, signal and image processing, acoustics, and so on [1–8]. Owing to widely applications and further properties in various fields of natural sciences, seeking the solutions of fractional PDEs has drawing the attention of scholars. Analyzing their solutions can help us understand and explain the nonlinear phenomena.

The generalized KdV equation is an important mathematical model used to describe long wave motion in shallow water, one-dimensional nonlinear lattice, hydrodynamics, quantum mechanics, plasma physics, and optics [1, 3, 4, 9–11]. The generalized KdV equation with higher order nonlinearity is put forward for internal solitary waves in a density and current stratified shear flow with a free surface. As a classical model, long has generated the steady-state version of the KdV equation [12] and an integral expression for the coefficients of the KdV equation in fluid is given by Benney [13].

Up to present, some effective methods have been put forward to search for exact solutions of the fractional KdV equation. Saha Ray and Gupta use the two-dimensional Legendre wavelet method to obtain the traveling wave solutions of the fractional seventh-order KdV equation [14]. In [10], the author applies the modified fractional subequation method to obtain the exact solution of the fractional coupled KdV equation [10]. (G'/G)-expansion method is given to obtain the solitary wave solutions of the fractional KdV equation in [15–17]. Applying the numerical technique based on the generalized Taylor series formula is original and convenient to obtain explicit and approximate solutions of the nonlinear fractional KdV-Burgers equation with time-space fractional derivatives [18]. Application of modified sine-cosine method to solve the fractional fifth-order KdV equation's traveling wave solutions is shown in [19]. In addition, the author draws a comparison between the generalized Kudryashov method and exp-function method to solve the exact solution of fractional KdV equation [20, 21]. Lie group analysis method [22] is also common and fundamental and much more [23–29]. This article is committed to seeking the new exact solutions for nonlinear time fractional fifth-order KdV via the generalized $\exp(-\Phi(\xi))$-expansion method [30].

The whole paper consists of five sections and the detailed contents and structure are as follows. An introduction is shown in Section 1. In Section 2, the definition of Jumarie's modified Riemann-Liouville derivatives [31, 32] is introduced. In Section 3, an analysis of the generalized $\exp(-\Phi(\xi))$-expansion method is formulated. The exact solutions of the fractional fifth-order KdV equation are obtained in Section 4. Finally, conclusions have been drawn in Section 5.

2. The Definition of Jumarie's Modified Riemann-Liouville Derivatives

In this section, we introduce the definition of the modified Riemann-Liouville derivative by Jumarie. Let $f : R \to R, x \to f(x)$, denote a continuous (but not necessarily differentiable) function, and let $h > 0$ denote a constant discretization span. Define the forward operator $\mathrm{FW}(h)$ (the symbol $:=$ means that the left side by the right one).

$$\mathrm{FW}(h) f(x) := f(x+h); \tag{1}$$

then the fractional difference of order α, $0 < \alpha \le 1$, of $f(x)$ is defined by the expression

$$\begin{aligned}\Delta^{\alpha} f(x) &= (\mathrm{FW}-1)^{\alpha} f(x) \\ &= \sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} f[x+(\alpha-n)h],\end{aligned} \tag{2}$$

and its fractional derivative of order α is

$$f^{(\alpha)}(x) = \lim_{h\to 0}\left(\frac{\sum_{n=0}^{\infty}(-1)^n \binom{\alpha}{n} f[x+(\alpha-n)h]}{h^{\alpha}}\right). \tag{3}$$

The above can be expressed as

$$\begin{aligned}&\frac{1}{\Gamma(-\alpha)}\int_0^x (x-\xi)^{-\alpha-1} f(\xi)\,d\xi, \quad \alpha < 0, \\ &\frac{1}{\Gamma(1-\alpha)}\frac{d}{d_x}\int_0^x (x-\xi)^{-\alpha}[f(\xi)-f(0)]\,d\xi, \\ &\qquad 0 < \alpha < 1, \\ &\left(f^{(\alpha-n)}(x)\right)^{(n)}, \quad n \le \alpha \le n+1,\ n \ge 1.\end{aligned} \tag{4}$$

An important property and formula of Jumarie's modified Riemann-Liouville derivative [31] can be cataloged [32] as

$$D_x^{\alpha} x^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{(\beta-\alpha)}, \quad \beta > 0. \tag{5}$$

3. An Analysis of the Generalized $exp(-\Phi(\xi))$-Expansion Method

In this section, Let us consider the following nonlinear FDEs:

$$\begin{aligned}&f\left(u, u_t, u_x, D_t^{\alpha}u, D_{\alpha}^{\alpha}u, D_t^{\beta}u, D_x^{\beta}u, \ldots\right) = 0, \\ &\qquad 0 < \alpha \le 1,\end{aligned} \tag{6}$$

where $u = u(x,t)$ is an unknown function and $D_t^{\alpha}u$, $D_{\alpha}^{\alpha}u, D_t^{\beta}u, D_x^{\beta}u$ are Jumarie's modified Riemann-Liouville derivatives of u and f is a function involving nonlinear terms and higher order derivatives. Then we propose the following four steps of the generalized $\exp(-\Phi(\xi))$-expansion method for seeking the solutions of nonlinear FDEs.

Step 1. Applying the traveling wave transformation in (6)

$$u(x,t) = u(\xi), \quad \xi = \frac{kx^{\beta}}{\Gamma(1+\beta)} + \frac{vt^{\alpha}}{\Gamma(1+\alpha)}, \tag{7}$$

where k and v are constants. Equation (6) is reduced to a nonlinear ODE for $u = u(\xi)$ with the help of (7)

$$F\left(u, u', u'', \ldots\right) = 0, \tag{8}$$

where F is a function of $u = u(\xi)$ and its derivatives with respect to ξ.

Step 2. Assuming (8) has the following traveling wave solution,

$$u(\xi) = \sum_{i=0}^{M} a_i \left(e^{-\Phi(\xi)}\right)^i, \quad a_M \ne 0, \tag{9}$$

where a_i $(1 \le i \le M)$ are coefficients to be determined later, and $\Phi = \Phi(\xi)$ satisfies the following nonlinear ODE:

$$\Phi'(\xi) = p\exp(-\Phi(\xi)) + q\exp(\Phi(\xi)) + r. \tag{10}$$

Step 3. Balancing the higher order derivative term and the higher order nonlinear term of (8), the value of the positive integer M is consequently obtained. The detailed expression is the same as the following formula: assuming $D[u(\xi)] = n$,

$$\begin{aligned}&D\left[\frac{d^M u(\xi)}{d\xi^M}\right] = M + n, \\ &D\left[\left(u^M \frac{d^K u(\xi)}{d\xi^K}\right)^S\right] = nM + S(n+K).\end{aligned} \tag{11}$$

Step 4. Substituting (9) and (10) into (8), we will obtain a function of $\exp(-\Phi(\xi))^i$; the parameters a_i $(1 \le i \le M)$, k, v, p, q, and r can be determined. The general solutions of (9) have been listed as the following.

Type 1 (when $p = 1$).

$$\Phi(\xi) = \ln\left(-\frac{\sqrt{r^2-4q}\tanh\left[0.5\sqrt{r^2-4q}(\xi+\xi_0)\right]}{2q} - \frac{r}{2q}\right), \quad q \ne 0,\ r^2-4q > 0,$$

$$\Phi(\xi) = \ln\left(\frac{\sqrt{4q-r^2}\tan\left[0.5\sqrt{4q-r^2}(\xi+\xi_0)\right]}{2q} - \frac{r}{2q}\right), \quad q \ne 0,\ r^2-4q < 0,$$

$$\Phi(\xi) = -\ln\left(\frac{r}{\exp(r(\xi+\xi_0))-1}\right), \quad q=0,\ r\neq 0,\ r^2-4q>0,$$
$$\Phi(\xi) = \ln\left(-\frac{2(r(\xi+\xi_0)+2)}{r^2(\xi+\xi_0)}\right), \quad q\neq 0,\ r\neq 0,\ r^2-4q=0. \tag{12}$$

Type 2 (when $r = 0$).

$$\Phi(\xi) = \ln\left(\sqrt{\frac{p}{q}}\tan\left[\sqrt{pq}(\xi+\xi_0)\right]\right), \quad p>0,\ q>0,$$
$$\Phi(\xi) = \ln\left(-\sqrt{\frac{p}{q}}\tan\left[\sqrt{pq}(\xi-\xi_0)\right]\right), \quad p<0,\ q<0,$$
$$\Phi(\xi) = \ln\left(\sqrt{\frac{-p}{q}}\tanh\left[\sqrt{-pq}(\xi-\xi_0)\right]\right), \quad p>0,\ q<0,$$
$$\Phi(\xi) = \ln\left(-\sqrt{\frac{-p}{q}}\tanh\left[\sqrt{-pq}(\xi+\xi_0)\right]\right), \quad p<0,\ q>0. \tag{13}$$

Type 3 (when $q = 0$ and $r = 0$).

$$\Phi(\xi) = \ln\left[p(\xi+\xi_0)\right], \tag{14}$$

where ξ_0 is the integrating constant.

4. The Application to the Time Fractional Generalized Fifth-Order KdV Equation

The following is a given space-time fractional generalized fifth-order KdV equation as

$$D_t^\alpha + uu_x - uu_{xxx} + u_{xxxxx} = 0. \tag{15}$$

Utilize the traveling wave transformation (7) of Step 2:

$$u(x,t) = u(\xi), \quad \xi = k\left(x + \frac{ct^\alpha}{\Gamma(1+\alpha)}\right), \tag{16}$$

where k and v are constants. Then (15) can be reduced to following nonlinear ODE:

$$kcu' + kuu' - k^3uu''' + k^5u^{(5)} = 0. \tag{17}$$

Integrating once with the respect ξ and letting the integration constant equal to zero [21], then we get (15) for simplicity,

$$cu + \frac{1}{2}u^2 + k^2uu'' - \frac{k^2}{2}(u')^2 + k^4u^{(4)} = 0. \tag{18}$$

Owing to the balancing principle, let $M = 2$; (9) can be written as

$$u(\xi) = a_0 + a_1e^{-\Phi(\xi)} + a_2e^{-2\Phi(\xi)}, \tag{19}$$

where a_0, a_1, and a_2 are constants to be determined later and $\Phi(\xi)$ satisfies (10). Substituting (19) into (18) and setting the coefficients of $(\exp(-\Phi(\xi)))^i (i = 0,1,2,3,4,5,6)$ to zero, we get a system of algebraic equation. Solving (18), we get

$$\begin{aligned} a_0 &= (5/2)(1+8k^2pq+k^2r^2),\\ a_1 &= 30k^2pr,\\ a_2 &= 30k^2p^2,\\ k &= k,\\ c &= (1/2)(-5+48k^4p^2q^2-24k^4pqr^2+3k^4r^4). \end{aligned}$$

Case 1 ($p = 1$).

$$u_1 = \frac{5}{2}\left(1+8k^2q+k^2r^2\right) - \frac{60k^2rq}{r+\sqrt{r^2-4q}\tanh\left[0.5\sqrt{r^2-4q}(\xi+\xi_0)\right]} + \frac{120k^2q^2}{\left(r+\sqrt{r^2-4q}\tanh\left[0.5\sqrt{r^2-4q}(\xi+\xi_0)\right]\right)^2}, \quad q\neq 0,\ r^2-4q>0,$$

$$u_2 = \frac{5}{2}\left(1+8k^2q+k^2r^2\right) + \frac{60k^2rq}{-r+\sqrt{r^2-4q}\tanh\left[0.5\sqrt{r^2-4q}(\xi+\xi_0)\right]} + \frac{120k^2q^2}{\left(-r+\sqrt{r^2-4q}\tanh\left[0.5\sqrt{r^2-4q}(\xi+\xi_0)\right]\right)^2}, \quad q\neq 0,\ r^2-4q=0,$$

$$u_3 = \frac{5}{2}\left(1+8k^2q+k^2r^2\right) + \frac{30k^2r^2}{e^{r(\xi+\xi_0)}-1} + \frac{30k^2r^2}{\left(e^{r(\xi+\xi_0)}-1\right)^2}, \quad r\neq 0,\ r^2-4q>0,$$

$$u_4 = \frac{5}{2}\left(1+8k^2q+k^2r^2\right) - \frac{30k^2r^2}{2(r(\xi+\xi_0)+2)} + \frac{30k^2r^2}{4(r(\xi+\xi_0)+2)^2}, \quad q\neq 0,\ r\neq 0,\ r^2-4q=0, \tag{20}$$

where $\xi = k(x + (48k^4q^2 - 24k^4qr^2 + 3k^4r^4 - 5)t^\alpha/2\Gamma(1+\alpha))$.

Figure 1(a) shows the dark solitary wave solutions of u_1 at $k = 0.5$, $q = 1$, $r = 3$, and $\alpha = 1$ in the interval $[-5, 5]$ and time in the interval $[0, 1]$.

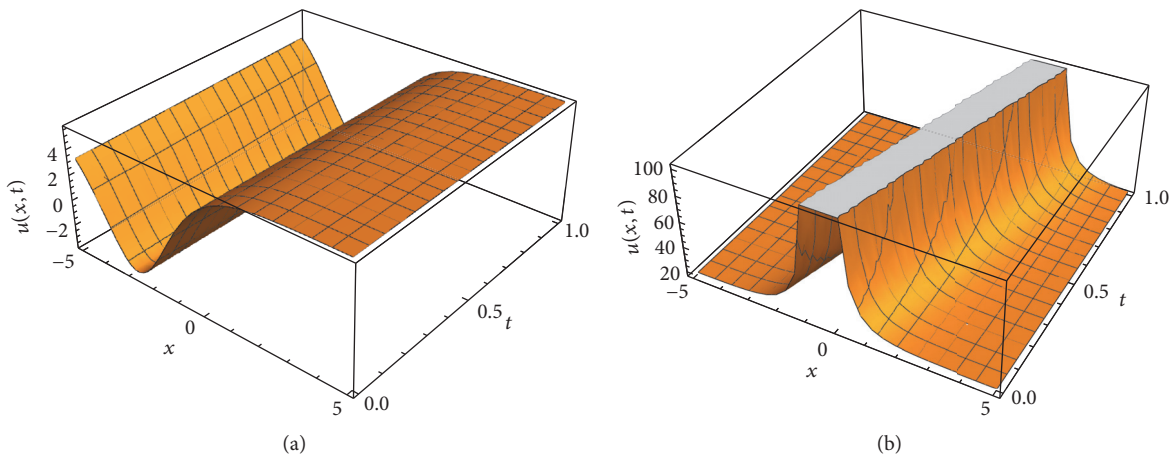

Figure 1: Exact traveling wave solutions of (20) are plotted in different shape.

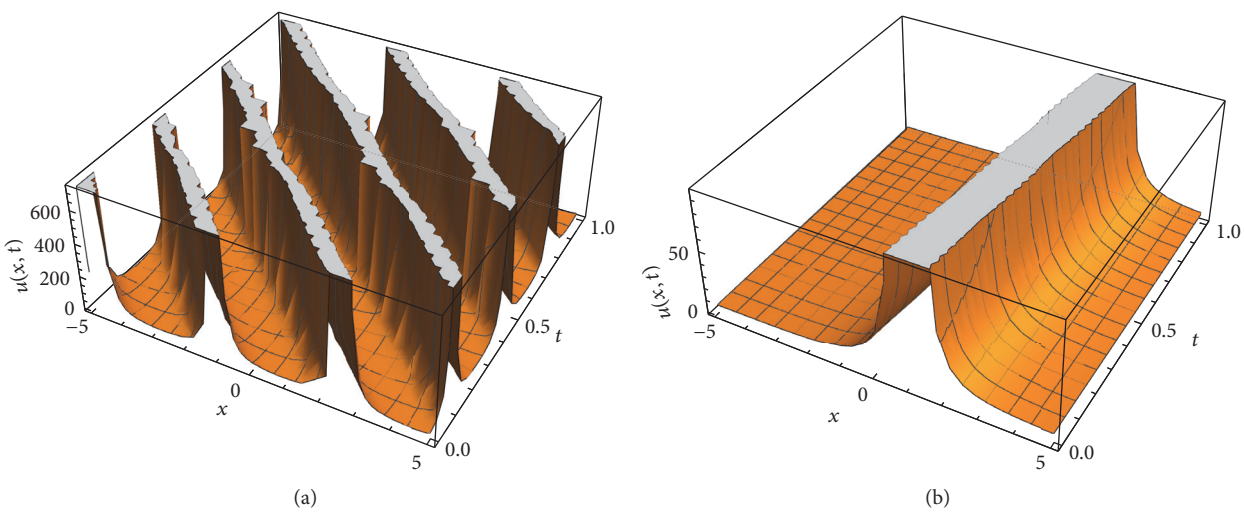

Figure 2: Exact traveling wave solutions of (21) in different shapes are drawn.

Figure 1(b) shows the solitary wave solutions of u_3 at $k = 1$, $q = 0.5$, $r = 1$, and $\alpha = 1$ in the interval $[-5, 5]$ and time in the interval $[0, 1]$.

Case 2 ($r = 0$).

$$u_5 = \frac{5}{2}\left(1 + 8k^2 pq\right) + \frac{30k^2 pq}{\tan^2\left(\sqrt{pq}\left(\xi + \xi_0\right)\right)}, \quad p > 0,\ q > 0,$$

$$u_6 = \frac{5}{2}\left(1 + 8k^2 pq\right) + \frac{30k^2 pq}{\tan^2\left(\sqrt{pq}\left(\xi - \xi_0\right)\right)}, \quad p < 0,\ q < 0,$$

$$u_7 = \frac{5}{2}\left(1 + 8k^2 pq\right) + \frac{30k^2 pq}{-\tanh^2\left(\sqrt{-pq}\left(\xi - \xi_0\right)\right)}, \quad p > 0,\ q < 0,$$

$$u_8 = \frac{5}{2}\left(1 + 8k^2 pq\right) + \frac{30k^2 pq}{-\tanh^2\left(\sqrt{-pq}\left(\xi + \xi_0\right)\right)}, \quad p < 0,\ q > 0, \tag{21}$$

where $\xi = k(x + (-5 + 48k^4 p^2 q^2)t^\alpha / 2\Gamma(1 + \alpha))$.

Figure 2(a) shows the periodic solitary wave solutions of u_5 at $k = 0.5$, $q = 1$, and $\alpha = 0.5$, in the interval $[-10, 10]$ and time in the interval $[0, 1]$.

Figure 2(b) shows the solitary wave solutions of u_7 at $k = 1$, $q = 1$, and $\alpha = 1$ in the interval $[-5, 5]$ and time in the interval $[-2, 2]$.

Case 3 ($q = 0$, $r = 0$).

$$u_9 = \frac{5}{2} + \frac{30k^2}{\left(\xi + \xi_0\right)^2}, \tag{22}$$

where ξ_0 is the integrating constant and $\xi = k(x - 5t^\alpha / 2\Gamma(1 + \alpha))$.

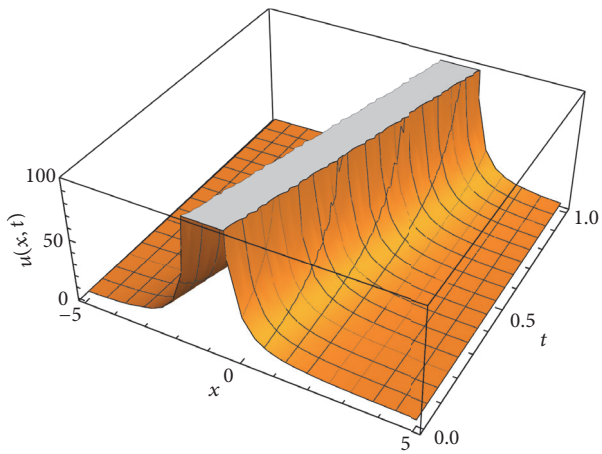

Figure 3: Exact traveling wave solutions of (22).

Figure 3 shows the solitary wave solutions of u_9 at $k = 0.5$, and $\alpha = 1$, in the interval $[-5, 5]$ and time in the interval $[0, 1]$.

5. Conclusion

In this article, we obtain the exact traveling wave solutions of space-time fractional generalized fifth-order KdV equation by utilizing the generalized $\exp(-\Phi(\xi))$-expansion method along with Jumarie's modified Riemann-Liouville derivatives. Three types of exact solutions are originated in terms of the hyperbolic, trigonometric, and rational function with some parameters and they have great potential which can been further researched. The generalized $\exp(-\Phi(\xi))$-expansion method has some advantages, such as efficiency, conciseness, and briefness. In addition, it is obvious that this method is rather well-organized and practically applied to many other nonlinear fractional PDEs from natural sciences.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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