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Research Article

Remarks on Pressure Blow-Up Criterion of the 3D Zero-Diffusion Boussinesq Equations in Margin Besov Spaces

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This study is focused on the pressure blow-up criterion for a smooth solution of three-dimensional zero-diffusion Boussinesq equations. With the aid of Littlewood-Paley decomposition together with the energy methods, it is proved that if the pressure satisfies the following condition on margin Besov spaces, $\pi(x,t) \in L^{2/(2+r)}(0,T;\dot{B}^{r}_{\infty,\infty})$ for $r = \pm 1$, then the smooth solution can be continually extended to the interval $(0,T^*)$ for some $T^* > T$. The findings extend largely the previous results.

1. Introduction and Main Results

It is well known that mathematical models in fluid dynamics have attracted more and more attention in the past ten years [1]. In this paper, we consider the dynamical models of the ocean or the atmosphere which arise from the density dependent incompressible Navier-Stokes equations by using the so-called Boussinesq approximation [2]. The so-called Boussinesq system is governed by the following nonlinear partial differential equations:

$$\begin{aligned} \partial_t u - \mu\Delta u + (u\cdot\nabla)u + \nabla\pi &= \theta e_3, \\ \partial_t \theta - \kappa\Delta\theta + (u\cdot\nabla)\theta &= 0, \\ \nabla\cdot u &= 0, \end{aligned} \tag{1}$$

associated with the initial conditions

$$\begin{aligned} u(x,0) &= u_0, \\ \theta(x,0) &= \theta_0. \end{aligned} \tag{2}$$

Here, $u(x,t)$, $\theta(x,t)$, and $\pi(x,t)$ represent the unknown velocity vector field, temperature field, and the unknown pressure scalar field, respectively. The constant $\mu > 0$ is the kinematic viscosity and the constant $\kappa \geq 0$ is the thermal diffusivity. The quantities $u_0(x)$ and $\theta_0(x)$ are the given initial velocity and initial temperature, respectively.

As an important mathematical model in many geophysical applications, the Boussinesq system attracted more and more attention in the past ten years [3–6]. Moreover, when the temperature field $\theta = 0$, it reduces to the classic Navier-Stokes equations (see [7])

$$\begin{aligned} \partial_t u - \mu\Delta u + (u\cdot\nabla)u + \nabla\pi &= 0, \\ \nabla\cdot u &= 0. \end{aligned} \tag{3}$$

In the three-dimensional case, the same as the classic Navier-Stokes equations, the issue on the global smooth solution with large initial data is still a challenging open problem. The development of blow-up criteria is of importance for both theoretical and practical situations [8–10]. When $\nu, \kappa > 0$, the first blow-up criterion of 3D Boussinesq equations in Lebesgue space was considered by Ishimura and Morimoto [11]; they proved that if the velocity satisfies

$$\nabla u \in L^1\left(0,T;L^\infty\left(\mathbf{R}^3\right)\right), \tag{4}$$

then the smooth solution can be continually extended to the interval $(0,T^*)$ for some $T^* > T$. For the zero-viscosity case,

that is, $\mu = 0$, $\kappa > 0$, Fan and Zhou [12] studied the blow-up criterion:

$$\nabla \times u \in L^1\left(0, T; \dot{B}^0_{\infty,\infty}\left(\mathbf{R}^3\right)\right). \tag{5}$$

In the zero-diffusion case, that is, $\mu > 0$, $\kappa = 0$, the situation becomes more difficult. The main obstacle is that the temperature function $\theta(x,t)$ in the transport equation does not gain any smoothness. Hence, the blow-up issue of the zero-diffusive Boussinesq equations (1) with $\kappa = 0$ is more difficult than that of full viscous Boussinesq system (1). Jia et al. [13] recently studied the blow-up criterion for local smooth solutions of zero-diffusive Boussinesq equations (1) in the large critical Besov space

$$\int_0^T \|u\|^p_{B^s_{q,\infty}(\mathbf{R}^3)}\, ds < \infty \tag{6}$$

with

$$\begin{gathered} \frac{2}{p} + \frac{3}{q} = 1 + s, \\ \frac{3}{1+s} < p \le \infty, \quad -1 < s \le 1, \ (p,s) \ne (\infty, 1). \end{gathered} \tag{7}$$

Recently, Wang [14] also proved the blow-up criterion for the zero-diffusive Boussinesq equation when the velocity components satisfy

$$\begin{gathered} \int_0^T \frac{\left\| |u_1| + |u_2| \right\|^q_{L^{p,\infty}}}{1 + \ln\left(e + \|\nabla u\|^2_{L^2}\right)} ds < \infty, \\ \frac{2}{q} + \frac{3}{p} = 1, \ 3 < p < \infty. \end{gathered} \tag{8}$$

For the pressure blow-up criterion of the zero-diffusive Boussinesq equations (1), Dong et al. [15] investigated the pressure regular criterion when

$$\pi(x,t) \in L^{2/(2+r)}\left(0, T; \dot{B}^r_{\infty,\infty}\left(\mathbf{R}^3\right)\right) \quad \text{for } -1 < r < 1. \tag{9}$$

However, the methods in [15] are not available for the margin case $r = 1$ or $r = -1$. One may also refer to some important regularity criteria on the fluid dynamics [16, 17].

The aim of the present paper is to improve the pressure blow-up criterion for smooth solution of three-dimensional zero-diffusion Boussinesq equations in the margin Besov spaces $r = \pm 1$ in (9); more precisely, we will prove the following result.

Theorem 1. *Suppose $\mu > 0$, $\kappa = 0$. Let $T > 0$; (u,θ) is a smooth solution of zero-diffusion Boussinesq equations (1) with $(u_0, \theta_0) \in H^m(\mathbf{R}^3)$, $m > 3/2$. If the pressure π satisfies*

$$\pi(x,t) \in L^{2/(2+r)}\left(0, T; \dot{B}^r_{\infty,\infty}\left(\mathbf{R}^3\right)\right) \quad \textit{for } r = \pm 1, \tag{10}$$

then the smooth solution can be continually extended to the interval $(0, T^*)$ for some $T^* > T$.

Theorem 1 also implies the following corollary.

Corollary 2. *Suppose (u,θ) is the smooth solution of zero-diffusion Boussinesq equations satisfying*

$$\begin{gathered} u \in C\left([0,T_1); H^m\left(\mathbf{R}^3\right)\right) \cap L^2\left(0, T_1; H^{m+1}\left(\mathbf{R}^3\right)\right), \\ \theta \in C\left([0,T_1); H^m\left(\mathbf{R}^3\right)\right). \end{gathered} \tag{11}$$

If T is the maximal existence time of the smooth solution, then

$$\begin{gathered} T < \infty \Longrightarrow \\ \int_0^T \|\pi\|^{2/(2+r)}_{\dot{B}^r_{\infty,\infty}}\, ds = +\infty, \end{gathered} \tag{12}$$

where $r = \pm 1$.

Remark 3. It should be mentioned that since our work spaces here are margin cases in Besov spaces, the methods used by Dong et al. [15] where the finding is mainly based on the function spaces decomposition cannot be available any more. Furthermore, compared with the many previous results on the pressure regularity criterion for full viscous fluid dynamical models such as Navier-Stokes equations and MHD equations (see [18]), the zero-diffusion Boussinesq equations (1) do not have the important inequality

$$\|\pi\|_{L^p} \le C \|u\|^2_{L^{2p}}, \quad 1 < p < \infty, \tag{13}$$

due to the appearance of θe_3. In order to overcome the difficulty of the absence of the above estimate, we will give some more explicit estimates on the gradient of the pressure in this paper. Additionally, our results here more or less extended the previous results by Gala et al. [19, 20] and others [14]. Since it is also interesting and important to consider this issue in some working space such as Morrey Space (see [17, 21, 22]), we will focus on this problem in the forthcoming paper.

2. Preliminary

In this section, C stands for a generic positive constant which may vary from line to line. $L^p(\mathbf{R}^3)$ with $1 \le p \le \infty$ denotes the usual Lebesgue space and $H^s(\mathbf{R}^3)$ with $s \in \mathbf{R}$ is the inhomogeneous fractional Sobolev space with the norm

$$\|f\|_{H^s} = \left(\int_{\mathbf{R}^3} \left|(I - \Delta)^{s/2} f\right|^2 d\xi\right)^{1/2}. \tag{14}$$

We now provide the Littlewood-Paley decomposition (see [23]). Let $\mathcal{S}(\mathbf{R}^3)$ be the Schwartz class of a rapidly decreasing function. Given $f \in \mathcal{S}(\mathbf{R}^3)$, we define the Fourier transformation $\mathcal{F} f = \widehat{f}$ as

$$\widehat{f}(\xi) = \int_{\mathbf{R}^3} e^{-ix\cdot\xi} f(x)\, dx. \tag{15}$$

Take two nonnegative radial functions $\chi, \psi \in \mathcal{S}(\mathbf{R}^3)$ supported, respectively, in $\mathcal{B} = \{\xi \in \mathbf{R}^3 : |\xi| \le 4/3\}$ and $\mathcal{C} = \{\xi \in \mathbf{R}^3 : 3/4 \le |\xi| \le 8/3\}$, such that

$$\sum_{j \in Z} \psi\left(2^{-j}\xi\right) = 1, \quad \xi \in \mathbf{R}^3 \setminus \{0\}. \tag{16}$$

Let $h = \mathcal{F}^{-1}\psi$ and $\tilde{h} = \mathcal{F}^{-1}\chi$. The frequency localization operator is defined by

$$\begin{aligned} \Delta_j f &= \psi\left(2^{-j}D\right) f = 2^{3j}\int_{\mathbf{R}^3} h\left(2^j y\right) f(x-y)\,dy, \\ &\qquad\qquad\qquad\qquad\qquad\qquad j \geq 0, \\ S_j f &= \chi\left(2^{-j}D\right) f = \sum_{-1\leq k\leq j-1} \Delta_k f \\ &= 2^{3j}\int_{\mathbf{R}^3} \tilde{h}\left(2^j y\right) f(x-y)\,dy, \\ &\qquad \Delta_1 f = S_0 f,\ \Delta_j f = 0 \text{ for } j \leq -2. \end{aligned} \tag{17}$$

Formally, Δ_j is a frequency projection to the annulus $\{|\xi| \approx 2^j\}$, and S_j is a frequency projection to the ball $\{|\xi| \lesssim 2^j\}$. The aforementioned dyadic decomposition has nice quasi-orthogonality, with the choice of χ and ψ; namely, given any $\chi, \psi \in \mathcal{S}(\mathbf{R}^3)$, we have the following properties:

$$\begin{aligned} \Delta_i \Delta_j f &= 0, \quad \text{if } |i-j| \geq 2, \\ \Delta_i\left(S_{j-1}\Delta_j g\right) &= 0, \quad \text{if } |i-j| \geq 5. \end{aligned} \tag{18}$$

With the full-dyadic decomposition, we define the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbf{R}^3)$:

$$\dot{B}^s_{p,q}\left(\mathbf{R}^3\right) = \left\{ f \in \frac{\mathcal{S}'\left(\mathbf{R}^3\right)}{\mathcal{P}\left(\mathbf{R}^3\right)} : \|f\|_{\dot{B}^s_{p,q}} < \infty \right\}, \tag{19}$$

where

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} \left(\displaystyle\sum_{j=-\infty}^{\infty} 2^{jsq}\left\|\Delta_j f\right\|_{L_p}^q\right)^{1/q}, & 1 \leq q < \infty, \\ \displaystyle\sup_{j\in Z} 2^{js}\left\|\Delta_j f\right\|_{L_p}, & q = \infty. \end{cases} \tag{20}$$

The set $\mathcal{S}'(\mathbf{R}^3)$ of tempered distributions is the dual set and $\mathcal{P}(\mathbf{R}^3)$ is the polynomials space.

In Lemma 4, we recall the Bernstein inequalities which will be applied frequently.

Lemma 4 (Chemin [24]). *Suppose $k, j \in Z$ and $1 \leq p \leq q \leq \infty$; one has, for all $f \in \mathcal{S}(\mathbf{R}^3)$,*

$$\sup_{|\alpha|=k}\left\|\nabla^\alpha \Delta_j f\right\|_{L_q} \leq C 2^{jk+3j(1/p-1/q)}\left\|\Delta_j f\right\|_{L_p}, \tag{21}$$

where C are the positive constants independent of j, k.

Lemma 5 (Meyer [25]). *For any function f belonging to the homogeneous Sobolev space $\dot{H}^1(\mathbf{R}^3)$, one has*

$$\|f\|_{L^4}^2 \leq C\|f\|_B\|\nabla f\|_{L^2}, \tag{22}$$

where B is the homogeneous Besov space $\dot{B}^{-1}_{\infty,\infty}(\mathbf{R}^3)$.

Lemma 6 (Kato-Ponce [26]). *Suppose $f, g \in W^{k,p}(\mathbf{R}^3)$, $1 < p < \infty$, for $1 \leq s \leq k$; then,*

$$\begin{aligned} \left\|J^s(fg) - fJ^s g\right\|_{L^p} &\leq C\|\nabla f\|_{L^q}\left\|J^{s-1}g\right\|_{L^r} \\ &\quad + C\|g\|_{L^q}\left\|J^s f\right\|_{L^r}, \end{aligned} \tag{23}$$

where $J^s = (I-\Delta)^{s/2}$, $1/p = 1/q + 1/r$, and $1 < q < \infty$, $1 < r < \infty$.

3. Proof of Theorem 1

Step 1 (energy estimate). Multiplying both sides of the second equation of zero-diffusion Boussinesq equations (1) by $|\theta|^{p-2}\theta$ and integrating in $\mathbf{R}^3$ give

$$\frac{1}{p}\frac{d}{dt}\|\theta(t)\|_{L^p}^p = 0. \tag{24}$$

Integrating in time,

$$\|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p}, \tag{25}$$

where we have used $\int_{\mathbf{R}^3}(u\cdot\nabla)\theta\,\theta dx = 0$.

Taking the inner product to the first equation of (1) by u and applying the integration by parts yield

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \mu\|\nabla u\|_{L^2}^2 &\leq \|\theta\|_{L^2}\|u\|_{L^2} \\ &\leq \|\theta_0\|_{L^2}\|u\|_{L^2}; \end{aligned} \tag{26}$$

therefore, it follows from Gronwall inequality that

$$\sup_{0<t<T}\|u(t)\|_{L^2}^2 + \mu\int_0^T\|\nabla u\|_{L^2}^2\,dt \leq C. \tag{27}$$

Since the pressure here plays an important role in our argument, we apply the operator $\nabla\mathrm{div}$ to the first equation of (1) to get

$$\begin{aligned} &\nabla\pi \\ &\quad = (-\Delta)^{-1}\sum_{i,j=1}^{3}\frac{\partial^2}{\partial x_i\partial x_j}\left(\nabla\left(u_i u_j\right) - (-\Delta)^{-1}\left(\nabla\partial_3\theta_3\right)\right), \end{aligned} \tag{28}$$

and then, according to Calderon-Zygmund inequality and (27), we obtain

$$\begin{aligned} \|\nabla\pi\|_{L^p} &\leq C\||u|\nabla u\|_{L^p} + C\|\theta\|_{L^p} \leq C\||u|\nabla u\|_{L^p} + C, \\ &\qquad\qquad\qquad\qquad 1 < p < \infty. \end{aligned} \tag{29}$$

Step 2 (L^4 estimate of u). Taking the inner product to the first equation, the second equation of (1) by $|u|^2u$ and $|\theta|^2\theta$, respectively, employing Hölder inequality and Young inequality, we have

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\left(\|u\|_{L^4}^4+\|\theta\|_{L^4}^4\right)+\mu\,\||u|\,\nabla u\|_{L^2}^2\\
&\quad\le\int_{\mathbf{R}^3}\theta e_3\cdot u\,|u|^2\,dx-\int_{\mathbf{R}^3}u\cdot\nabla\pi\,|u|^2\,dx\\
&\quad\le C\,\|\theta\|_{L^4}\,\|u\|_{L^4}^3-\int_{\mathbf{R}^3}u\cdot\nabla\pi\,|u|^2\,dx\\
&\quad\le C\left(\|u\|_{L^4}^4+\|\theta\|_{L^4}^4\right)-\int_{\mathbf{R}^3}u\cdot\nabla\pi\,|u|^2\,dx\\
&\quad=C\left(\|u\|_{L^4}^4+\|\theta\|_{L^4}^4\right)+I,
\end{aligned}\tag{30}$$

where we have used

$$\begin{aligned}
&\int_{\mathbf{R}^3}(u\cdot\nabla u)\cdot|u|^2\,u\,dx=0,\\
&\int_{\mathbf{R}^3}(u\cdot\nabla\theta)\cdot|\theta|^2\,\theta\,dx=0.
\end{aligned}\tag{31}$$

Case 1 ($r=1$). That is,

$$\begin{aligned}
&\int_0^T\|\pi(t)\|_{\dot{B}^1_{\infty,\infty}}^{2/3}\,dt<\infty,\\
&I\le\left|\int_{\mathbf{R}^3}u\cdot\nabla\pi\,|u|^2\,dx\right|\\
&\quad=\left|\int_{\mathbf{R}^3}\left(\sum_{j<-N}+\sum_{j=-N}^{N}+\sum_{j>N}\right)\Delta_j(\nabla\pi)\cdot|u|^2\,u\,dx\right|\\
&\quad\le\left|\int_{\mathbf{R}^3}\sum_{j<-N}\Delta_j(\nabla\pi)\cdot|u|^2\,u\,dx\right|\\
&\qquad+\left|\int_{\mathbf{R}^3}\sum_{j=-N}^{N}\Delta_j(\nabla\pi)\cdot|u|^2\,u\,dx\right|\\
&\qquad+\left|\int_{\mathbf{R}^3}\sum_{j>N}\Delta_j(\nabla\pi)\cdot|u|^2\,u\,dx\right|=I_1+I_2+I_3
\end{aligned}\tag{32}$$

for an arbitrary positive integer N.

In order to estimate the term I_1, we apply Hölder inequality, Lemma 5, and Young inequality:

$$\begin{aligned}
I_1&\le\sum_{j<-N}\left\|\Delta_j\nabla\pi\right\|_{L^4}\|u\|_{L^4}^3\\
&\le\sum_{j<-N}2^{3j(1/2-1/4)}\left\|\Delta_j\nabla\pi\right\|_{L^2}\|u\|_{L^4}^3\\
&\le\left(\sum_{j<-N}2^{(3/2)j}\right)^{1/2}\sum_{j<-N}\left(\left\|\Delta_j\nabla\pi\right\|_{L^2}^2\right)^{1/2}\|u\|_{L^4}^3\\
&\le 2^{-3N/4}\,\|\nabla\pi\|_{L^2}\,\|u\|_{L^4}^3\\
&\le 2^{-3N/4}\left(\||u|\,\nabla u\|_{L^2}+C\right)\|u\|_{L^4}^3\\
&\le C\left(2^{-N/4}\,\|u\|_{L^4}\right)^3\||u|\,\nabla u\|_{L^2}+C\left(2^{-N/4}\,\|u\|_{L^4}\right)^3.
\end{aligned}\tag{33}$$

In order to estimate the term I_2, we apply Hölder inequality and Young inequality:

$$\begin{aligned}
I_2&\le\sum_{j=-N}^{N}\left\|\Delta_j\nabla\pi\right\|_{L^4}\|u\|_{L^4}^3\\
&\le N^{3/4}\,\|\nabla\pi\|_{B^0_{\infty,\infty}}^{1/2}\,\|\nabla\pi\|_{L^2}^{1/2}\,\|u\|_{L^4}^3\\
&\le C\left(N\,\|\nabla\pi\|_{B^0_{\infty,\infty}}^{2/3}\right)^{3/4}\left(\||u|\,\nabla u\|_{L^2}^{1/2}+C\right)\|u\|_{L^4}^3\\
&\le C\left(N\,\|\nabla\pi\|_{B^0_{\infty,\infty}}^{2/3}\right)\|u\|_{L^4}^4+\frac{\mu}{8}\,\||u|\,\nabla u\|_{L^2}^2+C.
\end{aligned}\tag{34}$$

For I_3, we apply Hölder inequality, Lemma 5, and (29):

$$\begin{aligned}
I_3&\le\sum_{j>N}\left\|\Delta_j\nabla\pi\right\|_{L^4}\|u\|_{L^4}^3\le 2^{-3N/4}\,\|\nabla\pi\|_{L^2}\,\|u\|_{L^4}^3\\
&\le C\left(2^{-N/4}\,\|u\|_{L^4}\right)^3\left(\||u|\,\nabla u\|_{L^2}+C\right)\\
&\le C\left(2^{-N/4}\,\|u\|_{L^4}\right)^3\||u|\,\nabla u\|_{L^2}+C\left(2^{-N/4}\,\|u\|_{L^4}\right)^3.
\end{aligned}\tag{35}$$

We only need to choose the integer N satisfying

$$C\left(2^{-N/4}\,\|u\|_{L^4}\right)^3\le\frac{\mu}{8}.\tag{36}$$

For example, we may set

$$N=\left[\frac{(1/3)\ln\left(\mu/8C\right)+\ln\left(\|u\|_{L^4}+e\right)}{\ln 2}\right]+1.\tag{37}$$

Combining I_1, I_2, and I_3 and the above inequality yields

$$\begin{aligned}
&\frac{d}{dt}\left(\|u\|_{L^4}^4+\|\theta\|_{L^4}^4\right)+\mu\,\||u|\,\nabla u\|_{L^2}^2\\
&\quad\le C\left(\|u\|_{L^4}^4+\|\theta\|_{L^4}^4\right)+CN\,\|\nabla\pi\|_{B^0_{\infty,\infty}}^{2/3}\,\|u\|_{L^4}^4+C,
\end{aligned}\tag{38}$$

and the above inequality reduces to

$$\begin{aligned}\frac{d}{dt}\left(\|u\|_{L^4}^4+\|\theta\|_{L^4}^4\right)&\le C\left(\|u\|_{L^4}^4+\|\theta\|_{L^4}^4\right)\\&+CN\|\nabla\pi\|_{B^0_{\infty,\infty}}^{2/3}\|u\|_{L^4}^4+C=\left\{C\right.\\&\left.+C\left(\frac{\ln C+\ln\left(\|u\|_{L^4}^4+e\right)}{\ln 2}+1\right)\|\nabla\pi\|_{B^0_{\infty,\infty}}^{2/3}\right\}\\&\cdot\left(\|u\|_{L^4}^4+\|\theta\|_{L^4}^4\right)+C.\end{aligned}\tag{39}$$

Taking Gronwall inequality into consideration,

$$\begin{aligned}\|u\|_{L^4}^4+\|\theta\|_{L^4}^4&\le\left(\|u_0\|_{L^4}^4+\|\theta_0\|_{L^4}^4+CT\right)\exp\left\{CT\right.\\&+\int_0^T C\left(\frac{\ln C+\ln\left(\|u\|_{L^4}^4+e\right)}{\ln 2}+1\right)\\&\left.\cdot\|\nabla\pi\|_{B^0_{\infty,\infty}}^{2/3}\,d\tau\right\}\le\left(\|u_0\|_{L^4}^4+\|\theta_0\|_{L^4}^4+CT\right)\\&\cdot\exp\left\{CT+\int_0^T C\left[\ln\left(\|u\|_{L^4}^4+e\right)+C\right]\right.\\&\left.\cdot\|\nabla\pi\|_{B^0_{\infty,\infty}}^{2/3}\,d\tau\right\},\end{aligned}\tag{40}$$

for a suitable constant C.

Thus, we also have

$$\begin{aligned}&\ln\left(\|u\|_{L^4}^4+\|\theta\|_{L^4}^4+e\right)\\&\quad\le\ln\left(\|u_0\|_{L^4}^4+\|\theta_0\|_{L^4}^4+CT\right)+CT\\&\qquad+\int_0^T C\left[\ln\left(\|u\|_{L^4}^4+e\right)+C\right]\|\nabla\pi\|_{B^0_{\infty,\infty}}^{2/3}\,d\tau.\end{aligned}\tag{41}$$

Employing Gronwall inequality again, one shows that

$$\begin{aligned}&\ln\left(\|u\|_{L^4}^4+\|\theta\|_{L^4}^4+e\right)\\&\quad\le\left(\ln\left(\|u_0\|_{L^4}^4+\|\theta_o\|_{L^4}^4+CT\right)+C\right)\\&\quad\cdot\exp\left(\int_0^T C\|\nabla\pi\|_{B^0_{\infty,\infty}}^{2/3}\,d\tau\right)<\infty.\end{aligned}\tag{42}$$

Case 2 ($r=-1$). That is,

$$\int_0^T\|\pi(t)\|_{\dot B^{-1}_{\infty,\infty}}^2\,dt<\infty.\tag{43}$$

Employing Hölder inequality, Young inequality, and Lemma 5, one shows that

$$\begin{aligned}I&=2\int_{\mathbf{R}^3}\pi u\,|u|\cdot\nabla u\,dx\\&\le C\|\pi\|_{L^4}^2\|u\|_{L^4}^2+\frac{\mu}{8}\||u|\nabla u\|_{L^2}^2\\&\le C\|\pi\|_{B^{-1}_{\infty,\infty}}\|\nabla p\|_{L^2}\|u\|_{L^4}^2+\frac{\mu}{8}\||u|\nabla u\|_{L^2}^2\\&\le C\|\pi\|_{B^{-1}_{\infty,\infty}}\left(C+\||u|\nabla u\|_{L^2}\right)\|u\|_{L^4}^2+\frac{\mu}{8}\||u|\nabla u\|_{L^2}^2\\&\le C\|\pi\|_{B^{-1}_{\infty,\infty}}^2\|u\|_{L^4}^4+\frac{\mu}{4}\||u|\nabla u\|_{L^2}^2+C.\end{aligned}\tag{44}$$

Inserting it into (30), we get

$$\begin{aligned}&\frac{d}{dt}\left(\|u\|_{L^4}^4+\|\theta\|_{L^4}^4\right)+\mu\||u|\nabla u\|_{L^2}^2\\&\quad\le C\left(\|\theta\|_{L^4}^4+\|u\|_{L^4}^4\right)+C\|\pi\|_{B^{-1}_{\infty,\infty}}^2\|u\|_{L^4}^4\\&\quad\le\left(C+C\|\pi\|_{B^{-1}_{\infty,\infty}}^2\right)\left(\|u\|_{L^4}^4+\|\theta\|_{L^4}^4\right).\end{aligned}\tag{45}$$

Taking Gronwall inequality into consideration, one shows that

$$\begin{aligned}&\sup_{0<t<T}\left(\|u(t)\|_{L^4}^4+\|\theta\|_{L^4}^4\right)+\mu\int_0^T\||u|\nabla u\|_{L^2}^2\,dt\\&\quad\le\left(\|u_0\|_{L^4}^4+\|\theta_0\|_{L^4}^4\right)\exp\left\{\int_0^T C+C\|\pi(t)\|_{\dot B^{-1}_{\infty,\infty}}^2\,dt\right\}\\&\quad\le C\left(\|u_0\|_{L^4}^4+\|\theta_0\|_{L^4}^4\right)\exp\left\{\int_0^T\|\pi(t)\|_{\dot B^{-1}_{\infty,\infty}}^2\,dt\right\}\\&\quad<\infty.\end{aligned}\tag{46}$$

From the estimates of I in both Cases 1 and 2, we now derive the L^4 bound of the velocity u:

$$\|u\|_{L^\infty(0,T_1;L^4)}\le C.\tag{47}$$

Step 3 (H^1 estimates of u). Taking the inner product to the first equation of (1) by Δu and applying the integral by parts yield

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2+\mu\|\Delta u\|_{L^2}^2\\&\quad=\int_{\mathbf{R}^3}(u\cdot\nabla)u\cdot\Delta u\,dx-\int_{\mathbf{R}^3}\theta e_3\cdot\Delta u\,dx\\&\quad=:J_1+J_2.\end{aligned}\tag{48}$$

For J_1, J_2, we employ Hölder inequality, Young inequality, and Gagliardo-Nirenberg inequality:

$$\begin{aligned} J_1 &= \int_{\mathbf{R}^3} (u\cdot\nabla)\, u\cdot\Delta u\, dx \le C\|u\|_{L^4}\|\nabla u\|_{L^4}\|\Delta u\|_{L^2} \\ &\le C\|u\|_{L^4}\|\nabla u\|_{L^2}^{1/4}\|\Delta u\|_{L^2}^{7/4} \\ &\le \frac{\mu}{4}\|\Delta u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2, \end{aligned} \tag{49}$$

$$\begin{aligned} J_2 &\le C\|\theta\|_{L^2}\|\Delta u\|_{L^2} \le \frac{\mu}{4}\|\Delta u\|_{L^2}^2 + C\|\theta\|_{L^2}^2 \\ &\le \frac{\mu}{4}\|\Delta u\|_{L^2}^2 + C. \end{aligned} \tag{50}$$

Plugging (49) and (50) into (48) to deduce

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \mu\|\Delta u\|_{L^2}^2 \le C\|\nabla u\|_{L^2}^2 + C, \tag{51}$$

taking Gronwall inequality into consideration together with (27), it follows that

$$\sup_{0<t<T}\|\nabla u(t)\|_{L^2}^2 + \int_0^T \|\Delta u\|_{L^2}^2 \le C; \tag{52}$$

that is

$$\sup_{0<t<T}\|u(t)\|_{H^1}^2 + \int_0^T \|\nabla u\|_{H^1}^2 \le C. \tag{53}$$

Step 4 (H^m estimates of u, θ). To prove the H^m estimates of u, θ, we need L^∞ estimates of ∇u, $\nabla\theta$. To do so, we first rewrite the first equations of (1):

$$\partial_t u - \mu\Delta u = \theta e_3 - u\cdot\nabla u - \nabla\pi. \tag{54}$$

Thanks to

$$\begin{aligned} &\int_0^T \|\theta e_3\|_{L^3}^2\, ds \le \|\theta_0\|_{L^3}^2\, T, \\ &\int_0^T \|u\cdot\nabla u\|_{L^3}^2\, ds \le \int_0^T \|u\|_{L^6}^2\|\nabla u\|_{L^2}^2\, ds \\ &\qquad \le \operatorname*{ess\,sup}_{0<s<T}\|\nabla u(s)\|_{L^2}^2 \int_0^T \|\Delta u\|_{L^2}^2\, ds \\ &\qquad \le C, \\ &\int_0^T \|\nabla p\|_{L^3}^2\, ds \le \int_0^T \|u\cdot\nabla u\|_{L^3}^2\, ds + CT \\ &\qquad \le \operatorname*{ess\,sup}_{0<s<T}\|\nabla u(s)\|_{L^2}^2 \int_0^T \|\Delta u\|_{L^2}^2\, ds \\ &\qquad\quad + CT \le CT, \end{aligned} \tag{55}$$

the maximal regularity properties of the heat equation allow us to derive

$$\int_0^T \|u\|_{W^{2,3}}^2\, ds \le C \tag{56}$$

implied by Sobolev imbedding inequality

$$\int_0^T \|\nabla u\|_{L^\infty}\, ds \le C. \tag{57}$$

Now, we take the operator ∇ into the second equation of (1) and then take the inner product by $|\nabla\theta|^{p-2}\nabla\theta$ to give

$$\frac{d}{dt}\|\nabla\theta\|_{L^p} \le C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^p}. \tag{58}$$

Employing Gronwall inequality yields

$$\|\nabla\theta\|_{L^p} \le \|\nabla\theta_0\|_{L^p}\exp\left\{\int_0^T \|\nabla u\|_{L^\infty}\, d\tau\right\}. \tag{59}$$

For any given $\epsilon, \rho > 0$, we define the set $A_{\epsilon,\rho}(t)$:

$$A_{\epsilon,\rho}(t) = \left\{x\in\mathbf{R}^3;\ |\nabla\theta| > \|\nabla\theta\|_{L^\infty} - \varepsilon,\ |x| > \rho\right\}, \tag{60}$$

and then

$$\begin{aligned} &\left(\|\nabla\theta(t)\|_{L^\infty} - \epsilon\right)\left|A_{\epsilon,\rho}(t)\right| \le \|\nabla\theta\|_{L^p} \\ &\quad \le \|\nabla\theta_0\|_{L^2}^{2/p}\|\nabla\theta_0\|_{L^\infty}^{1-2/p}\exp\left(\int_0^T \|\nabla u\|_{L^\infty}\, d\tau\right) \le C, \end{aligned} \tag{61}$$

where $|A_{\epsilon,\rho}(t)|$ is the Lebesgue measure of the set $A_{\epsilon,\rho}(t)$. Since C is independent of t, ϵ, the above uniform bound allows us to take the limitation $t\to\infty$, $\epsilon\to 0$; then, we have

$$\|\nabla\theta(t)\|_{L^\infty} \le \|\nabla\theta_0\|_{L^\infty}\exp\left(\int_0^T \|\nabla u\|_{L^\infty}\, d\tau\right). \tag{62}$$

Taking the derivative operator $J^m = (I-\Delta)^{m/2}$ into the first equation of (1) and taking the inner product by $J^m u$ and then applying Lemma 6, Hölder inequality, and Young inequality,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u\|_{H^m}^2 + \mu\|\nabla u\|_{H^m}^2 \\ &\quad \le \left|\int_{\mathbf{R}^3}\left(J^m(u\cdot\nabla u) - u\cdot\nabla J^m u\right)J^m u\, dx\right| \\ &\qquad + \left|\int_{\mathbf{R}^3} J^m(\theta e_3)\, J^m u\, dx\right| \\ &\quad \le C\|\nabla u\|_{L^\infty}\|J^m u\|_{L^2}^2 + C\|J^m\theta\|_{L^2}\|J^m u\|_{L^2} \\ &\quad \le C\left(\|\nabla u\|_{L^\infty}\|u\|_{H^m}^2 + \|\theta\|_{H^m}^2 + \|u\|_{H^m}^2\right). \end{aligned} \tag{63}$$

Besides, we employ similar methods to (63) to the second equation of (1), yielding

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\theta\|_{H^m}^2 &\le \left|\int_{\mathbf{R}^3}\left(J^m(u\cdot\nabla\theta) - u\cdot\nabla J^m\theta\right)J^m\theta\, dx\right| \\ &\le C\|\nabla u\|_{L^\infty}\|J^m\theta\|_{L^2}^2 \\ &\quad + C\|\nabla\theta\|_{L^\infty}\|J^m\theta\|_{L^2}\|J^m u\|_{L^2} \\ &\le C\left(\|\nabla u\|_{L^\infty} + \|\nabla\theta\|_{L^\infty}\right)\|\theta\|_{H^m}^2 \\ &\quad + C\|\nabla\theta\|_{L^\infty}\|u\|_{H^m}^2. \end{aligned} \tag{64}$$

Combining (63) and (64),

$$\begin{aligned}&\frac{d}{dt}\left(\|u\|_{H^m}^2+\|\theta\|_{H^m}^2\right)+2\mu\|\nabla u\|_{H^m}^2\\&\quad\le C\left(\|\nabla u\|_{L^\infty}+\|\nabla\theta\|_{L^\infty}+C\right)\left(\|u\|_{H^m}^2+\|\theta\|_{H^m}^2\right),\end{aligned}\tag{65}$$

and taking Gronwall inequality into consideration give

$$\begin{aligned}&\sup_{0<t<T}\left(\|u(t)\|_{H^m}^2+\|\theta(t)\|_{H^m}^2\right)+2\mu\int_0^T\|\nabla u\|_{H^m}^2\,d\tau\\&\quad\le C.\end{aligned}\tag{66}$$

Thus, we complete the proof of Theorem 1.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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