Research Article
Calculations on Lie Algebra of the Group of Affine Symplectomorphisms

Zuhier Altawallbeh
Department of Mathematics, Tafila Technical University, Tafila 66110, Jordan

Correspondence should be addressed to Zuhier Altawallbeh; zuhier1980@gmail.com

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We find the image of the affine symplectic Lie algebra \(\mathfrak{g}_n\) from the Leibniz homology \(H\), to the Lie algebra homology \(H^\text{Lie} (\mathfrak{g}_n)\). The result shows that the image is the exterior algebra \(\Lambda^\bullet (\omega_n)\) generated by the forms \(\omega_n = \sum_{i=1}^n (\partial/\partial x^i \wedge \partial/\partial y^i)\). Given the relevance of Hochschild homology to string topology and to get more interesting applications, we show that such a map is of potential interest in string topology and homological algebra by taking into account that the Hochschild homology \(HH_{s-1}(U(\mathfrak{g}_n))\) is isomorphic to \(H^\text{Lie}_s(\mathfrak{g}_n, U(\mathfrak{g}_n))\). Explicitly, we use the alternation of multilinear map, in our elements, to do certain calculations.

1. Introduction

Recall that the group of affine symplectomorphisms, which is the affine symplectic group \(\text{ASp}_n\), is given by all transformations \(\Psi: \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}\) of the form \(\Psi(x) = x + Ax\), where \(A\) is a \(2n \times 2n\) symplectic matrix and \(x_0\) a fixed element of \(\mathbb{R}^{2n}\) [1].

The Lie algebra \(\mathfrak{g}_n\) of \(\text{ASp}_n\) is called the affine symplectic Lie algebra by using some details and facts from [2, 3], Lodder [4] has proved that the structure of the Leibniz homology of \(\mathfrak{g}_n\) is determined by the exterior algebra of the forms \(w_n = \sum_{i=1}^n (\partial/\partial x^i \wedge \partial/\partial y^i)\), as follows:

\[
\text{HL}_s(\mathfrak{g}_n) = \Lambda^s (w_n),
\]

where \(\partial/\partial x^i\) and \(\partial/\partial y^i\) are the unit vector fields parallel to \(x_i\) and \(y_i\) axes, respectively, and the Lie algebra homology \(H^\text{Lie}_s(\mathfrak{g}_n)\) has been proved to have an isomorphic vector space as follows:

\[
H^\text{Lie}_s(\mathfrak{g}_n) = H_s(s\mathfrak{p}_n, \mathbb{R}) \otimes \Lambda^s (w_n),
\]

where \(H_s(s\mathfrak{p}_n, \mathbb{R})\) is the singular homology of the real symplectic Lie algebra \(s\mathfrak{p}_n\) and \(\Lambda^s (w_n)\) is the exterior algebra generated by \(w_n\).

Here, we find the image of the affine symplectic Lie algebra \(\mathfrak{g}_n\) from the Leibniz homology \(H\), to the Lie algebra homology \(H^\text{Lie} (\mathfrak{g}_n)\). The result shows that the image is the tensor of a real number with the exterior algebra \(\Lambda^s (w_n)\). We use the alternation of multilinear map, in our elements, to do certain calculations.

Any advances for computations in Hochschild homology are fundamental in string topology because of the high connection between both Hochschild homology and free loop spaces [5]. In this paper, we show the image of \(H\) is important to find the image in Hochschild homology \(HH_{s-1}(U(\mathfrak{g}_n))\). In particular, we find the image via the map

\[
j_* \circ \pi_* : H\rightarrow HH_{s-1}(U(\mathfrak{g}_n))
\]

where the maps \(j_*\) and \(\pi_*\) are induced by the chain maps, on the chain level, \(C(\mathfrak{g}_n) \rightarrow C^{\text{Lie}}_{s-1}(\mathfrak{g}_n, \mathfrak{g}_n)\) and \(\pi_*: C^{\text{Lie}}_{s-1}(\mathfrak{g}_n, \mathfrak{g}_n) \rightarrow C^{\text{Lie}}_{s-1}(\mathfrak{g}_n, U(\mathfrak{g}_n)),\) respectively, and \(U(\mathfrak{g}_n)\) is isomorphic to the Hochschild homology \(HH_{s-1}(U(\mathfrak{g}_n))\) [6].

In symmetric geometry, the study of symplectic algebras is important in manifolds because of the structure of the symplectic group \(\text{SP}(2n, \mathbb{R})\) preserving the transformations of the symplectic vector space at any point of symplectic manifolds, and the reader is kindly requested to refer to [7] to get more applications and interactions with classical mechanics. Moreover, considering position and momentum in the frame of quantum state in physics, symplectic group
can be considered as an important tool in the phase space. Thus, in the paper, both the source about symplectic algebras and the target related to Hochschild (co)homology make the paper in the intersection field from mathematics to physics.

By referring to [6], we recall that Leibniz homology is a noncommutative theory for Lie algebras, while Hochschild homology is a noncommutative theory for algebras, in the sense that Leibniz homology does not require the skew-symmetry of the bracket for a Lie algebra, while Hochschild homology does not require commutativity of the product in an algebra.

2. Preliminaries

For any Lie algebra \( \mathfrak{g} \) over a ring \( k \), the Lie algebra homology of \( \mathfrak{g} \), written \( H^\text{Lie}_\ast (\mathfrak{g}, k) \), is the homology of the chain complex \( \wedge^\ast (\mathfrak{g}) \) which was introduced by Chevalley and Eilenberg in [8]; namely,

\[
k \leftarrow \mathfrak{g} \leftarrow \mathfrak{g}^2 \leftarrow \cdots \leftarrow \mathfrak{g}^\ast \leftarrow \mathfrak{g}^n \leftarrow \cdots ,
\]

where

\[
d (g_1 \wedge g_2 \wedge \cdots \wedge g_n) = \sum_{1 \leq i < j \leq n} (-1)^{ij} [g_i, g_j] \wedge g_1 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge \hat{g}_j \wedge \cdots \wedge g_n ,
\]

where the notation \( \hat{g}_i \) means that element has been deleted. In this paper \( H^\text{Lie}_\ast (\mathfrak{g}) \) denotes homology with real coefficients, where \( k = \mathbb{R} \). Lie homology, with coefficients in the adjoint \( \mathfrak{g}^\ast \mathfrak{g} \), is the homology of the chain complex \( U(\mathfrak{g}) \otimes \wedge^\ast (\mathfrak{g}) \):

\[
U(\mathfrak{g}) \leftarrow U(\mathfrak{g}) \otimes \mathfrak{g} \leftarrow U(\mathfrak{g}) \otimes \mathfrak{g}^2 \leftarrow \cdots \\
\leftarrow U(\mathfrak{g}) \otimes \wedge^{\ast (n-1)} \mathfrak{g} \leftarrow U(\mathfrak{g}) \otimes \mathfrak{g}^n \leftarrow \cdots,
\]

where

\[
d (g_0 \otimes g_1 \wedge g_2 \wedge \cdots \wedge g_n) = \sum_{1 \leq j \leq n} (-1)^{j+1} [g_0, g_j] \otimes g_1 \wedge g_2 \wedge \cdots \hat{g}_j \wedge \cdots \wedge g_n + \sum_{1 \leq i < j \leq n} (-1)^{ij} g_0 \otimes [g_i, g_j] \wedge g_1 \wedge \cdots \hat{g}_i \wedge \cdots \hat{g}_j \wedge \cdots \wedge g_n 
\]

The canonical projection \( \rho : \mathfrak{g} \otimes \wedge^\ast (\mathfrak{g}) \rightarrow \wedge^{\ast +1} (\mathfrak{g}) \) given by \( \mathfrak{g} \otimes \mathfrak{g}^n \rightarrow \mathfrak{g}^{\ast (n+1)} \) is a map of chain complexes and induces a \( k \)-linear map on homology

\[
\rho_* : H^\text{Lie}_\ast (\mathfrak{g}, \mathfrak{g}) \rightarrow H^\text{Lie}_{\ast +1} (\mathfrak{g}, k).
\]

To see more details, the reader is kindly requested to look at [9].

3. Leibniz and Hochschild Homology

Returning to the general setting of any Lie algebra \( \mathfrak{g} \) over a ring \( k \), we recall that the Leibniz homology [10] of \( \mathfrak{g} \), written \( HL_\ast (\mathfrak{g}) \), is the homology of the chain complex

\[
k \leftarrow \mathfrak{g} \leftarrow \mathfrak{g}^2 \leftarrow \cdots \leftarrow \mathfrak{g}^{\ast (n-1)} \leftarrow \mathfrak{g}^n \leftarrow \cdots ,
\]

where

\[
\partial (g_1 \otimes g_2 \otimes \cdots \otimes g_n) = \sum_{1 \leq i < j \leq n} (-1)^{ij} g_1 \otimes g_2 \otimes \cdots \otimes g_i \otimes g_{i+1} \otimes \cdots \otimes g_j \otimes \cdots \otimes g_n
\]

\[
= mg_1 \otimes g_2 \otimes \cdots \otimes g_n + \sum_{i=1}^{n-1} (-1)^i m \otimes g_i \otimes \cdots \otimes g_n \otimes \cdots \otimes g_{n-1}
\]

for \( m \in M \) and \( g_i \in \mathfrak{g} \) for all \( i = 1, \ldots, n \). The homology groups of the Hochschild complex \( CH_\ast (\mathfrak{g}, M) \) are called the Hochschild homology groups \( HH_\ast (\mathfrak{g}, M) \). For \( \mathfrak{g} = M \), we write \( HH_\ast (\mathfrak{g}) \).

4. Affine Symplectic Lie Algebra

We begin by \( (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) \in \mathbb{R}^{2n} \), where \( \partial / \partial x^i, \partial / \partial y^i \) are the unit vector fields parallel to \( x_i \) and \( y_i \) axes, respectively. Then the real symplectic Lie algebra \( sp_n \) has a basis

\[
\mathcal{B}_1 = \left\{ x_k \frac{\partial}{\partial y^i}, y_k \frac{\partial}{\partial x^i} \mid \text{where } k = 1, 2, \ldots, n, \ x_i \frac{\partial}{\partial y^j}, \ y_i \frac{\partial}{\partial x^j}, \ \text{where } 1 \leq i < j \right\}
\]

\[
\leq n, \ y_j \frac{\partial}{\partial y^i} - x_i \frac{\partial}{\partial x^i} \mid \text{where } i = 1, 2, \ldots, n, \ j
\]

\[
= 1, 2, \ldots, n \right\}
\]

Let \( I_n \) be the abelian Lie algebra with the basis \( \mathcal{B}_2 = \{ \partial / \partial x^1, \partial / \partial x^2, \ldots, \partial / \partial x^n, \partial / \partial y^1, \partial / \partial y^2, \ldots, \partial / \partial y^n \} \). The affine
symplectic Lie algebra \( g_n \) has the basis \( B_1 \cup B_2 \). Thus, there is a short exact sequence of Lie algebras

\[
0 \rightarrow L_n \xrightarrow{i} g_n \xrightarrow{\pi} sp_n \rightarrow 0. \tag{12}
\]

In the following example, we find the Lie brackets of the elements in \( sp_2 \) by taking into account the basic elements illustrated above.

**Example 2.** The basis \( B_1 \) of the real symplectic Lie algebra \( sp_2 \) contains exactly these elements

\[
x_1 \frac{\partial}{\partial y^1},
\]

\[
x_2 \frac{\partial}{\partial y^2},
\]

\[
y_1 \frac{\partial}{\partial x^1},
\]

\[
y_2 \frac{\partial}{\partial x^2},
\]

\[
x_1 \frac{\partial}{\partial y^2} + x_2 \frac{\partial}{\partial y^1},
\]

\[
y_1 \frac{\partial}{\partial x^2} + y_2 \frac{\partial}{\partial x^1},
\]

\[
y_1 \frac{\partial}{\partial y^1} - x_1 \frac{\partial}{\partial x^1},
\]

\[
y_2 \frac{\partial}{\partial y^1} - x_1 \frac{\partial}{\partial x^2},
\]

\[
y_1 \frac{\partial}{\partial y^2} - x_2 \frac{\partial}{\partial x^1},
\]

\[
y_2 \frac{\partial}{\partial y^2} - x_2 \frac{\partial}{\partial x^2},
\]

which can be denoted by \( e_1, e_2, \ldots, e_{10} \), respectively. It is known that \([e_i, e_j] = 0\) and \([e_i, e_j] = -[e_j, e_i]\) for all \( i, j = 1, 2, \ldots, n \). By taking the Lie brackets of the others, it follows that

\[
[e_1, e_2] = 0,
\]

\[
[e_1, e_j] = x_1 \frac{\partial}{\partial x^j} - y_1 \frac{\partial}{\partial y^j} = -e_j,
\]

\[
[e_1, e_4] = 0,
\]

\[
[e_1, e_5] = 0,
\]

\[
[e_1, e_6] = x_1 \frac{\partial}{\partial x^2} - y_2 \frac{\partial}{\partial y^1} = -e_6.
\]

Now we take \([e_1, e_2] = 2x_1 (\partial/\partial y^1) = 2e_1\), which means that \( e_1 \) is the Eigenvector of \([e_1, e_2]\). Similarly, if we continue the computations, we get that \( e_i \) is the Eigenvector not only for the bracket \([e_i, e_j]\) but also for \([e_i, e_1]\) for all \( i = 1, \ldots, n \).

The above example shows that the Cartan subalgebra of \( sp_2 \) is \( \{e_7, e_{10}\} \) which is the tangent of the maximal torus subset in the Lie group \( SP(2, \mathbb{R}) \).

**5. The Image of \( HL_*(g_n) \) in \( H^*_L(g_n) \)**

By convention, we denote the affine symplectic Lie algebra by \( g_n \). There is a canonical projection \( T(g_n) \rightarrow \Lambda^*(g_n) \), where \( T(g_n) \) is the tensor algebra of \( g_n \) and \( \Lambda^*(g_n) \) is the exterior algebra of \( g_n \), which is naturally defined by \( \pi' : g_n^* \rightarrow g_n^* \) for \( l \geq 0 \). Thus, the map \( \pi' \) induces a \( k \)-linear map on homology

\[
\pi' : HL_*(g_n) \rightarrow H^L_*(g_n). \tag{15}
\]

From [4], there are these two vector spaces isomorphisms \( HL_*(g_n) = \Lambda^*(w_n) \) and \( H^L_*(g_n) = H_*(sp_n, \mathbb{R}) \otimes \Lambda^*(w_n) \). Let us start with the element \( w_n = \sum_{i=1}^n (\partial/\partial x^i) \otimes \partial/\partial y^i \) \( \in HL_*(g_n) \).

By using the alternation multilinear form, we can rewrite the elements from the wedge notation into tensor product by taking into account the signs of the permutations, so

\[
\pi'(w_n) = \pi'(\sum_{i=1}^n \left( \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^i} \right))
\]

\[
= \frac{1}{n} \sum_{i=1}^n \left( \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^i} - \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial x^i} \right)
\]

\[
= \frac{1}{2} \sum_{i=1}^n \left( \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^i} \right) = w_n.
\]

For more general setting, let us take \( \Lambda^2 w_n \in HL_*(g_n) \), so we get

\[
\pi'(\Lambda^2 w_n) = \pi'(\sum_{i=1}^n \left( \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^i} \right) \wedge \sum_{i=1}^n \left( \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^i} \right) + \cdots + \left( \frac{\partial}{\partial x^n} \wedge \frac{\partial}{\partial y^n} \right) \wedge \sum_{i=1}^n \left( \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^i} \right))
\]

\[
= \pi' \left( 2 \sum_{1 \leq i, j \leq n} \left( \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial y^j} \right) \right)
\]
Thus

\[ \pi'_* (\wedge^2 w_n) = \wedge^2 w_n \in H_{*}^{\text{Lie}} (g_n). \]  

(18)

The result makes sense because \( H_{*}^{\text{Lie}} (g_n) = H_*(sp_n, \mathbb{R}) \otimes \wedge^*(w_n) \) and \( H_*(sp_n, \mathbb{R}) \otimes \wedge^*(w_n) = \mathbb{R} \otimes \wedge^*(w_n) = \wedge^*(w_n) \).

6. The Image in the Hochschild Homology

Hochschild homology plays a significant role in string topology, so any progress on computations of this kind of homology will be interesting for mathematicians and for those who are working in theoretical physics. First, we find the nonzero images of Liebniz homology \( H_*(g_n) \) in the Lie algebra homology \( \text{H}_*^{\text{Lie}} (g_n, U(g_n))^{\text{ad}} \) of the adjoint universal enveloping algebra \( U(g_n)^{\text{ad}} \). In particular, we find the image via the map

\[ j_* \circ \pi_* : H_*(g_n) \xrightarrow{\pi_*} \text{H}_*^{\text{Lie}} (g_n, g_n) \xrightarrow{j_*} \text{H}_*^{\text{Lie}} (g_n, U(g_n)^{\text{ad}}), \]  

(19)

where the maps \( j_* \) and \( \pi_* \) are induced by the chain maps \( \pi \) and \( j \) on the chain level. Naturally \( \pi \) and \( j \) can be defined as follows: \( \pi : g_n^{(0)} \rightarrow g_n \otimes g_n^{(0)} \) and the inclusion \( j : g_n \otimes g_n^{(0)} \rightarrow U(g_n) \otimes g_n^{(0)} \). It is not difficult to prove that \( \pi \) and \( j \) are chain maps. Now if we are trying to find \( \pi_*(w_n) \), where \( w_n \in H_*(g_n) \), we get similar procedure steps as we have done above, by taking into account that \( \pi \) is different a little bit from the map \( \pi' \) and we will get the result. I mean \( \pi_*(\wedge^iw_n) = \wedge^i w_n \). The image result \( \wedge^i w_n \) is in \( \text{H}_*^{\text{Lie}} (g_n, g_n) \) because in [4] it was proven that \( H_{*}^{\text{Lie}} (g_n, g_n) = H_*(sp_n, \mathbb{R}) \otimes \wedge^*(w_n) \), where \( \wedge^*(w_n) = \sum_{i \geq 1} \wedge^i w_n \). After taking the induced map \( j_* \), we get \( j_* (\pi_*(\wedge^i w_n)) = \wedge^i w_n \in \text{H}_*^{\text{Lie}} (g_n, U(g_n)^{\text{ad}}) \). If we put the mentioned homological algebras in more general setting as operadic theory and generalize the above result in category theory, it will be more and more applicable in many different fields of study. To see how the homological algebra meets operad, we can read [11].

Definition 3 (the antisymmetrization map \( \epsilon_n \)). Suppose that \( g \) and \( M \) are as in the previous definition. We define [6] the antisymmetrization map \( \epsilon_n : M \otimes \wedge^n g \rightarrow M \otimes g^n \) as one that sends the element \( m \otimes g_1 \wedge \cdots \wedge g_n \) (for \( m \in M \) and \( g_i \in g \) for all \( i = 1, \ldots, n \)) to \( \epsilon_n (m \otimes g_1 \wedge \cdots \wedge g_n) \) by \( \epsilon_n = \sum_{\sigma \in S_n} \text{sgn} (\sigma) \sigma \), where \( \sigma \) is a permutation in the symmetric group \( S_n \) on the set of indices \( \{1, \ldots, n\} \), and \( \sigma \) acts on (the left of) \( m \otimes g_1 \otimes \cdots \otimes g_n \) by \( \sigma \cdot (m \otimes g_1 \otimes \cdots \otimes g_n) = m \otimes g_{\sigma^{-1}(1)} \otimes \cdots \otimes g_{\sigma^{-1}(n)} \).

Theorem 4. From page 98 of [6], we know that if \( g \) is a Lie \( K \)-algebra and \( M \) is a \( U(g) \)-bimodule, then we have the following isomorphism: \( \text{HH}_* (U(g), M) \cong \text{HH}_*^{\text{Lie}} (g, M^{ad}) \). For \( M = U(g) \), we get \( \text{HH}_* (U(g)) \cong \text{HH}_*^{\text{Lie}} (g, U(g)^{ad}) \).

By applying the above theorem, we get the image of \( H_*(g_n) \) in the Hochschild homology \( \text{H}_*^{\text{Lie}} (g_n, U(g_n))^{\text{ad}} \) of the adjoint universal enveloping algebra \( U(g_n)^{\text{ad}} \). In other words, \( \text{HH}_* (U(g_n)) \) contains \( \wedge^*(w_n) \) as a direct summand. Now, we know that

\[ \text{HH}_*^{\text{Lie}} (U(g_n)) = \text{Hom} (\text{HH}_*^{\text{Lie}} (U(g_n)), \mathbb{R}). \]  

(20)

Taking into account that \( \wedge^*(w_n) \) is the dual space of \( \wedge^*(w_n) \), where \( w_n = dx^1 \wedge dy^1 \) and \( dx^i \) is dual of \( \partial/\partial x^i \) and \( dy^i \) is dual of \( \partial/\partial y^i \), we end up with this following result about the image in Hochschild cohomology of the given algebra.

Corollary 6. The Hochschild cohomology \( \text{HH}_* (U(g_n)) \) contains \( \wedge^*(w_n) \) as a direct summand.
As an algebraic point of departure and theoretical physics point of view, the Hochschild cohomology $HH^*(A)$ of an associative algebra $A$ has natural product with a Lie type bracket of degree $-1$, satisfying Jacobi identity and graded anticommutativity such that both natural product and Lie type bracket are compatible to make $HH^*(A)$ a Gerstenhaber algebra. Furthermore, the Gerstenhaber algebra structure can be viewed as algebraic properties of the loop homology algebra of a manifold. Here, we concentrate our work by setting $A = U(g)$. 

**Competing Interests**

The author declares no competing interests.

**References**


