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Research Article

An Iterative Method for Time-Fractional Swift-Hohenberg Equation

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We study a type of iterative method and apply it to time-fractional Swift-Hohenberg equation with initial value. Using this iterative method, we obtain the approximate analytic solutions with numerical figures to initial value problems, which indicates that such iterative method is effective and simple in constructing approximate solutions to Cauchy problems of time-fractional differential equations.

1. Introduction

In 1695, L'Hopital wrote a letter to Leibniz, and he proposed a problem: "What is the result of $d^n y/dx^n$ if $n = 1/2$?" Lcebniz answered to L'Hopital "$d^{1/2}x$ will be equal to $x\sqrt{dx : x}$. This is an apparent paradox, from which, one day useful consequences will be drawn." [1, 2] Later, as the development of mathematics, especially, the theory of operator, researchers started to have a new recognition of the fractional derivative. They found the fractional derivative has a wide applications in many fields, such as physics, chemistry, and many other sciences [3, 4]. It should be emphasised that the fractional derivative is defined by integral and it is a nonlocal operator with a singular kernel; hence it can provide an excellent instrument for description of memory and hereditary properties of various physical processes. For example, half-order derivatives and integrals proved to be more useful for the formulation of certain electrochemical problems than the classical models [5]. However, the definition of fractional derivative has not been unified; there are many kinds of fractional integral and fractional derivative, such as in the sense of Riemann-CLiouville, Caputo, Riesz, and Weyl [6–9]. The Riemann-Liouville fractional integral and the Caputo fractional derivative are the most commonly used.

For a fractional system with respect to some initial or boundary conditions, one of the fundamental problems is naturally that what is the exact solution or approximate solution to such system. Solving fractional system is usually more difficult than the classical system, for its operator is defined by integral. Luckily, there are some different effective methods which have been developed to construct approximate solutions of fractional systems and even obtain the exact solutions such as the homotopy analysis method [10–12], the residual power series method [13–16], the differential transform method [17, 18], the Laplace transform method [19], the perturbation method [20, 21]. In addition, using polynomials to approximate the fractional system is an effective method as well, such as Jacobi polynomials [22], Bernstein polynomials [23], and Chebyshev and Legendre polynomials [24]. In this paper, we introduce a type of iterative method, based on decomposing the nonlinearity term, for solving a class of functional equations [25–30].

Outline of Paper. In Section 2, we introduce some necessary concepts and lemmas on fractional differential equations. In Section 3, a type of iterative method for solving a class of functional equation is presented. Also, we obtain the convergence analysis of this iterative method. In Sections 4 and 5, we take the linear time-fractional S-H equations, including the term of dispersion, with respect to differential initial conditions as examples to illustrate the strong power of such iterative method, respectively.

2. Notations of Fractional Calculus

In subsection, we introduce some concepts and lemmas we need in this paper, such as the Gamma function, the Mittag-Leffler function, the Riemann-Liouville fractional integral, and the Caputo fractional derivative. It should be emphasised that there are many kinds of fractional integral and fractional derivative, such as in the sense of Riemann-Liouville, Caputo, Riesz, and Weyl [6–9]. The Riemann-Liouville fractional integral and Caputo fractional derivative are the most commonly used version.

Definition 1. The Gamma function is defined by [6, 29, 30]

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt \qquad \operatorname{Re}(x) > 0. \tag{1}$$

Definition 2. The Mittag-Leffler function is defined by [31, 32]

$$E_\alpha(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)} \qquad \alpha > 0. \tag{2}$$

Definition 3. The (left sided) Riemann-Liouville fractional integral of order $\beta (\beta > 0)$ of a function $u(x,t) \in C_p (p \geq -1)$ is denoted by $\mathrm{I}^\beta u(x,t)$ (with respect to t) and defined as [29, 30]

$$\mathrm{I}^\beta u(x,t) := \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} u(x,\tau)\, d\tau. \tag{3}$$

Definition 4. The (left sided) Caputo fractional derivative of order $\beta (\beta > 0)$ of a function $u(x,t) \in C_l^m$ is denoted by $\mathrm{D}^\beta u(x,t)$ (with respect to t) and defined as [29, 30]

$$\mathrm{D}^\beta u(x,t) := \begin{cases} \partial_t^m u(x,t) & \beta = m \in \mathbb{N}^* \\ \mathrm{I}^{m-\beta} \partial_t^m u(x,t) & m-1 < \beta < m. \end{cases} \tag{4}$$

Lemma 5. *For any $m-1 < \beta \leq m \in \mathbb{N}^*$, one has [6–9, 29, 30]*

$$\mathrm{I}^\beta \mathrm{D}^\beta u(x,t) = u(x,t) - \sum_{k=0}^{m-1} \partial^k u(x,0) \frac{t^k}{k!}. \tag{5}$$

Lemma 6. *For any β, γ, one has [6–9, 29]*

$$\mathrm{I}^\beta t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\beta+\gamma+1)} t^{\beta+\gamma}. \tag{6}$$

3. A Type of Iterative Method

In this section, we introduce a generalized iterative method for solving a class of functional equation (7) (see below). Some more specific details about this type of iterative method could be found in [25–30] and the references therein.

Now we state this iterative method as the following lemma together convergence analysis.

Lemma 7. *Consider the nonlinear functional equation*

$$u(t,x) = f(t,x) + \mathscr{L}(u(t,x)) + \mathscr{N}(u(t,x)), \tag{7}$$

where $u(t,x)$ is an unknown function, i.e., solution of functional equation (7), $f(t,x)$ is a known function, $(t,x) \in D = \{(t,x) : t \in \mathbb{R}, x \in \mathbb{R}^n, n \in \mathbb{N}\}$, and $\mathscr{L}$ and $\mathscr{N}$ are linear and nonlinear operators from a Banach space B to itself. We are looking for a solution of functional equation (7) having a series form

$$u(t,x) = \sum_{n=0}^{\infty} u_n(t,x) = u_0 + u_1 + u_2 + \cdots, \tag{8}$$

where u_n are defined by letting

$$\begin{aligned} u_0 &= f, \\ u_1 &= \mathscr{L}(u_0) + \mathscr{N}(u_0), \end{aligned} \tag{9}$$

$$\begin{aligned} u_{m+1} = \mathscr{L}(u_m) + \left\{ \mathscr{N}\left(\sum_{n=0}^{m} u_n\right) - \mathscr{N}\left(\sum_{n=0}^{m-1} u_n\right) \right\}, \\ m = 1, 2 \cdots. \end{aligned} \tag{10}$$

Furthermore, if the operator $\mathscr{L}$ and $\mathscr{N}$ are contractive, then the series $\sum_{n=0}^{\infty} u_n(t,x)$ converges absolutely and uniformly.

Proof. Obviously, the nonlinear operator $\mathscr{N}$ can be decomposed

$$\begin{aligned} \mathscr{N}(u(t,x)) &= \mathscr{N}\left(\sum_{n=0}^{\infty} u_n(t,x)\right) \\ &= \mathscr{N}(u_0) \\ &\quad + \sum_{n=1}^{\infty} \left\{ \mathscr{N}\left(\sum_{j=0}^{n} u_j\right) - \mathscr{N}\left(\sum_{j=0}^{n-1} u_j\right) \right\}. \end{aligned} \tag{11}$$

Similarly, linear operator $\mathscr{L}$ can also be decomposed

$$\begin{aligned} \mathscr{L}(u(t,x)) &= \mathscr{L}\left(\sum_{n=0}^{\infty} u_n(t,x)\right) \\ &= \mathscr{L}(u_0) \\ &\quad + \sum_{n=1}^{\infty} \left\{ \mathscr{L}\left(\sum_{j=0}^{n} u_j\right) - \mathscr{L}\left(\sum_{j=0}^{n-1} u_j\right) \right\}. \end{aligned} \tag{12}$$

Set

$$\mathscr{M}(u(t,x)) := \mathscr{L}(u(t,x)) + \mathscr{N}(u(t,x)). \tag{13}$$

then

$$\begin{aligned} \mathscr{M}(u(t,x)) &= \mathscr{M}(u_0) \\ &\quad + \sum_{n=1}^{\infty} \left\{ \mathscr{M}\left(\sum_{j=0}^{n} u_j\right) - \mathscr{M}\left(\sum_{j=0}^{n-1} u_j\right) \right\}. \end{aligned} \tag{14}$$

Define the following recurrence equations:

$$\begin{aligned} u_0 &= f, \\ u_1 &= \mathscr{M}(u_0), \\ u_{m+1} &= \mathscr{M}\left(\sum_{n=0}^{m} u_n\right) - \mathscr{M}\left(\sum_{n=0}^{m-1} u_n\right), \quad m = 1, 2 \cdots . \end{aligned} \tag{15}$$

Since operators $\mathscr{L}$ and $\mathscr{N}$ are contractive, then $\mathscr{M}$ is also contractive; i.e., there exists a constant $0 < K < 1$, such that

$$\left\| \mathscr{M}(v_i) - \mathscr{M}(v_j) \right\| \le K \left\| v_i - v_j \right\|, \quad \forall v_i, v_j \in B, \tag{16}$$

where $\|\cdot\|$ denotes the usual norm on Banach space B. What is more, for u_{m+1}, one can obtain

$$\begin{aligned} \|u_{m+1}\| &= \left\| \mathscr{M}\left(\sum_{n=0}^{m} u_n\right) - \mathscr{M}\left(\sum_{n=0}^{m-1} u_n\right) \right\| \le K \|u_m\| \\ &\le K^{m+1} \|u_0\| . \end{aligned} \tag{17}$$

Since $0 < K < 1$, the series $\sum_{m=1}^{\infty} K^{m+1} \|u_0\|$ converges absolutely as well as uniformly. □

According to the Weierstrass M-test, one can obtain that the series $\sum_{i=0}^{\infty} u_i$ converges absolutely as well as uniformly.

4. Linear Swift-Hohenberg Equation

The Swift-Hohenberg (for short S-H) equation

$$\partial_t u = ru - \left(1 + \nabla^2\right)^2 u + N(u) \tag{18}$$

is a model pattern-forming equation which was derived from the equations for thermal convection by Jack Swift and Pierre Hohenberg [33]. Here $u = u(x,t)$ is a scalar function defined on the line or the plane, r is a real bifurcation parameter, and $N(u)$ is some smooth nonlinearity. The S-H equation plays an important role in pattern formation theory. In [34], Braaksma et al. proved the existence of quasipatterns for the S-H equation. Also, wave process described by the S-H equation is important as well. For example, it describes the patterns inside thin vibrated granular layers, the mechanism of the amplitude of optical electric field inside the cavity, and so on [35].

Fractional S-H equation

$$\mathrm{D}^{\alpha} u = -2u_{xx} - u_{xxxx} - (1 - r) u - u^3 \quad 0 < \alpha \le 1 \tag{19}$$

was firstly introduced in [36], and Khan et al. obtained analytical approximation of this equation. Later, Vishal et al. in [37] constructed the approximate analytic solution with respect initial value $u(x,0) = 1/10 \sin(\pi x/l)$ using the homotopy analysis method. Furthermore, Vishal et al. considered the time-fractional S-H equation with dispersion [38]

$$\begin{aligned} \mathrm{D}^{\alpha} &= ru - \left(1 + \partial_{xx}\right)^2 u + \sigma \partial_{xxx} u + 2u^2 - u^3 \\ & \qquad 0 < \alpha \le 1 \end{aligned} \tag{20}$$

and obtained the approximate analytic solution. Here σ is the dispersive parameter. Lately, in [39], Merdan applied the fractional variational iteration method to obtain the approximate solution to time-fractional S-H equation with respect to initial condition $u(x,0) = 1/10 \sin(\pi x/l)$. Also, homotopy analysis method is valid for S-H equation as well [40].

In this subsection, we apply the iterative method introduced in Section 3 to linear time-fractional S-H equation with different initial values, such as e^x, $\sin x$, and $\cos x$.

4.1. Linear Time-Fractional S-H Equation with Initial Value $u(x,0) = e^x$. Consider the following linear time-fractional S-H equation:

$$\mathrm{D}^{\alpha} u + (1 - r) u + 2u_{xx} + u_{xxx} = 0 \quad 0 < \alpha \le 1 \tag{21}$$

with initial condition

$$u(x,0) = e^x. \tag{22}$$

Clearly, applying I^{α} to both sides of (21), then initial value problem (21)-(22) is equivalent to the following integral equation:

$$u(x,t) = f(x) + (\mathscr{L}u)(x,t) + (\mathscr{N}u)(x,t), \tag{23}$$

where

$$\begin{aligned} f(x) &= e^x, \\ \mathscr{L}(u) &= \mathrm{I}^{\alpha}\left((r - 1) u - 2u_{xx} - u_{xxxx}\right), \\ \mathscr{N}(u) &= 0. \end{aligned} \tag{24}$$

The solution to system (21)-(22) what we are looking for has the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{25}$$

According to iterative scheme (9)-(10), one can obtain

$$\begin{aligned} u_0 &= f = e^x; \\ u_1 &= \mathscr{L}(u_0) + \mathscr{N}(u_0) \\ &= \left((r - 1) u_0 - 2(u_0)_{xx} - (u_0)_{xxxx}\right) \mathrm{I}^{\alpha} 1 \\ &= (r - 4) e^x \frac{1}{\Gamma(\alpha + 1)} t^{\alpha}; \\ u_2 &= \mathscr{L}(u_1) + \mathscr{N}(u_0 + u_1) - \mathscr{N}(u_0) \\ &= \left((r - 1) u_1 - 2(u_1)_{xx} - (u_1)_{xxxx}\right) \mathrm{I}^{\alpha} \frac{1}{\Gamma(\alpha + 1)} t^{\alpha} \\ &= (r - 4)^2 e^x \frac{1}{\Gamma(2\alpha + 1)} t^{2\alpha} \\ &\vdots \\ u_n &= \mathscr{L}(u_{n-1}) + \mathscr{N}(u_0 + u_1 + \cdots + u_{n-1}) \\ &\quad - \mathscr{N}(u_0 + u_1 + \cdots + u_{n-2}) \end{aligned}$$

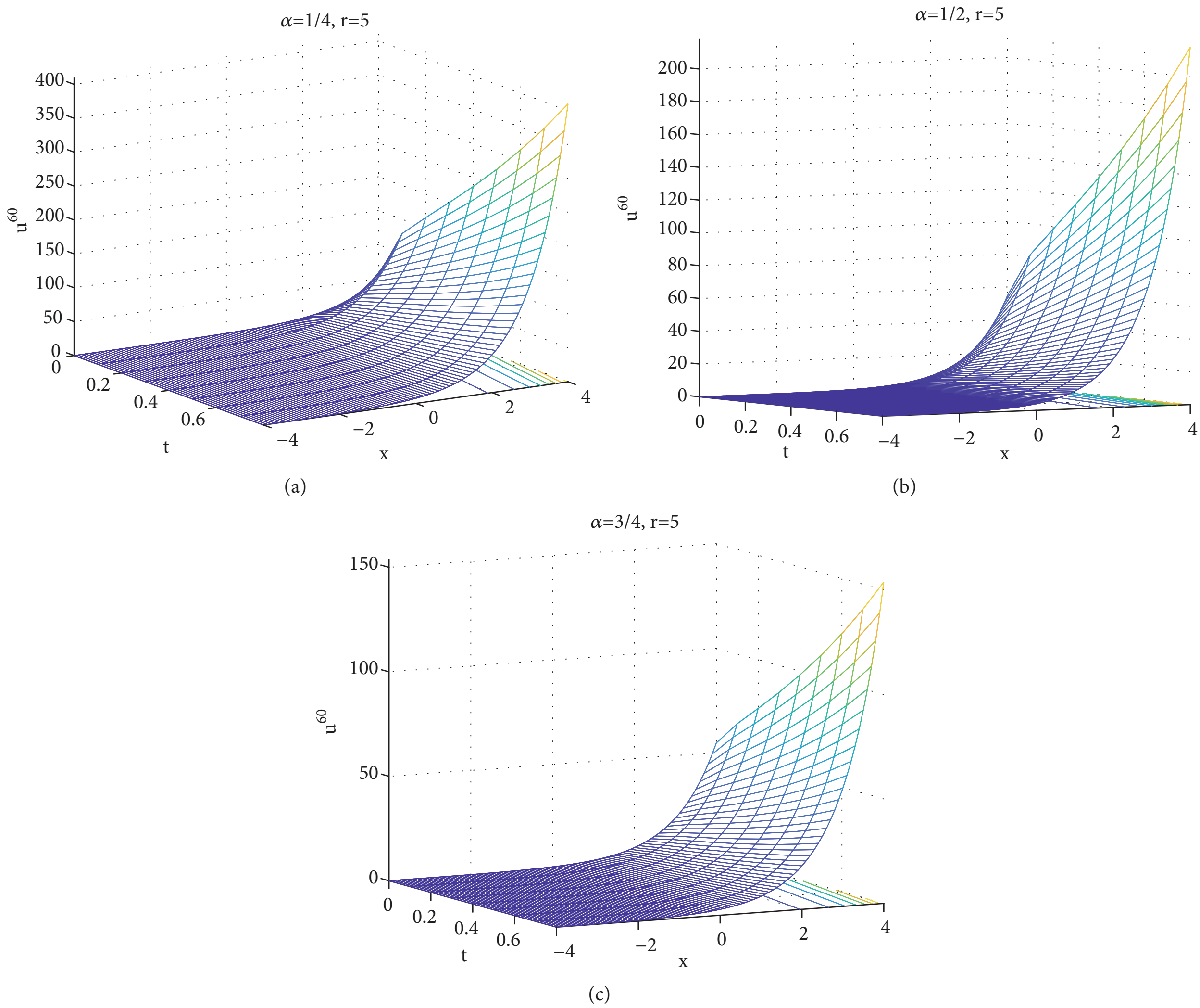

Figure 1

$$\begin{aligned}&= \left((r-1)\,u_{n-1} - 2\left(u_{n-1}\right)_{xx} - \left(u_{n-1}\right)_{xxxx}\right) \mathrm{I}^{\alpha} \\ &\quad \cdot \frac{1}{\Gamma\left((n-1)\,\alpha+1\right)} t^{(n-1)\alpha} \\ &= (r-4)^{n} e^{x} \frac{1}{\Gamma(n\alpha+1)} t^{n\alpha} \qquad (n = 2, 3, \cdots)\end{aligned} \tag{26}$$

Hence the N-th approximate solution to (21)-(22) is

$$u^{N}(x,t) = e^{x} \sum_{n=0}^{N} (r-4)^{n} \frac{1}{\Gamma(n\alpha+1)} t^{n\alpha} \tag{27}$$

and the exact solution to (21)-(22) is

$$u(x,t) = \lim_{N\longrightarrow\infty} u^{N}(x,t) = e^{x} \mathrm{E}_{\alpha}\left((r-4)\,t^{\alpha}\right). \tag{28}$$

Numerical Simulation. See Figure 1.

4.2. Linear Time-Fractional S-H Equation with Initial Value $u(x,0) = \sin x$. Consider the following linear time-fractional S-H equation:

$$\mathrm{D}^{\alpha} u + (1-r)\,u + 2u_{xx} + u_{xxxx} = 0 \quad 0 < \alpha \leq 1 \tag{29}$$

with initial condition

$$u(x,0) = \sin x. \tag{30}$$

Applying I^{α} to both sides of (29), then initial value problem (29)-(30) is equivalent to the following integral equation:

$$u(x,t) = f(x) + (\mathscr{L}u)(x,t) + (\mathscr{N}u)(x,t), \tag{31}$$

where

$$\begin{aligned}f(x) &= \sin x, \\ \mathscr{L}(u) &= \mathrm{I}^{\alpha}\left((r-1)\,u - 2u_{xx} - u_{xxxx}\right), \\ \mathscr{N}(u) &= 0.\end{aligned} \tag{32}$$

The solution of system (29)-(30) what we are looking for has the form

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t). \tag{33}$$

We shall distinguish the following two cases.

Case 1 ($r=0$). According to iterative scheme (9)-(10), we have

$$\begin{aligned} u_0 &= f=\sin x;\\ u_1 &= \mathscr{L}(u_0)+\mathscr{N}(u_0)\\ &= \left(-u_0-2(u_0)_{xx}-(u_0)_{xxxx}\right)\mathrm{I}^{\alpha}1\\ &= 0;\\ u_n &= 0,\quad n=2,3,\cdots. \end{aligned} \tag{34}$$

Hence the solution to (29)-(30) is

$$u(x,t)=\sin x. \tag{35}$$

Case 2 ($r\neq 0$). By iterative scheme (9)-(10), we have

$$\begin{aligned} u_0 &= f=\sin x;\\ u_1 &= \mathscr{L}(u_0)+\mathscr{N}(u_0)\\ &= \left((r-1)u_0-2(u_0)_{xx}-(u_0)_{xxxx}\right)\mathrm{I}^{\alpha}1\\ &= \left((r-1)+1\right)\sin x\frac{1}{\Gamma(\alpha+1)}t^{\alpha};\\ u_2 &= \mathscr{L}(u_1)+\mathscr{N}(u_0+u_1)-\mathscr{N}(u_0)\\ &= \left((r-1)u_1-2(u_1)_{xx}-(u_1)_{xxxx}\right)\mathrm{I}^{\alpha}\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\\ &= \left((r-1)^2+(r-1)+1\right)\sin x\frac{1}{\Gamma(2\alpha+1)}t^{2\alpha};\\ u_3 &= \mathscr{L}(u_2)+\mathscr{N}(u_0+u_1+u_2)-\mathscr{N}(u_0+u_1)\\ &= \left((r-1)u_2-2(u_2)_{xx}-(u_2)_{xxxx}\right)\mathrm{I}^{\alpha}\frac{1}{\Gamma(2\alpha+1)}t^{2\alpha}\\ &= \left((r-1)^3+(r-1)^2+(r-1)+1\right)\sin x\frac{1}{\Gamma(3\alpha+1)}\\ &\quad\cdot t^{3\alpha};\\ u_4 &= \mathscr{L}(u_3)+\mathscr{N}(u_0+u_1+u_2+u_3)\\ &\quad-\mathscr{N}(u_0+u_1+u_2)\\ &= \left((r-1)u_3-2(u_3)_{xx}-(u_3)_{xxxx}\right)\mathrm{I}^{\alpha}\frac{1}{\Gamma(3\alpha+1)}t^{3\alpha}\\ &= \left((r-1)^4+(r-1)^3+(r-1)^2+(r-1)+1\right)\sin x\\ &\quad\cdot\frac{1}{\Gamma(4\alpha+1)}t^{4\alpha}. \end{aligned} \tag{36}$$

Generally, for $n=1,2,\cdots$, by induction, one can derive that

$$\begin{aligned} u_n &= \mathscr{L}(u_{n-1})+\mathscr{N}(u_0+u_1+\cdots+u_{n-1})-\mathscr{N}(u_0\\ &\quad+u_1+\cdots+u_{n-2})\\ &= \left((r-1)u_{n-1}-2(u_{n-1})_{xx}-(u_{n-1})_{xxxx}\right)\mathrm{I}^{\alpha}\\ &\quad\cdot\frac{1}{\Gamma((n-1)\alpha+1)}t^{(n-1)\alpha}\\ &= \left((r-1)^n+(r-1)^{n-1}+\cdots+(r-1)^2+(r-1)\right.\\ &\quad\left.+1\right)\sin x\frac{1}{\Gamma((n-1)\alpha+1)}t^{(n-1)\alpha}\\ &= -\frac{1}{r}\left(1-(r-1)^{n+1}\right)\sin x\frac{1}{\Gamma(n\alpha+1)}t^{n\alpha} \end{aligned} \tag{37}$$

Hence the N-th approximate solution to (29)-(30) is

$$u^N(x,t)=-\frac{1}{r}\sin x\sum_{n=0}^{N}\left(1-(r-1)^{n+1}\right)\frac{1}{\Gamma(n\alpha+1)}t^{n\alpha} \tag{38}$$

and the exact solution to (29)-(30) is

$$\begin{aligned} u(x,t) &= \lim_{N\longrightarrow\infty}u^N(x,t)\\ &= -\frac{1}{r}\sin x\lim_{N\longrightarrow\infty}\sum_{n=0}^{N}\left(1-(r-1)^{n+1}\right)\frac{1}{\Gamma(n\alpha+1)}t^{n\alpha}. \end{aligned} \tag{39}$$

As a consequence, the solution of (29)-(30) is

$$\begin{aligned} &u(x,t)\\ &=\begin{cases}\sin x & r=0;\\ -\dfrac{1}{r}\sin x\displaystyle\lim_{N\longrightarrow\infty}\sum_{n=0}^{N}\left(1-(r-1)^{n+1}\right)\frac{1}{\Gamma(n\alpha+1)}t^{n\alpha} & r\neq 0.\end{cases} \end{aligned} \tag{40}$$

Numerical Simulation. See Figure 2.

4.3. Linear Time-Fractional S-H Equation with Initial Value $u(x,0)=\cos x$. Consider the following linear time-fractional S-H equation:

$$\mathrm{D}^{\alpha}u+(1-r)u+2u_{xx}+u_{xxxx}=0\quad 0<\alpha\leq 1 \tag{41}$$

with initial condition

$$u(x,0)=\cos x. \tag{42}$$

Applying I^{α} to both sides of (41), then initial value problem (41)-(42) is equivalent to the following integral equation:

$$u(x,t)=f(x)+(\mathscr{L}u)(x,t)+(\mathscr{N}u)(x,t), \tag{43}$$

where

$$\begin{aligned} f(x) &= \sin x,\\ \mathscr{L}(u) &= \mathrm{I}^{\alpha}\left((r-1)u-2u_{xx}-u_{xxxx}\right),\\ \mathscr{N}(u) &= 0. \end{aligned} \tag{44}$$

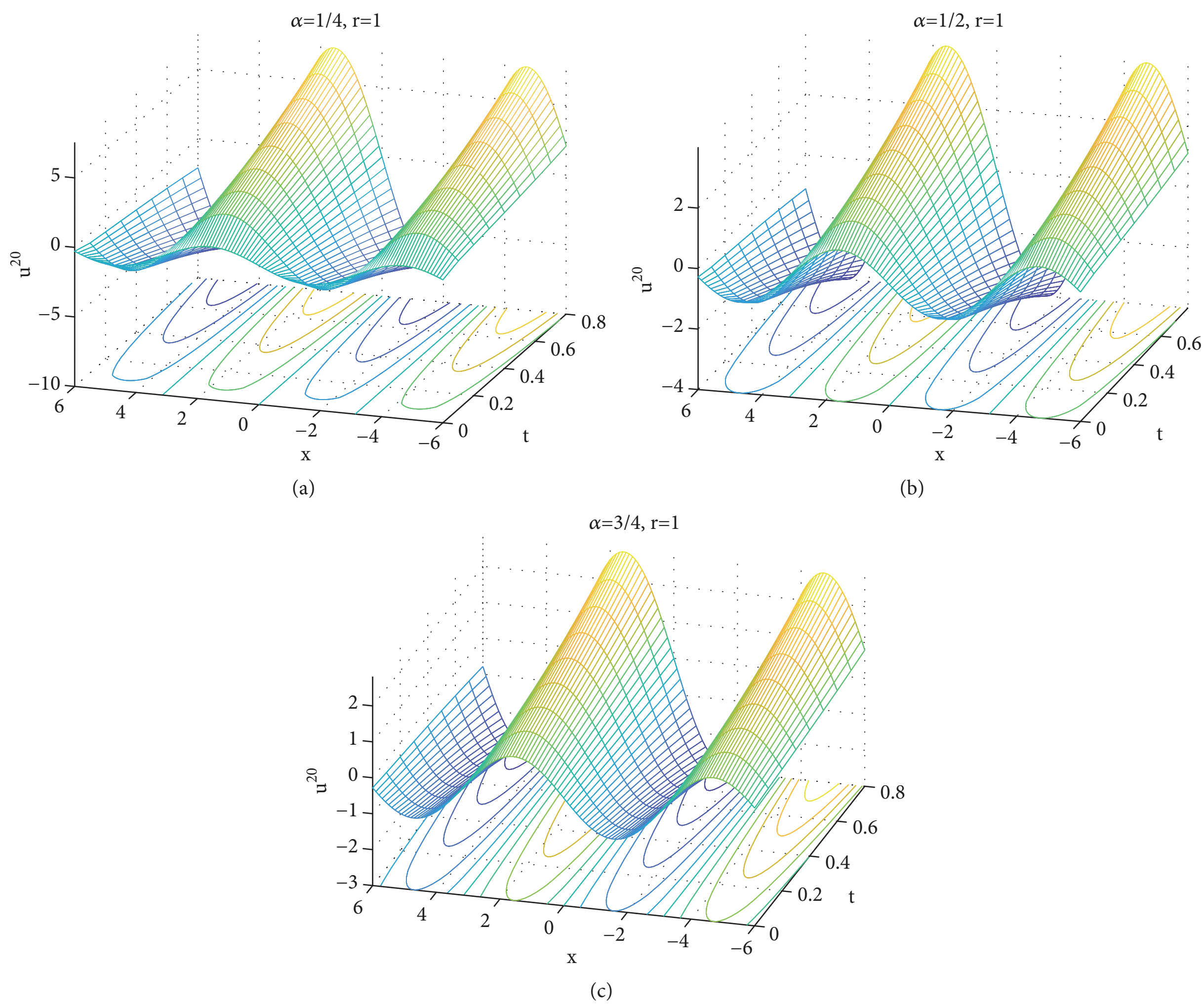

FIGURE 2

The solution to system (41)-(42) what we are looking for has the form

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t). \tag{45}$$

We shall distinguish the following two cases.

Case 1 ($r=0$). According to iterative scheme (9)-(10), we have

$$\begin{aligned}
u_0 &= f=\cos x;\\
u_1 &= \mathscr{L}(u_0)+\mathscr{N}(u_0)\\
&= \left(-u_0-2(u_0)_{xx}-(u_0)_{xxxx}\right)\mathrm{I}^{\alpha}1\\
&= 0;\\
u_n &= 0,\quad n=2,3,\cdots.
\end{aligned} \tag{46}$$

Hence the solution to (41)-(42) is

$$u(x,t)=\cos x. \tag{47}$$

Case 2 ($r\neq 0$). According to iterative scheme (9)-(10), we have

$$\begin{aligned}
u_0 &= f=\cos x;\\
u_1 &= \mathscr{L}(u_0)+\mathscr{N}(u_0)\\
&= \left((r-1)u_0-2(u_0)_{xx}-(u_0)_{xxxx}\right)\mathrm{I}^{\alpha}1\\
&= ((r-1)+1)\cos x\frac{1}{\Gamma(\alpha+1)}t^{\alpha};\\
u_2 &= \mathscr{L}(u_1)+\mathscr{N}(u_0+u_1)-\mathscr{N}(u_0)\\
&= \left((r-1)u_1-2(u_1)_{xx}-(u_1)_{xxxx}\right)\mathrm{I}^{\alpha}\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\\
&= \left((r-1)^2+(r-1)+1\right)\cos x\frac{1}{\Gamma(2\alpha+1)}t^{2\alpha};\\
u_3 &= \mathscr{L}(u_2)+\mathscr{N}(u_0+u_1+u_2)-\mathscr{N}(u_0+u_1)\\
&= \left((r-1)u_2-2(u_2)_{xx}-(u_2)_{xxxx}\right)\mathrm{I}^{\alpha}\frac{1}{\Gamma(2\alpha+1)}t^{2\alpha}\\
&= \left((r-1)^3+(r-1)^2+(r-1)+1\right)\cos x\frac{1}{\Gamma(3\alpha+1)}
\end{aligned}$$

$$\begin{aligned}
&\cdot t^{3\alpha};\\
u_4 &= \mathscr{L}(u_3) + \mathscr{N}(u_0 + u_1 + u_2 + u_3)\\
&\quad - \mathscr{N}(u_0 + u_1 + u_2)\\
&= \left((r-1)u_3 - 2(u_3)_{xx} - (u_3)_{xxxx}\right) \mathrm{I}^{\alpha} \frac{1}{\Gamma(3\alpha+1)} t^{3\alpha}\\
&= \left((r-1)^4 + (r-1)^3 + (r-1)^2 + (r-1) + 1\right)\cos x\\
&\quad \cdot \frac{1}{\Gamma(4\alpha+1)} t^{4\alpha}.
\end{aligned} \tag{48}$$

Generally, for $n = 1, 2, \cdots$, one can derive that

$$\begin{aligned}
u_n &= \mathscr{L}(u_{n-1}) + \mathscr{N}(u_0 + u_1 + \cdots + u_{n-1}) - \mathscr{N}(u_0\\
&\quad + u_1 + \cdots + u_{n-2})\\
&= \left((r-1)u_{n-1} - 2(u_{n-1})_{xx} - (u_{n-1})_{xxxx}\right)\mathrm{I}^{\alpha}\\
&\quad \cdot \frac{1}{\Gamma((n-1)\alpha+1)} t^{(n-1)\alpha}\\
&= \left((r-1)^n + (r-1)^{n-1} + \cdots + (r-1)^2 + (r-1)\right.\\
&\quad \left. + 1\right)\cos x \frac{1}{\Gamma((n-1)\alpha+1)} t^{(n-1)\alpha}\\
&= -\frac{1}{r}\left(1 - (r-1)^{n+1}\right)\cos x \frac{1}{\Gamma(n\alpha+1)} t^{n\alpha}
\end{aligned} \tag{49}$$

Hence the N-th approximate solution to (41)-(42) is

$$\begin{aligned}
&u^N(x,t)\\
&\quad = -\frac{1}{r}\cos x \sum_{n=0}^{N}\left(1 - (r-1)^{n+1}\right)\frac{1}{\Gamma(n\alpha+1)} t^{n\alpha}
\end{aligned} \tag{50}$$

and the exact solution to (41)-(42) is

$$\begin{aligned}
u(x,t) &= \lim_{N\longrightarrow\infty} u^N(x,t)\\
&= -\frac{1}{r}\cos x \lim_{N\longrightarrow\infty}\sum_{n=0}^{N}\left(1 - (r-1)^{n+1}\right)\frac{1}{\Gamma(n\alpha+1)} t^{n\alpha}.
\end{aligned} \tag{51}$$

As a consequence, the solution of (41)-(42) is

$$\begin{aligned}
&u(x,t)\\
&= \begin{cases} \cos x & r = 0;\\ -\dfrac{1}{r}\cos x \lim\limits_{N\longrightarrow\infty}\sum\limits_{n=0}^{N}\left(1 - (r-1)^{n+1}\right)\dfrac{1}{\Gamma(n\alpha+1)} t^{n\alpha} & r \neq 0. \end{cases}
\end{aligned} \tag{52}$$

Numerical Simulation. See Figure 3.

4.4. Linear S-H Equation with Dispersion and Initial Value $u(x,0) = e^x$. Consider the linear S-H equation with dispersion [41] as follows:

$$\begin{aligned}
&\mathrm{D}^{\alpha}u + (1-r)u + 2u_{xx} - \sigma u_{xxx} + u_{xxxx} = 0\\
&\qquad\qquad\qquad\qquad 0 < \alpha \leq 1
\end{aligned} \tag{53}$$

with initial value

$$u(x,0) = e^x. \tag{54}$$

Here $\sigma \in \mathbb{R}$ is a parameter. Clearly, applying I^{α} on both sides of (53), then initial value problem (53)-(54) is equivalent to the following integral equation:

$$u(x,t) = f(x) + (\mathscr{L}u)(x,t) + (\mathscr{N}u)(x,t), \tag{55}$$

where

$$\begin{aligned}
f(x) &= e^x,\\
\mathscr{L}(u) &= \mathrm{I}^{\alpha}\left((r-1)u - 2u_{xx} + \sigma u_{xxx} - u_{xxxx}\right),\\
\mathscr{N}(u) &= 0.
\end{aligned} \tag{56}$$

The solution to system (53)-(54) what we are looking for has the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{57}$$

According to iterative scheme (9)-(10), one can obtain

$$\begin{aligned}
u_0 &= f = e^x;\\
u_1 &= \mathscr{L}(u_0) + \mathscr{N}(u_0)\\
&= \left((r-1)u_0 - 2(u_0)_{xx} + \sigma(u_1)_{xxx} - (u_0)_{xxxx}\right)\mathrm{I}^{\alpha}1\\
&= (r-4+\sigma)e^x \frac{1}{\Gamma(\alpha+1)} t^{\alpha};\\
u_2 &= \mathscr{L}(u_1) + \mathscr{N}(u_0 + u_1) - \mathscr{N}(u_0)\\
&= \left((r-1)u_1 - 2(u_1)_{xx} + \sigma(u_1)_{xx} - (u_1)_{xxxx}\right)\mathrm{I}^{\alpha}\\
&\quad \cdot \frac{1}{\Gamma(\alpha+1)} t^{\alpha}\\
&= (r-4+\sigma)^2 e^x \frac{1}{\Gamma(2\alpha+1)} t^{2\alpha}\\
&\quad \vdots\\
u_n &= \mathscr{L}(u_{n-1}) + \mathscr{N}(u_0 + u_1 + \cdots + u_{n-1}) - \mathscr{N}(u_0\\
&\quad + u_1 + \cdots + u_{n-2})\\
&= \left((r-1)u_{n-1} - 2(u_{n-1})_{xx} + \sigma(u_{n-1})_{xxx}\right.\\
&\quad \left. - (u_{n-1})_{xxxx}\right)\mathrm{I}^{\alpha}\frac{1}{\Gamma((n-1)\alpha+1)} t^{(n-1)\alpha}\\
&= (r-4+\sigma)^n e^x \frac{1}{\Gamma(n\alpha+1)} t^{n\alpha} \quad (n = 2, 3, \cdots)
\end{aligned} \tag{58}$$

Hence the N-th approximate solution to (53)-(54) is

$$u^N(x,t) = e^x \sum_{n=0}^{N}(r-4+\sigma)^n \frac{1}{\Gamma(n\alpha+1)} t^{n\alpha} \tag{59}$$

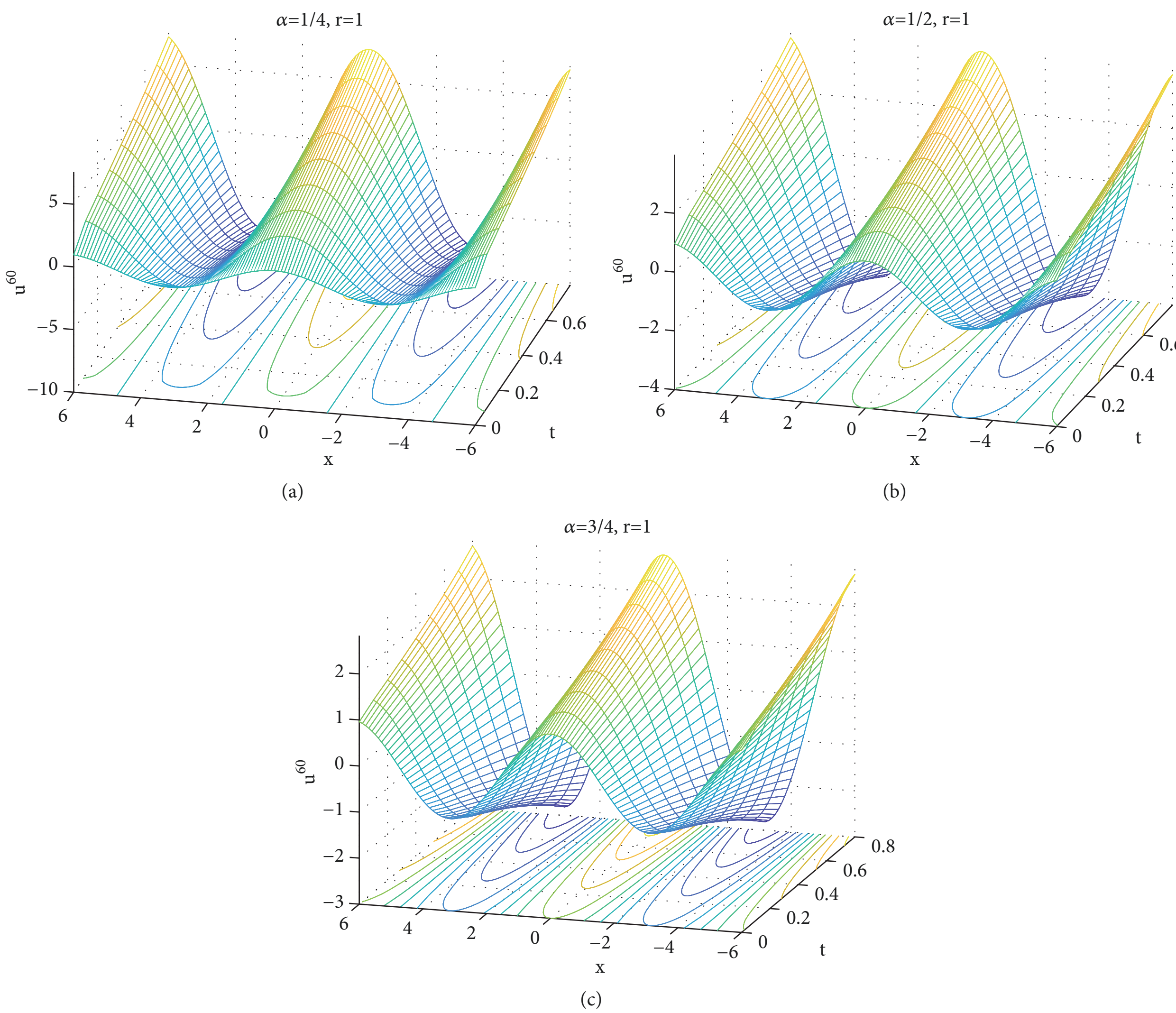

FIGURE 3

and the exact solution to (53)-(54) is

$$u(x,t) = \lim_{N\longrightarrow\infty} u^N(x,t) = e^x \mathrm{E}_\alpha\left((r-4+\sigma)\,t^\alpha\right). \tag{60}$$

Numerical Simulation. See Figure 4.

5. Nonlinear Time-Fractional Swift-Hohenberg Equation

In this subsection, we apply the iterative method introduced in Section 3 to nonlinear time-fractional S-H equation with such as e^x.

5.1. Nonlinear S-H Equation with Initial Value $u(x,0) = e^x$. Consider the following nonlinear S-H equation with dispersion:

$$\mathrm{D}^\alpha u + (1-r)\,u + 2u_{xx} + u_{xxxx} = u^\ell - (u_x)^\ell \qquad 0 < \alpha \le 1 \tag{61}$$

with initial value

$$u(x,0) = e^x. \tag{62}$$

Clearly, applying I^α on both sides of (61), then initial value problem (61)-(62) is equivalent to the following integral equation:

$$u(x,t) = f(x) + (\mathscr{L}u)(x,t) + (\mathscr{N}u)(x,t), \tag{63}$$

where

$$\begin{aligned} f(x) &= e^x, \\ \mathscr{L}(u) &= \mathrm{I}^\alpha\left((r-1)\,u - 2u_{xx} - u_{xxxx}\right), \\ \mathscr{N}(u) &= \mathrm{I}^\alpha\left(u^\ell - (u_x)^\ell\right). \end{aligned} \tag{64}$$

The solution to system (61)-(62) what we are looking for has the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{65}$$

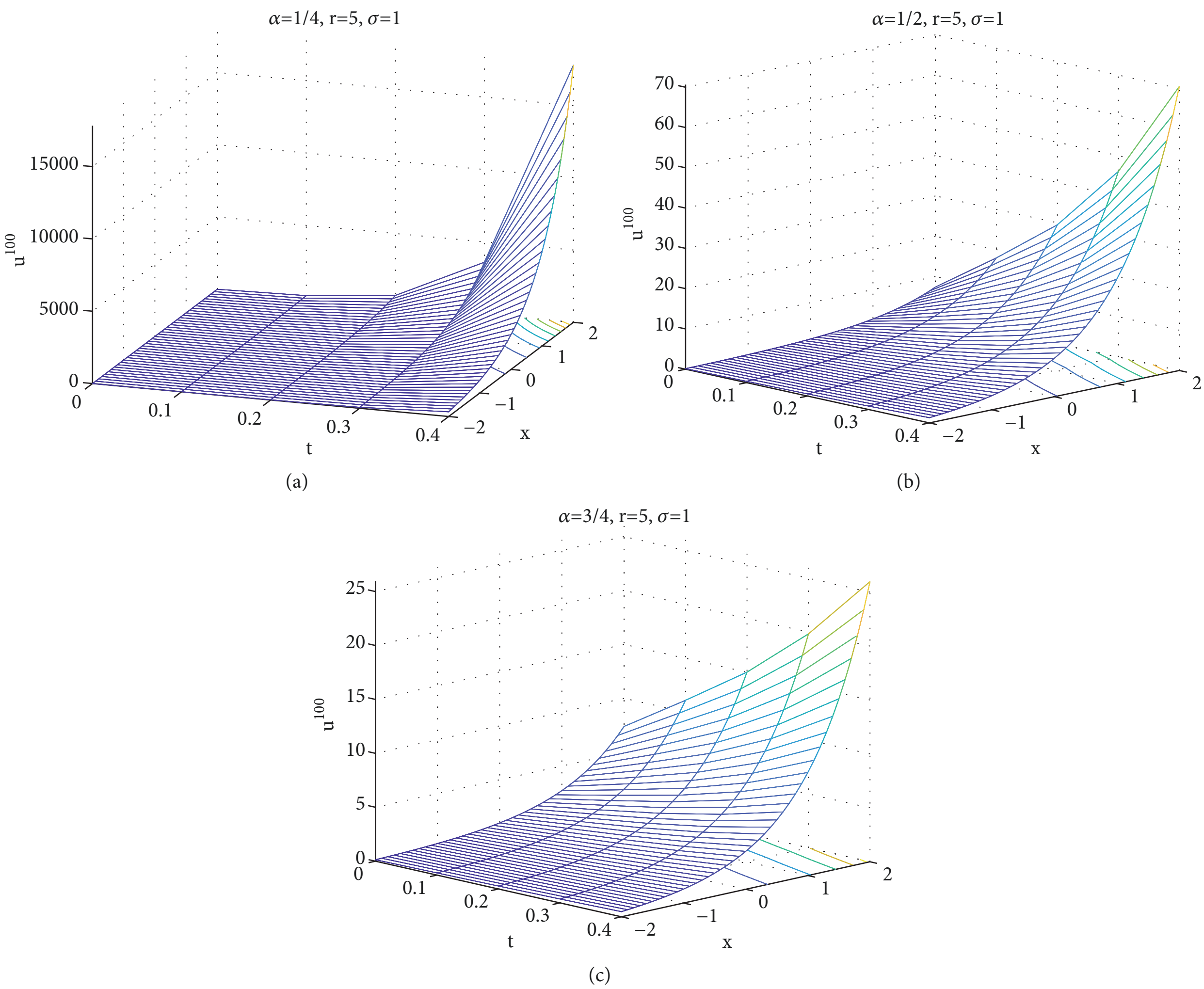

Figure 4

According to iterative scheme (9)-(10), one can obtain

$$
\begin{aligned}
u_0 &= f = e^x;\\
u_1 &= \mathscr{L}(u_0) + \mathscr{N}(u_0)\\
&= \left((r-1)u_0 - 2(u_0)_{xx} - (u_0)_{xxxx}\right)\mathrm{I}^{\alpha}1 + \left(u_0^{\ell} - \left((u_0)_x\right)^{\ell}\right)\mathrm{I}^{\alpha}1\\
&= (r-4)e^x \frac{1}{\Gamma(\alpha+1)}t^{\alpha};\\
u_2 &= \mathscr{L}(u_1) + \mathscr{N}(u_0+u_1) - \mathscr{N}(u_0)\\
&= \left((r-1)u_1 - 2(u_1)_{xx} - (u_1)_{xxxx}\right)\mathrm{I}^{\alpha}\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\\
&\quad + \mathrm{I}^{\alpha}\left((u_0+u_1)^2 - \left((u_0+u_1)_x\right)^2 - (u_0)^2 + \left((u_0)_x\right)^2\right)\\
&= (r-4)^2 e^x \frac{1}{\Gamma(2\alpha+1)}t^{2\alpha}\\
&\ \ \vdots\\
u_n &= \mathscr{L}(u_{n-1}) + \mathscr{N}(u_0+u_1+\cdots+u_{n-1}) - \mathscr{N}(u_0 + u_1 + \cdots + u_{n-2})\\
&= \left((r-1)u_{n-1} - 2(u_{n-1})_{xx} - (u_{n-1})_{xxxx}\right)\mathrm{I}^{\alpha} \cdot \frac{1}{\Gamma((n-1)\alpha+1)}t^{(n-1)\alpha} +\\
&\quad + \mathrm{I}^{\alpha}\left((u_0+\cdots+u_{n-1})^{\ell} - \left((u_0+\cdots+u_{n-1})_x\right)^{\ell} - (u_0+\cdots+u_{n-2})^{\ell} + \left((u_0+\cdots+u_{n-2})_x\right)^{\ell}\right)\\
&= (r-4)^n e^x \frac{1}{\Gamma(n\alpha+1)}t^{n\alpha} \quad (n = 2, 3, \cdots).
\end{aligned} \tag{66}
$$

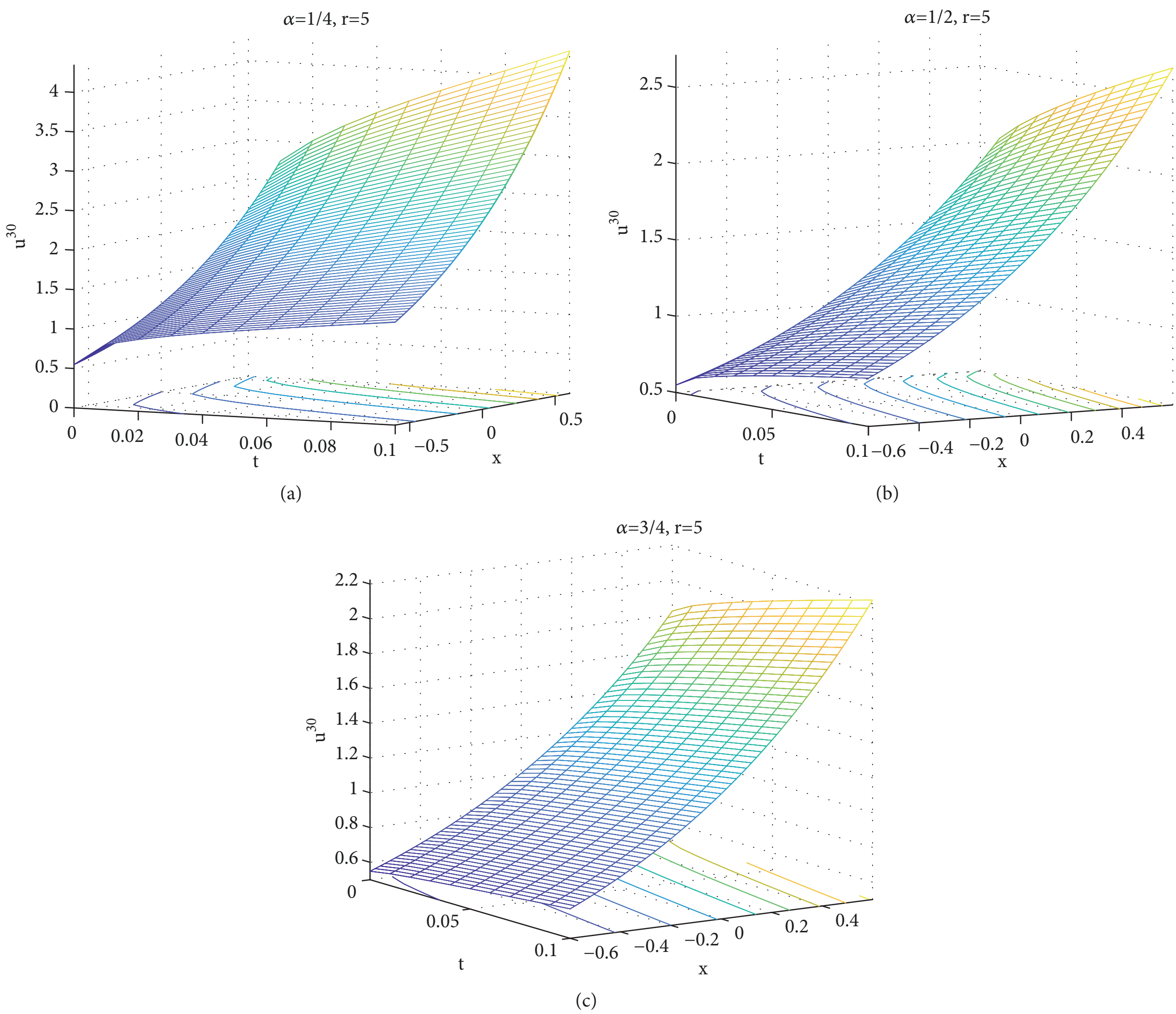

Figure 5

Hence the N-th approximate solution to (61)-(62) is

$$u^N(x,t) = e^x \sum_{n=0}^{N} (r-4)^n \frac{1}{\Gamma(n\alpha+1)} t^{n\alpha} \tag{67}$$

and the exact solution to (61)-(62) is

$$u(x,t) = \lim_{N\longrightarrow\infty} u^N(x,t) = e^x \mathrm{E}_\alpha\left((r-4)t^\alpha\right). \tag{68}$$

Numerical Simulation. See Figure 5.

5.2. Nonlinear Time-Fractional S-H Equation with Dispersion and Initial Value $u(x,0) = e^x$. Consider the following nonlinear S-H equation with dispersion:

$$\mathrm{D}^\alpha u + (1-r)u + 2u_{xx} - \sigma u_{xxx} + u_{xxxx} = u^\ell - (u_x)^\ell \quad 0 < \alpha \le 1 \tag{69}$$

with initial value

$$u(x,0) = e^x. \tag{70}$$

Clearly, applying I^α to both sides of (69), then initial value problem (69)-(70) is equivalent to the following integral equation:

$$u(x,t) = f(x) + (\mathscr{L}u)(x,t) + (\mathscr{N}u)(x,t), \tag{71}$$

where

$$\begin{aligned} f(x) &= e^x, \\ \mathscr{L}(u) &= \mathrm{I}^\alpha\left((r-1)u - 2u_{xx} + \sigma u_{xxx} - u_{xxxx}\right), \\ \mathscr{N}(u) &= \mathrm{I}^\alpha\left(u^\ell - (u_x)^\ell\right). \end{aligned} \tag{72}$$

The solution to system (69)-(70) what we are looking for has the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{73}$$

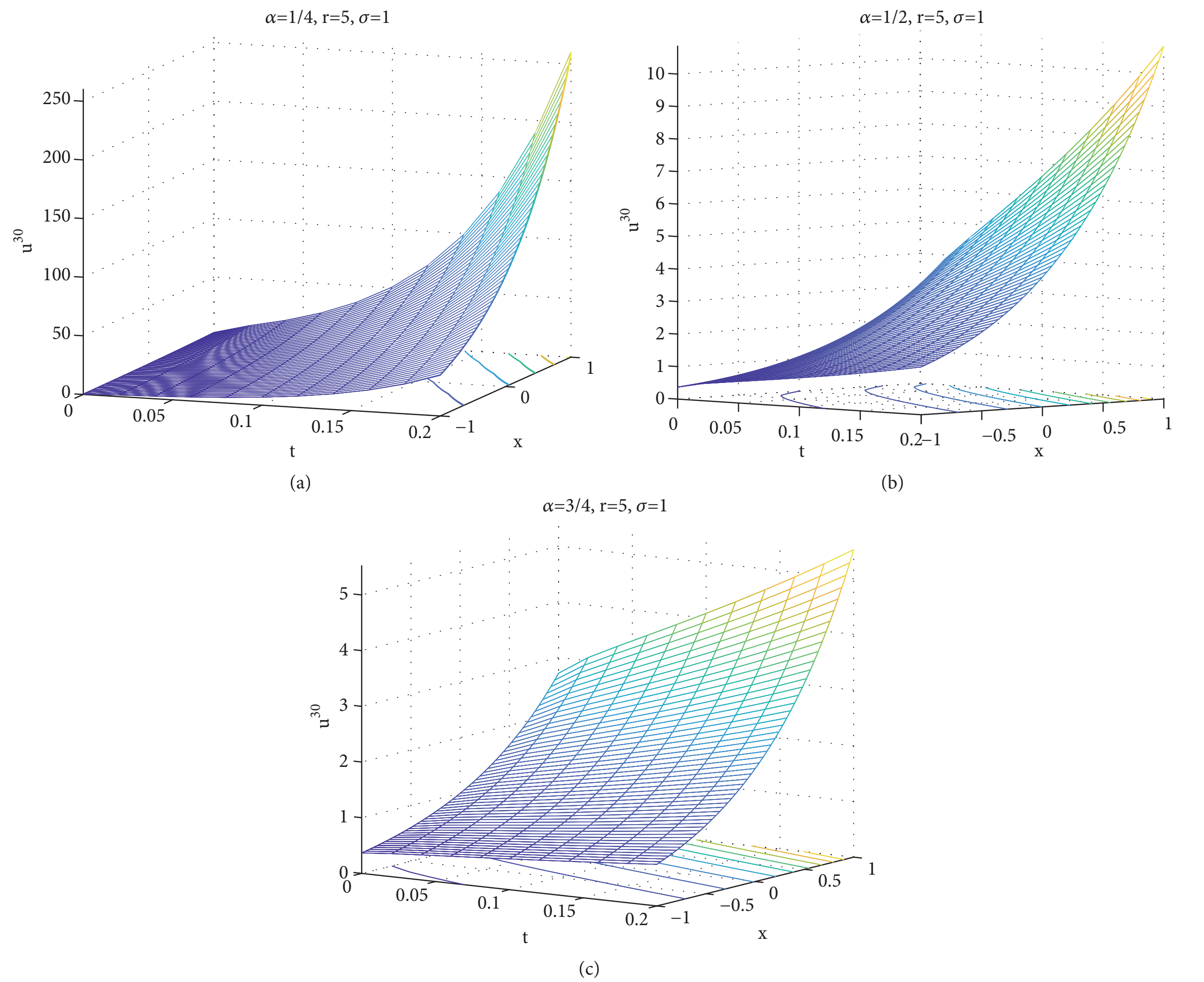

FIGURE 6

According to iterative scheme (9)-(10), one can obtain

$$\begin{aligned}
u_0 &= f = e^x;\\
u_1 &= \mathscr{L}(u_0) + \mathscr{N}(u_0)\\
&= \left((r-1)u_0 - 2(u_0)_{xx} + \sigma(u_0)_{xxx} - (u_0)_{xxxx}\right)\mathrm{I}^\alpha 1\\
&\quad + \mathrm{I}^\alpha\left(u_0^\ell - \left((u_0)_x\right)^\ell\right)\\
&= (r-4+\sigma)e^x \frac{1}{\Gamma(\alpha+1)}t^\alpha;\\
u_2 &= \mathscr{L}(u_1) + \mathscr{N}(u_0+u_1) - \mathscr{N}(u_0)\\
&= \left((r-1)u_1 - 2(u_1)_{xx} - (u_1)_{xxxx}\right)\mathrm{I}^\alpha \frac{1}{\Gamma(\alpha+1)}t^\alpha\\
&\quad + \mathrm{I}^\alpha\left((u_0+u_1)^\ell - \left((u_0+u_1)_x\right)^\ell - (u_0)^\ell\right.\\
&\quad \left. + \left((u_0)_x\right)^\ell\right)\\
&= (r-4+\sigma)^2 e^x \frac{1}{\Gamma(2\alpha+1)}t^{2\alpha}\\
&\ \ \vdots\\
u_n &= \mathscr{L}(u_{n-1}) + \mathscr{N}(u_0+u_1+\cdots+u_{n-1}) - \mathscr{N}(u_0\\
&\quad + u_1+\cdots+u_{n-2})\\
&= \left((r-1)u_{n-1} - 2(u_{n-1})_{xx} - (u_{n-1})_{xxxx}\right)\mathrm{I}^\alpha\\
&\quad \cdot \frac{1}{\Gamma((n-1)\alpha+1)}t^{(n-1)\alpha} +\\
&\quad + \mathrm{I}^\alpha\left((u_0+\cdots+u_{n-1})^\ell - \left((u_0+\cdots+u_{n-1})_x\right)^\ell\right.\\
&\quad \left. - (u_0+\cdots+u_{n-2})^\ell + \left((u_0+\cdots+u_{n-2})_x\right)^\ell\right)\\
&= (r-4+\sigma)^n e^x \frac{1}{\Gamma(n\alpha+1)}t^{n\alpha} \quad (n = 2,3,\cdots)
\end{aligned} \tag{74}$$

Hence the N-th approximate solution to (69)-(70) is

$$u^N(x,t) = e^x \sum_{n=0}^{N} (r-4+\sigma)^n \frac{1}{\Gamma(n\alpha+1)}t^{n\alpha} \tag{75}$$

and the exact solution to (61)-(70) is

$$u(x,t) = \lim_{N\longrightarrow\infty} u^N(x,t) = e^x \mathrm{E}_\alpha\left((r-4+\sigma)t^\alpha\right). \tag{76}$$

Numerical Simulation. See Figure 6.

6. Concluding Remark

This paper introduce an iterative method which has considerable power in constructing approximated solutions and even exact solutions to time-fractional differential equation. We take the linear and nonlinear Swift-Hohenberg equations with different initial conditions to illustrate the effectiveness of such method.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

[1] B. Ross, "The development of fractional calculus 1695–1900," *Historia Mathematica*, vol. 4, no. 1, pp. 75–89, 1977.

[2] G. W. Leibniz, "Letter from hanover, Germany, deptember 30, 1695 to ga l' hospital," *JLeibnizen Mathematische Schriften*, vol. 2, pp. 301-302, 1849.

[3] V. V. Uchaikin, *Fractional Derivatives for Physicists and Engineers. Volume I. Nonlinear Physical Science. Background and Theory*, Higher Education Press, Beijing, China; Springer, Heidelberg, Germany, 2013.

[4] V. V. Uchaikin, *Fractional Derivatives for Physicists and Engineers. Volume II. Nonlinear Physical Science. Applications*, Higher Education Press, Beijing, China; Springer, Heidelberg, Germany, 2013.

[5] A. El-Ajou, O. Abu Arqub, Z. Al Zhour, and S. Momani, "New results on fractional power series: theories and applications," *Entropy*, vol. 15, no. 12, pp. 5305–5323, 2013.

[6] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier Science B.V., Amsterdam, the Netherlands, 2006.

[7] M. D. Ortigueira and J. A. Tenreiro Machado, "What is a fractional derivative?" *Journal of Computational Physics*, vol. 293, pp. 4–13, 2015.

[8] U. N. Katugampola, "Correction to "What is a fractional derivative?" by Ortigueira and Machado [Journal of Computational Physics, Volume 293, 15 July 2015, Pages 4–13. Special issue on fractional PDEs]," *Journal of Computational Physics*, vol. 321, pp. 1255–1257, 2016.

[9] L. K. B. Kuroda, A. V. Gomes, R. Tavoni, P. F. D. A. Mancera, N. Varalta, and R. D. F. Camargo, "Unexpected behavior of Caputo fractional derivative," *Computational & Applied Mathematics*, vol. 36, no. 3, pp. 1173–1183, 2017.

[10] S. Liao, "On the homotopy analysis method for nonlinear problems," *Applied Mathematics and Computation*, vol. 147, no. 2, pp. 499–513, 2004.

[11] I. Hashim, O. Abdulaziz, and S. Momani, "Homotopy analysis method for fractional IVPs," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 3, pp. 674–684, 2009.

[12] Sh. J. Liao, *The proposed homotopy analysis technique for the solution of nonlinear problems [Ph.D. thesis]*, Shanghai Jiao Tong University, 1992.

[13] O. Abu Arqub, "Series solution of fuzzy differential equations under strongly generalized differentiability," *Journal of Advanced Research in Applied Mathematics*, vol. 5, no. 1, pp. 31–52, 2013.

[14] A. El-Ajou, O. Abu Arqub, S. Momani, D. Baleanu, and A. Alsaedi, "A novel expansion iterative method for solving linear partial differential equations of fractional order," *Applied Mathematics and Computation*, vol. 257, pp. 119–133, 2015.

[15] F. Xu, Y. Gao, X. Yang, and H. Zhang, "Construction of fractional power series solutions to fractional Boussinesq equations using residual power series method," *Mathematical Problems in Engineering*, vol. 2016, Article ID 5492535, 15 pages, 2016.

[16] W. Yin, F. Xu, W. Zhang, and Y. Gao, "Asymptotic expansion of the solutions to time-space fractional Kuramoto-Sivashinsky equations," *Advances in Mathematical Physics*, vol. 2016, Article ID 4632163, 9 pages, 2016.

[17] A. Arikoglu and I. Ozkol, "Solution of fractional differential equations by using differential transform method," *Chaos, Solitons & Fractals*, vol. 34, no. 5, pp. 1473–1481, 2007.

[18] Z. Odibat, S. Momani, and V. S. Erturk, "Generalized differential transform method: application to differential equations of fractional order," *Applied Mathematics and Computation*, vol. 197, no. 2, pp. 467–477, 2008.

[19] K. Mosaleheh and A. Vakilzadeh, "The Laplace transform method for linear ordinary differential equations of fractional order," *Journal of Mathematical Extension*, vol. 2, no. 1-2, pp. 93–102, 154, 2007-2008.

[20] J.-H. He, "Homotopy perturbation technique," *Computer Methods Applied Mechanics and Engineering*, vol. 178, no. 3-4, pp. 257–262, 1999.

[21] S. Momani and Z. Odibat, "Homotopy perturbation method for nonlinear partial differential equations of fractional order," *Physics Letters A*, vol. 365, no. 5-6, pp. 345–350, 2007.

[22] H. Singh, "Approximate solution of fractional vibration equation using Jacobi polynomials," *Applied Mathematics and Computation*, vol. 317, pp. 85–100, 2018.

[23] M. M. Khader and R. T. Alqahtani, "Approximate solution for system of fractional non-linear dynamical marriage model using Bernstein polynomials," *Journal of Nonlinear Sciences and Applications*, vol. 10, no. 3, pp. 865–873, 2017.

[24] E. Akbari Kojabad and S. Rezapour, "Approximate solutions of a sum-type fractional integro-differential equation by using Chebyshev and Legendre polynomials," *Advances in Difference Equations*, vol. 2017, article 351, 18 pages, 2017.

[25] V. Daftardar-Gejji and H. Jafari, "An iterative method for solving nonlinear functional equations," *Journal of Mathematical Analysis and Applications*, vol. 316, no. 2, pp. 753–763, 2006.

[26] M. A. Noor, K. I. Noor, S. T. Mohyud-Din, and A. Shabbir, "An iterative method with cubic convergence for nonlinear equations," *Applied Mathematics and Computation*, vol. 183, no. 2, pp. 1249–1255, 2006.

[27] S. Bhalekar and V. Daftardar-Gejji, "New iterative method: application to partial differential equations," *Applied Mathematics and Computation*, vol. 203, no. 2, pp. 778–783, 2008.

[28] V. Daftardar-Gejji and S. Bhalekar, "Solving fractional boundary value problems with Dirichlet boundary conditions using a new iterative method," *Computers & Mathematics with Applications*, vol. 59, no. 5, pp. 1801–1809, 2010.

[29] C. D. Dhaigude and V. R. Nikam, "Solution of fractional partial differential equations using iterative method," *Fractional Calculus and Applied Analysis*, vol. 15, no. 4, pp. 684–699, 2012.

[30] F. Xu, Y. Gao, and W. Zhang, "Construction of analytic solution for time-fractional boussinesq equation using iterative method," *Advances in Mathematical Physics*, vol. 2015, Article ID 506140, 7 pages, 2015.

[31] R. Figueiredo Camargo, E. Capelas de Oliveira, and J. Vaz, "On the generalized Mittag-Leffler function and its application in a fractional telegraph equation," *Mathematical Physics, Analysis and Geometry*, vol. 15, no. 1, pp. 1–16, 2012.

[32] R. Gorenflo, A. A. Kilbas, F. Mainardi, and S. V. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications*, Springer Monographs in Mathematics, Springer, Heidelberg, Germany, 2014.

[33] J. B. Swift and P. C. Hohenberg, "Hydrodynamic fluctuations at the convective instability," *Physical Review A: Atomic, Molecular and Optical Physics*, vol. 15, no. 1, article 319, 1977.

[34] B. Braaksma, G. Iooss, and L. Stolovitch, "Proof of quasipatterns for the Swift-Hohenberg equation," *Communications in Mathematical Physics*, vol. 353, no. 1, pp. 37–67, 2017.

[35] P. N. Ryabov and N. A. Kudryashov, "Nonlinear waves described by the generalized Swift-Hohenberg equation," *Journal of Physics: Conference Series*, vol. 788, p. 012032, 2017.

[36] N. A. Khan, N.-U. Khan, M. Ayaz, and A. Mahmood, "Analytical methods for solving the time-fractional Swift-Hohenberg (S-H) equation," *Computers & Mathematics with Applications*, vol. 61, no. 8, pp. 2182–2185, 2011.

[37] K. Vishal, S. Kumar, and S. Das, "Application of homotopy analysis method for fractional Swift Hohenberg equation—revisited," *Applied Mathematical Modelling*, vol. 36, no. 8, pp. 3630–3637, 2012.

[38] K. Vishal, S. Das, S. H. Ong, and P. Ghosh, "On the solutions of fractional Swift Hohenberg equation with dispersion," *Applied Mathematics and Computation*, vol. 219, no. 11, pp. 5792–5801, 2013.

[39] M. Merdan, "A numeric-analytic method for time-fractional Swift-Hohenberg (S-H) equation with modified Riemann-Liouville derivative," *Applied Mathematical Modelling: Simulation and Computation for Engineering and Environmental Systems*, vol. 37, no. 6, pp. 4224–4231, 2013.

[40] S. Das and K. Vishal, "Homotopy analysis method for fractional Swift-Hohenberg equation," in *Advances in the Homotopy Analysis Method*, pp. 291–308, World Scientific Publishing, Hackensack, NJ, USA, 2014.

[41] N. A. Kudryashov and D. I. Sinelshchikov, "Exact solutions of the Swift-Hohenberg equation with dispersion," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 1, pp. 26–34, 2012.