Research Article

An Iterative Method for Time-Fractional Swift-Hohenberg Equation

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Abstract

We study a type of iterative method and apply it to time-fractional Swift-Hohenberg equation with initial value. Using this iterative method, we obtain the approximate analytic solutions with numerical figures to initial value problems, which indicates that such iterative method is effective and simple in constructing approximate solutions to Cauchy problems of time-fractional differential equations.

1. Introduction

In 1695, L’Hopital wrote a letter to Leibniz, and he proposed a problem: “What is the result of $\frac{d^n}{dx^n} y$ if $n = 1/2$?” Leibniz answered to L’Hopital “$d^{1/2} x$ will be equal to $x \sqrt{dx : x}$. This is an apparent paradox, from which, one day useful consequences will be drawn.” [1, 2] Later, as the development of mathematics, especially, the theory of operator, researchers started to have a new recognition of the fractional derivative. They found the fractional derivative has a wide applications in many fields, such as physics, chemistry, and many other sciences [3, 4]. It should be emphasised that the fractional derivative is defined by integral and it is a nonlocal operator with a singular kernel; hence it can provide an excellent instrument for description of memory and hereditary properties of various physical processes. For example, half-order derivatives and integrals proved to be more useful for the formulation of certain electrochemical problems than the classical models [5]. However, the definition of fractional derivative has not been unified; there are many kinds of fractional integral and fractional derivative, such as in the sense of Riemann-Liouville, Caputo, Riesz, and Weyl [6–9]. The Riemann-Liouville fractional integral and the Caputo fractional derivative are the most commonly used.

For a fractional system with respect to some initial or boundary conditions, one of the fundamental problems is naturally that what is the exact solution or approximate solution to such system. Solving fractional system is usually more difficult than the classical system, for its operator is defined by integral. Luckily, there are some different effective methods which have been developed to construct approximate solutions of fractional systems and even obtain the exact solutions such as the homotopy analysis method [10–12], the residual power series method [13–16], the differential transform method [17, 18], the Laplace transform method [19], and the perturbation method [20, 21]. In addition, using polynomials to approximate the fractional system is an effective method as well, such as Jacobi polynomials [22], Bernstein polynomials [23], and Chebyshev and Legendre polynomials [24]. In this paper, we introduce a type of iterative method, based on decomposing the nonlinearity term, for solving a class of functional equations [25–30].

Outline of Paper. In Section 2, we introduce some necessary concepts and lemmas on fractional differential equations. In Section 3, a type of iterative method for solving a class of functional equation is presented. Also, we obtain the convergence analysis of this iterative method. In Sections 4 and 5, we take the linear time-fractional S-H equations, including the term of dispersion, with respect to differential initial conditions as examples to illustrate the strong power of such iterative method, respectively.
2. Notations of Fractional Calculus

In subsection, we introduce some concepts and lemmas we need in this paper, such as the Gamma function, the Mittag-Leffler function, the Riemann-Liouville fractional integral, and the Caputo fractional derivative. It should be emphasised that there are many kinds of fractional integral and fractional derivative, such as in the sense of Riemann-Liouville, Caputo, Riesz, and Weyl [6–9]. The Riemann-Liouville fractional integral and Caputo fractional derivative are the most commonly used version.

Definition 1. The Gamma function is defined by [6, 29, 30]
\[ \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt \quad \text{Re}(x) > 0. \] (1)

Definition 2. The Mittag-Leffler function is defined by [31, 32]
\[ E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)} \quad \alpha > 0. \] (2)

Definition 3. The (left sided) Riemann-Liouville fractional integral of order \( \beta(\beta > 0) \) of a function \( u(t, x) \in C_p(p \geq -1) \) is denoted by \( I^\beta u(t, x) \) (with respect to \( t \)) and defined as [29, 30]
\[ I^\beta u(t, x) = \frac{1}{\Gamma(\beta)} \int_0^t (t-r)^{\beta-1} u(r, x) \, dr. \] (3)

Definition 4. The (left sided) Caputo fractional derivative of order \( \beta(\beta > 0) \) of a function \( u(t, x) \in C_p^m \) is denoted by \( D^\beta u(t, x) \) (with respect to \( t \)) and defined as [29, 30]
\[ D^\beta u(t, x) = \begin{cases} \frac{\partial^m u(t, x)}{\partial t^m} & \beta = m \in \mathbb{N}^* \\ \frac{\partial^{m-\beta} u(t, x)}{\partial t^{m-\beta}} & m-1 < \beta < m. \end{cases} \] (4)

Lemma 5. For any \( m-1 < \beta \leq m \in \mathbb{N}^* \), one has [6–9, 29, 30]
\[ I^\beta D^\beta u(t, x) = u(t, x) - \sum_{k=0}^{m-1} \frac{\partial^k u(0, x)}{k!} t^k. \] (5)

Lemma 6. For any \( \beta, \gamma \), one has [6–9, 29]
\[ I^\beta I^\gamma u(t, x) = \frac{\Gamma(\gamma+1)}{\Gamma(\beta+\gamma+1)} I^{\beta+\gamma} u(t, x). \] (6)

3. A Type of Iterative Method

In this section, we introduce a generalized iterative method for solving a class of functional equation (7) (see below). Some more details about this type of iterative method could be found in [25–30] and the references therein.

Now we state this iterative method as the following lemma together convergence analysis.

Lemma 7. Consider the nonlinear functional equation
\[ u(t, x) = f(t, x) + \mathcal{L}(u(t, x)) + \mathcal{N}(u(t, x)), \] (7)

where \( u(t, x) \) is an unknown function, i.e., solution of functional equation (7), \( f(t, x) \) is a known function, \( (t, x) \in D = \{(t, x) : t \in \mathbb{R}, x \in \mathbb{R}^n, n \in \mathbb{N}\} \), and \( \mathcal{L} \) and \( \mathcal{N} \) are linear and nonlinear operators from a Banach space \( B \) to itself. We are looking for a solution of functional equation (7) having a series form
\[ u(t, x) = \sum_{n=0}^{\infty} u_n(t, x) = u_0 + u_1 + u_2 + \cdots, \] (8)

where \( u_n \) are defined by letting
\[ u_0 = f, \]
\[ u_1 = \mathcal{L}(u_0) + \mathcal{N}(u_0), \]
\[ u_{m+1} = \mathcal{L}(u_m) + \left\{ \mathcal{N}\left( \sum_{n=0}^{m} u_n \right) - \mathcal{N}\left( \sum_{n=0}^{m-1} u_n \right) \right\}, \] (10)

Furthermore, if the operator \( \mathcal{L} \) and \( \mathcal{N} \) are contractive, then the series \( \sum_{n=0}^{\infty} u_n(t, x) \) converges absolutely and uniformly.

Proof. Obviously, the nonlinear operator \( \mathcal{N} \) can be decomposed
\[ \mathcal{N}(u(t, x)) = \mathcal{N}\left( \sum_{n=0}^{\infty} u_n(t, x) \right) = \mathcal{N}(u_0) \]
\[ + \sum_{n=1}^{\infty} \left\{ \mathcal{N}\left( \sum_{j=0}^{n} u_j \right) - \mathcal{N}\left( \sum_{j=0}^{n-1} u_j \right) \right\} . \] (11)

Similarly, linear operator \( \mathcal{L} \) can also be decomposed
\[ \mathcal{L}(u(t, x)) = \mathcal{L}\left( \sum_{n=0}^{\infty} u_n(t, x) \right) = \mathcal{L}(u_0) \]
\[ + \sum_{n=1}^{\infty} \left\{ \mathcal{L}\left( \sum_{j=0}^{n} u_j \right) - \mathcal{L}\left( \sum_{j=0}^{n-1} u_j \right) \right\} . \] (12)

Set
\[ \mathcal{M}(u(t, x)) = \mathcal{L}(u(t, x)) + \mathcal{N}(u(t, x)). \] (13)

then
\[ \mathcal{M}(u(t, x)) = \mathcal{M}(u_0) \]
\[ + \sum_{n=1}^{\infty} \left\{ \mathcal{M}\left( \sum_{j=0}^{n} u_j \right) - \mathcal{M}\left( \sum_{j=0}^{n-1} u_j \right) \right\} . \] (14)
Define the following recurrence equations:
\[ u_0 = f, \]
\[ u_t = \mathcal{M} (u_t), \]
\[ u_{m+1} = \mathcal{M} \left( \sum_{n=0}^{m} u_n \right) - \mathcal{M} \left( \sum_{n=0}^{m-1} u_n \right), \quad m = 1, 2, \ldots \] (15)

Since operators \( \mathcal{L} \) and \( \mathcal{N} \) are contractive, then \( \mathcal{M} \) is also contractive; i.e., there exists a constant \( 0 < K < 1 \), such that
\[ \| \mathcal{M} (v_1) - \mathcal{M} (v_2) \| \leq K \| v_1 - v_2 \|, \quad \forall v_1, v_2 \in B, \] (16)
where \( \| \cdot \| \) denotes the usual norm on Banach space \( B \). What is more, for \( u_{m+1} \), one can obtain
\[ \| u_{m+1} \| = \| \mathcal{M} \left( \sum_{n=0}^{m} u_n \right) - \mathcal{M} \left( \sum_{n=0}^{m-1} u_n \right) \| \leq K \| u_m \| \] (17)
\[ \leq K^{m+1} \| u_0 \|. \]

Since \( 0 < K < 1 \), the series \( \sum_{m=0}^{\infty} K^{m+1} \| u_0 \| \) converges absolutely as well as uniformly.

According to the Weierstrass M-test, one can obtain that the series \( \sum_{m=0}^{\infty} u_t \) converges absolutely as well as uniformly.

**4. Linear Swift-Hohenberg Equation**

The Swift-Hohenberg (for short S-H) equation
\[ \partial_t u = ru - \left( 1 + \nabla^2 \right)^2 u + N(u) \] (18)
is a model pattern-forming equation which was derived from the equations for thermal convection by Jack Swift and Pierre Hohenberg [33]. Here \( u = u(x, t) \) is a scalar function defined on the line or the plane, \( r \) is a real bifurcation parameter, and \( N(u) \) is some smooth nonlinearity. The S-H equation plays an important role in pattern formation theory. In [34], Braaksma et al. proved the existence of quasipatterns for the S-H equation. Also, wave process described by the S-H equation is important as well. For example, it describes the patterns inside thin vibrated granular layers, the mechanism of the amplitude of optical electric field inside the cavity, and so on [35].

Fractional S-H equation
\[ D^\alpha u = -2u_{xx} - u_{xxxx} - (1 - r) u - u^3 \quad 0 < \alpha \leq 1 \] (19)was firstly introduced in [36], and Khan et al. obtained analytical approximation of this equation. Later, Vishal et al. in [37] constructed the approximate analytic solution with respect initial value \( u(x, 0) = 1/10 \sin(\pi x/l) \) using the homotopy analysis method. Furthermore, Vishal et al. considered the time-fractional S-H equation with dispersion [38]
\[ D^\alpha u = ru - \left( 1 + \partial_{xx} \right)^2 u + \sigma \partial_{xxx} u + 2u^2 - u^3 \quad 0 < \alpha \leq 1 \] (20)and obtained the approximate analytic solution. Here \( \sigma \) is the dispersive parameter. Lately, in [39], Merdan applied the fractional variational iteration method to obtain the approximate solution to time-fractional S-H equation with respect to initial condition \( u(x, 0) = 1/10 \sin(\pi x/l) \). Also, homotopy analysis method is valid for S-H equation as well [40].

In this subsection, we apply the iterative method introduced in Section 3 to linear time-fractional S-H equation with different initial values, such as \( e^x \), \( \sin x \), and \( \cos x \).

**4.1. Linear Time-Fractional S-H Equation with Initial Value**

\[ u(x, 0) = e^x. \] (22)

Clearly, applying \( i^\alpha \) to both sides of (21), then initial value problem (21)-(22) is equivalent to the following integral equation:
\[ u(x, t) = f(x) + (\mathcal{L} u)(x, t) + (\mathcal{N} u)(x, t), \] (23)
where
\[ f(x) = e^x, \]
\[ \mathcal{L} (u) = i^\alpha \left( (r - 1) u - 2u_{xx} - u_{xxxx} \right), \]
\[ \mathcal{N} (u) = 0. \]

The solution to system (21)-(22) what we are looking for has the form
\[ u(x, t) = \sum_{n=0}^{\infty} u_n(x, t). \] (25)

According to iterative scheme (9)-(10), one can obtain
\[ u_0 = f = e^x; \]
\[ u_1 = \mathcal{L} (u_0) + \mathcal{N} (u_0) = ((r - 1) u_0 - 2 (u_0)_{xx} - (u_0)_{xxxx}) i^\alpha 1 \]
\[ = (r - 4) e^x \frac{1}{\Gamma (\alpha + 1)} i^\alpha; \]
\[ u_2 = \mathcal{L} (u_1) + \mathcal{N} (u_0 + u_1) - \mathcal{N} (u_0) = ((r - 1) u_1 - 2 (u_1)_{xx} - (u_1)_{xxxx}) i^\alpha 1 \frac{1}{\Gamma (\alpha + 1)} i^\alpha \]
\[ = (r - 4)^2 e^x \frac{1}{\Gamma (2\alpha + 1)} i^{2\alpha} \]
\[ \vdots \]
\[ u_n = \mathcal{L} (u_{n-1}) + \mathcal{N} (u_0 + u_1 + \cdots + u_{n-1}) - \mathcal{N} (u_0 + u_1 + \cdots + u_{n-1}) \]
\[ = \mathcal{N} (u_0 + u_1 + \cdots + u_{n-1}) \]
\[ \begin{align*} 
&= ((r-1)u_{n-1} - 2(u_{n-1})_{xx} - (u_{n-1})_{xxxx}) \Gamma \alpha \cdot \frac{1}{\Gamma((n-1)\alpha+1)} \\
&= (r-4)^n e^x \frac{1}{\Gamma(n\alpha+1)} t^{\rho_x} \quad (n = 2, 3, \cdots) 
\end{align*} \]

Hence the \( N \)-th approximate solution to (21)-(22) is

\[ u_N(x, t) = e^x \sum_{n=0}^{N} (r-4)^n \frac{1}{\Gamma(n\alpha+1)} t^{\rho_x} \] (27)

and the exact solution to (21)-(22) is

\[ u(x, t) = \lim_{N \to \infty} u_N(x, t) = e^x E_{\alpha} \left( (r-4) t^{\rho_x} \right). \] (28)

### 4.2. Linear Time-Fractional S-H Equation with Initial Value

\( u(x, 0) = \sin x \). Consider the following linear time-fractional S-H equation:

\[ D^\alpha u + (1-r) u + 2u_{xx} + u_{xxxx} = 0 \quad 0 < \alpha \leq 1 \] (29)

with initial condition

\[ u(x, 0) = \sin x. \] (30)

Applying \( I^\alpha \) to both sides of (29), then initial value problem (29)-(30) is equivalent to the following integral equation:

\[ u(x, t) = f(x) + (\mathcal{L}u)(x, t) + (\mathcal{N}u)(x, t), \] (31)

where

\[ f(x) = \sin x, \]

\[ \mathcal{L} (u) = I^\alpha \left( (r-1) u - 2u_{xx} - u_{xxxx} \right), \]

\[ \mathcal{N} (u) = 0. \] (32)
The solution of system (29)-(30) what we are looking for has the form
\[ u(x, t) = \sum_{n=0}^{\infty} u_n(x, t). \]  
(33)

We shall distinguish the following two cases.

**Case 1** \((r = 0)\). According to iterative scheme (9)-(10), we have
\[ u_0 = f = \sin x; \]
\[ u_1 = \mathcal{L}(u_0) + \mathcal{N}(u_0) \]
\[ = ((r - 1) u_0 - 2 (u_0)_{xx} - (u_0)_{xxxx}) \Gamma^a 1 \]
\[ = 0; \]
\[ u_n = 0, \quad n = 2, 3, \ldots. \]

Hence the solution to (29)-(30) is
\[ u(x, t) = \sin x. \]  
(35)

**Case 2** \((r \neq 0)\). By iterative scheme (9)-(10), we have
\[ u_0 = f = \sin x; \]
\[ u_1 = \mathcal{L}(u_0) + \mathcal{N}(u_0) \]
\[ = ((r - 1) u_0 - 2 (u_0)_{xx} - (u_0)_{xxxx}) \Gamma^a 1 \]
\[ = (r - 1) \sin x \frac{1}{\Gamma(\alpha + 1)} t^\alpha; \]
\[ u_2 = \mathcal{L}(u_1) + \mathcal{N}(u_0 + u_1) - \mathcal{N}(u_0) \]
\[ = ((r - 1)^2 + (r - 1) + 1) \sin x \frac{1}{\Gamma(2\alpha + 1)} t^{2\alpha}; \]
\[ u_3 = \mathcal{L}(u_2) + \mathcal{N}(u_0 + u_1 + u_2) - \mathcal{N}(u_0 + u_1) \]
\[ = ((r - 1)^3 + (r - 1)^2 + (r - 1) + 1) \sin x \frac{1}{\Gamma(3\alpha + 1)} t^{3\alpha}; \]
\[ u_4 = \mathcal{L}(u_3) + \mathcal{N}(u_0 + u_1 + u_2 + u_3) \]
\[ - \mathcal{N}(u_0 + u_1 + u_2 + u_3) \]
\[ = ((r - 1)^4 + (r - 1)^3 + (r - 1)^2 + (r - 1) + 1) \sin x \frac{1}{\Gamma(4\alpha + 1)} t^{4\alpha}. \]  
(34)

Generally, for \(n = 1, 2, \ldots\), by induction, one can derive that
\[ u_n = \mathcal{L}(u_{n-1}) + \mathcal{N}(u_0 + u_1 + \cdots + u_{n-1}) - \mathcal{N}(u_0 + u_1 + \cdots + u_{n-2}) \]
\[ = ((r - 1) u_{n-1} - 2 (u_{n-1})_{xx} - (u_{n-1})_{xxxx}) \Gamma^a \]
\[ - \frac{1}{\Gamma((n - 1)\alpha + 1)} t^{(n-1)\alpha} \]
\[ = (r - 1)^n + (r - 1)^{n-1} + \cdots + (r - 1)^2 + (r - 1) \]
\[ + 1 \sin x \frac{1}{\Gamma((n - 1)\alpha + 1)} t^{(n-1)\alpha} \]
\[ = \frac{1}{r} (1 - (r - 1)^{n+1}) \sin x \frac{1}{\Gamma(n\alpha + 1)} t^{n\alpha}. \]  
(37)

Hence the \(N\)-th approximate solution to (29)-(30) is
\[ u^N(x, t) = -\frac{1}{r} \sin x \sum_{n=0}^{N} (1 - (r - 1)^{n+1}) \frac{1}{\Gamma(n\alpha + 1)} t^{n\alpha} \]  
(38)

and the exact solution to (29)-(30) is
\[ u(x, t) = \lim_{N \to \infty} u^N(x, t) \]
\[ = \frac{1}{r} \sin x \lim_{N \to \infty} \sum_{n=0}^{N} (1 - (r - 1)^{n+1}) \frac{1}{\Gamma(n\alpha + 1)} t^{n\alpha}. \]  
(39)

As a consequence, the solution of (29)-(30) is
\[ u(x, t) \]
\[ = \begin{cases} \sin x & r = 0; \\ \frac{1}{r} \sin x \sum_{n=0}^{N} (1 - (r - 1)^{n+1}) \frac{1}{\Gamma(n\alpha + 1)} t^{n\alpha} & r \neq 0. \end{cases} \]  
(40)

**Numerical Simulation.** See Figure 2.

### 4.3. Linear Time-Fractional S-H Equation with Initial Value

Consider the following linear time-fractional S-H equation:
\[ D^\alpha u + (1 - r) u + 2u_{xx} + u_{xxxx} = 0 \quad 0 < \alpha \leq 1 \]  
(41)

with initial condition
\[ u(x, 0) = \cos x. \]  
(42)

Applying \(\mathcal{L}\) to both sides of (41), then initial value problem (41)-(42) is equivalent to the following integral equation:
\[ u(x, t) = f(x) + (\mathcal{L}u)(x, t) + (\mathcal{N}u)(x, t), \]  
(43)

where
\[ f(x) = \sin x, \]
\[ \mathcal{L}u = \mathcal{L}((r - 1) u - 2u_{xx} + u_{xxxx}), \]  
(44)
\[ \mathcal{N}u = 0. \]
The solution to system (41)-(42) what we are looking for has the form

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t).$$

We shall distinguish the following two cases.

**Case 1** ($r = 0$). According to iterative scheme (9)-(10), we have

$$u_0 = f = \cos x;$$

$$u_1 = \mathcal{L}(u_0) + \mathcal{N}(u_0)$$

$$= (-u_0 - 2 (u_0)_{xx} - (u_0)_{xxxx}) I^\alpha 1$$

$$= 0;$$

$$u_n = 0, \quad n = 2, 3, \ldots.$$  \hspace{1cm} (46)

Hence the solution to (41)-(42) is

$$u(x, t) = \cos x.$$ \hspace{1cm} (47)

**Case 2** ($r \neq 0$). According to iterative scheme (9)-(10), we have

$$u_0 = f = \cos x;$$

$$u_1 = \mathcal{L}(u_0) + \mathcal{N}(u_0)$$

$$= ((r-1) u_0 - 2 (u_0)_{xx} - (u_0)_{xxxx}) I^\alpha 1$$

$$= ((r-1) + 1) \cos x \frac{1}{\Gamma(\alpha+1)} t^\alpha;$$

$$u_2 = \mathcal{L}(u_1) + \mathcal{N}(u_0 + u_1) - \mathcal{N}(u_0)$$

$$= ((r-1) u_1 - 2 (u_1)_{xx} - (u_1)_{xxxx}) I^\alpha 1$$

$$= ((r-1)^2 + (r-1) + 1) \cos x \frac{1}{\Gamma(2\alpha+1)} t^{2\alpha};$$

$$u_3 = \mathcal{L}(u_2) + \mathcal{N}(u_0 + u_1 + u_2) - \mathcal{N}(u_0 + u_1)$$

$$= ((r-1) u_2 - 2 (u_2)_{xx} - (u_2)_{xxxx}) I^\alpha 1$$

$$= ((r-1)^3 + (r-1)^2 + (r-1) + 1) \cos x \frac{1}{\Gamma(3\alpha+1)} t^{3\alpha}.$$
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\[ u_4 = \mathcal{L}(u_3) + \mathcal{N}(u_0 + u_1 + u_2 + u_3) - \mathcal{N}(u_0 + u_1 + \cdots + u_{n-2}) = ((r-1)u_{n-1} - 2(u_{n-1})_{xx} - (u_{n-1})_{xxxx})I^\alpha \]

\[ = \frac{1}{\Gamma((n-1)\alpha + 1)} t^{(n-1)\alpha} \]

Generally, for \( n = 1, 2, \cdots \), one can derive that

\[ u_n = \mathcal{L}(u_{n-1}) + \mathcal{N}(u_0 + u_1 + \cdots + u_{n-1}) - \mathcal{N}(u_0 + u_1 + \cdots + u_{n-2}) = ((r-1)u_{n-1} - 2(u_{n-1})_{xx} - (u_{n-1})_{xxxx})I^\alpha \]

\[ = \frac{1}{\Gamma((n-1)\alpha + 1)} t^{(n-1)\alpha} \]

Hence the \( N \)-th approximate solution to (41)-(42) is

\[ u_N(x, t) = e^x \sum_{n=0}^N (r-4+\sigma)^n \frac{1}{\Gamma((n\alpha + 1))} t^{n\alpha} \]

with initial value

\[ u(x, 0) = e^x. \] (54)

Here \( \sigma \in \mathbb{R} \) is a parameter. Clearly, applying \( I^\alpha \) on both sides of (53), then initial value problem (53)-(54) is equivalent to the following integral equation:

\[ u(x, t) = f(x) + (\mathcal{L}u)(x, t) + (\mathcal{N}u)(x, t), \] (55)

where

\[ f(x) = e^x, \]

\[ \mathcal{L}(u) = I^\alpha ((r-1)u - 2u_{xx} + \sigma u_{xxx} - u_{xxxx}), \] (56)

\[ \mathcal{N}(u) = 0. \]

The solution to system (53)-(54) what we are looking for has the form

\[ u(x, t) = \lim_{N \to \infty} u_N(x, t). \] (57)

According to iterative scheme (9)-(10), one can obtain

\[ u_0 = f = e^x; \]

\[ u_1 = \mathcal{L}(u_0) + \mathcal{N}(u_0) = ((r-1)u_0 - 2(u_0)_{xx} + \sigma(u_1)_{xxx} - (u_0)_{xxxx})I^\alpha 1 \]

\[ = (r-4+\sigma) e^x \frac{1}{\Gamma(\alpha + 1)} t^\alpha; \]

\[ u_2 = \mathcal{L}(u_1) + \mathcal{N}(u_0 + u_1) - \mathcal{N}(u_0) = ((r-1)u_1 - 2(u_1)_{xx} + \sigma(u_1)_{xx} - (u_1)_{xxxx})I^\alpha \]

\[ = (r-4+\sigma) e^x \frac{1}{\Gamma(2\alpha + 1)} t^{2\alpha}; \]

\[ \vdots \]

\[ u_n = \mathcal{L}(u_{n-1}) + \mathcal{N}(u_0 + u_1 + \cdots + u_{n-1}) - \mathcal{N}(u_0 + u_1 + \cdots + u_{n-2}) = ((r-1)u_{n-1} - 2(u_{n-1})_{xx} + \sigma(u_{n-1})_{xxx} - (u_{n-1})_{xxxx})I^\alpha \]

\[ = (r-4+\sigma)^n e^x \frac{1}{\Gamma((n\alpha + 1))} t^{n\alpha} \]

(58)

With initial value

\[ u(x, 0) = e^x. \] (54)

Here, \( \sigma \in \mathbb{R} \) is a parameter. Clearly, applying \( I^\alpha \) on both sides of (53), then initial value problem (53)-(54) is equivalent to the following integral equation:

\[ u(x, t) = f(x) + (\mathcal{L}u)(x, t) + (\mathcal{N}u)(x, t), \] (55)

where

\[ f(x) = e^x, \]

\[ \mathcal{L}(u) = I^\alpha ((r-1)u - 2u_{xx} + \sigma u_{xxx} - u_{xxxx}), \] (56)

\[ \mathcal{N}(u) = 0. \]

The solution to system (53)-(54) what we are looking for has the form

\[ u(x, t) = \lim_{N \to \infty} u_N(x, t). \] (57)

According to iterative scheme (9)-(10), one can obtain

\[ u_0 = f = e^x; \]

\[ u_1 = \mathcal{L}(u_0) + \mathcal{N}(u_0) = ((r-1)u_0 - 2(u_0)_{xx} + \sigma(u_1)_{xxx} - (u_0)_{xxxx})I^\alpha 1 \]

\[ = (r-4+\sigma) e^x \frac{1}{\Gamma(\alpha + 1)} t^\alpha; \]

\[ u_2 = \mathcal{L}(u_1) + \mathcal{N}(u_0 + u_1) - \mathcal{N}(u_0) = ((r-1)u_1 - 2(u_1)_{xx} + \sigma(u_1)_{xx} - (u_1)_{xxxx})I^\alpha \]

\[ = (r-4+\sigma) e^x \frac{1}{\Gamma(2\alpha + 1)} t^{2\alpha}; \]

\[ \vdots \]

\[ u_n = \mathcal{L}(u_{n-1}) + \mathcal{N}(u_0 + u_1 + \cdots + u_{n-1}) - \mathcal{N}(u_0 + u_1 + \cdots + u_{n-2}) = ((r-1)u_{n-1} - 2(u_{n-1})_{xx} + \sigma(u_{n-1})_{xxx} - (u_{n-1})_{xxxx})I^\alpha \]

\[ = (r-4+\sigma)^n e^x \frac{1}{\Gamma((n\alpha + 1))} t^{n\alpha} \]

(58)
and the exact solution to (53)-(54) is
\[ u(x, t) = \lim_{N \to \infty} u^N (x, t) = e^x E_\alpha \left( (r - 4 + \sigma) t^\alpha \right). \]  
(60)

**Numerical Simulation.** See Figure 4.

### 5. Nonlinear Time-Fractional Swift-Hohenberg Equation

In this subsection, we apply the iterative method introduced in Section 3 to nonlinear time-fractional S-H equation with such as \( e^x \).

#### 5.1. Nonlinear S-H Equation with Initial Value \( u(x, 0) = e^x \).

Consider the following nonlinear S-H equation with dispersion:
\[
D^\alpha u + (1 - r) u + 2u_{xx} + u_{xxxx} = u^\ell - (u_x)^\ell,
\]
\[ 0 < \alpha \leq 1 \]  
with initial value
\[ u(x, 0) = e^x. \]  
(62)

Clearly, applying \( I^\alpha \) on both sides of (61), then initial value problem (61)-(62) is equivalent to the following integral equation:
\[
u(x, t) = f(x) + (\mathcal{L}u)(x, t) + (\mathcal{N}u)(x, t), \]  
(63)

where
\[
f(x) = e^x, \]
\[
\mathcal{L}u(x, t) = \Gamma^\alpha \left( (r - 1) u - 2u_{xx} - u_{xxxx} \right), \]  
(64)
\[
\mathcal{N}u(x, t) = \Gamma^\alpha \left( u^\ell - (u_x)^\ell \right). \]

The solution to system (61)-(62) what we are looking for has the form
\[ u(x, t) = \sum_{n=0}^{\infty} u_n(x, t). \]  
(65)
According to iterative scheme (9)-(10), one can obtain

\[ u_0 = f = e^x; \]
\[ u_1 = \mathcal{L}(u_0) + \mathcal{N}(u_0) \]
\[ = ((r - 1) u_0 - 2(u_0)_{xx} - (u_0)_{xxxx}) I^\alpha 1 + \left( u_0^\ell \right) \Gamma(\alpha + 1) t^\alpha \]
\[ = (r - 4) e^x \frac{1}{\Gamma(\alpha + 1)} t^\alpha; \]
\[ u_2 = \mathcal{L}(u_1) + \mathcal{N}(u_0 + u_1) - \mathcal{N}(u_0) \]
\[ = ((r - 1) u_1 - 2(u_1)_{xx} - (u_1)_{xxxx}) I^\alpha \frac{1}{\Gamma(\alpha + 1)} t^\alpha \]
\[ + \Gamma^\alpha \left( (u_0 + u_1)^2 - ((u_0 + u_1)_x^2) - (u_0)^2 \right) \]
\[ + \left( (u_0)_x^2 \right)^2 \]
\[ = (r - 4)^2 e^x \frac{1}{\Gamma(2\alpha + 1)} t^{2\alpha} \]
\[ \vdots \]
\[ u_n = \mathcal{L}(u_{n-1}) + \mathcal{N}(u_0 + u_1 + \cdots + u_{n-1}) - \mathcal{N}(u_0 + \cdots + u_{n-2}) \]
\[ = ((r - 1) u_{n-1} - 2(u_{n-1})_{xx} - (u_{n-1})_{xxxx}) I^\alpha \]
\[ + \Gamma^\alpha \left( (u_0 + \cdots + u_{n-1})^\ell - ((u_0 + \cdots + u_{n-1})_x^\ell) \right) \]
\[ - \left( (u_0 + \cdots + u_{n-2})^\ell + \left( (u_0 + \cdots + u_{n-2})_x^\ell \right) \right) \]
\[ = (r - 4)^n e^x \frac{1}{\Gamma(n\alpha + 1)} t^{n\alpha} \quad (n = 2, 3, \cdots). \]
Hence the $N$-th approximate solution to (61)-(62) is 

$$u^N(x, t) = e^x \sum_{n=0}^{N} (r-4)^n \frac{1}{\Gamma(na+1)} t^{na}$$

(67)

and the exact solution to (61)-(62) is 

$$u(x, t) = \lim_{N \to \infty} u^N(x, t) = e^x E_{a} ((r-4) t^a).$$

(68)

Numerical Simulation. See Figure 5.

5.2. Nonlinear Time-Fractional S-H Equation with Dispersion and Initial Value $u(x, 0) = e^x$. Consider the following nonlinear S-H equation with dispersion:

$$D^\alpha u + (1-r)u + 2u_{xx} - \sigma u_{xxx} + u_{xxxx} = u_x^f - (u_x^f)^r$$

with initial value 

$$u(x, 0) = e^x.$$

(69)

Clearly, applying $t^a$ to both sides of (69), then initial value problem (69)-(70) is equivalent to the following integral equation:

$$u(x, t) = f(x) + (L u)(x, t) + (N u)(x, t),$$

(71)

where

$$f(x) = e^x,$$

$$L (u) = \Gamma^a ((r-1) u - 2u_{xx} + \sigma u_{xxx} - u_{xxxx}),$$

(72)

$$N (u) = \Gamma^a (u_x^f - (u_x^f)^r).$$

The solution to system (69)-(70) what we are looking for has the form

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t).$$

(73)
According to iterative scheme (9)-(10), one can obtain
\[ u_0 = f = e^x; \]
\[ u_1 = \mathcal{L}(u_0) + \mathcal{N}(u_0) = \left((r-1)u_0 - 2(u_0)_{xx} + \sigma(u_0)_{xxxx} - (u_0)_{xxxxx}\right)I_1^a + \Gamma(1)u_0 \]
\[ u_2 = \mathcal{L}(u_1) + \mathcal{N}(u_0 + u_1) - \mathcal{N}(u_0) = \left((r-1)u_1 - 2(u_1)_{xx} - (u_1)_{xxxx}\right)I_1^a + \Gamma(1)\left((u_0 + u_1)_{xx} - (u_0 + u_1)_{xxxx}\right) - (u_0 + u_1)_{xx} + (u_0 + u_1)_{xxxx} \]
\[ \vdots \]
\[ u_n = \mathcal{L}(u_{n-1}) + \mathcal{N}(u_0 + u_1 + \cdots + u_{n-1}) - \mathcal{N}(u_0 + u_1 + \cdots + u_{n-2}) = \left((r-1)u_{n-1} - 2(u_{n-1})_{xx} - (u_{n-1})_{xxxxx}\right)I_1^a + \Gamma(1)\left((u_0 + u_1 + \cdots + u_{n-1})_{xx} - (u_0 + u_1 + \cdots + u_{n-2})_{xx}\right) - (u_0 + u_1 + \cdots + u_{n-2})_{xx} + (u_0 + u_1 + \cdots + u_{n-2})_{xxxx} \]
\[ \vdots \]
\[ u_n = \mathcal{L}(u_{n-1}) + \mathcal{N}(u_0 + u_1 + \cdots + u_{n-1}) - \mathcal{N}(u_0 + u_1 + \cdots + u_{n-2}) = \left((r-1)u_{n-1} - 2(u_{n-1})_{xx} - (u_{n-1})_{xxxxx}\right)I_1^a + \Gamma(1)\left((u_0 + u_1 + \cdots + u_{n-1})_{xx} - (u_0 + u_1 + \cdots + u_{n-2})_{xx}\right) - (u_0 + u_1 + \cdots + u_{n-2})_{xx} + (u_0 + u_1 + \cdots + u_{n-2})_{xxxx} \]

Hence the \(N\)-th approximate solution to (69)-(70) is
\[ u^N(x,t) = e^x\sum_{n=0}^N (r-4+\sigma)^n \frac{1}{\Gamma(n\alpha + 1)} I_1^a \]
\[ \text{and the exact solution to (61)-(70) is} \]
\[ u(x,t) = \lim_{N \to \infty} u^N(x,t) = e^{E_\alpha (1)} \left((r-4+\sigma)^\alpha \right) \]

**Numerical Simulation.** See Figure 6.
6. Concluding Remark

This paper introduce an iterative method which has considerable power in constructing approximated solutions and even exact solutions to time-fractional differential equation. We take the linear and nonlinear Swift-Hohenberg equations with different initial conditions to illustrate the effectiveness of such method.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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