Research Article

Coulomb Planar Periodic Motion of n Equal Charges in the Field of n Equal Positive Charges Fixed at a Line and Constant Magnetic Field

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We consider a neutral Coulomb planar system of n equal negative charges in a constant magnetic field and a field of n equal fixed positive charges and construct a periodic solution of the equation of motion such that each negative charge is close to its own positive fixed charge.

1. Introduction

Construction of solutions of the ML (Maxwell-Lorentz) equations of motion of point charges of classical electrodynamics is a fundamental task of mathematics. The simplest approximations of these equations are the Coulomb and Darwin equations, which do not take into account radiation of the charges. Solutions of the former and latter for arbitrary number of charges were proven to exist on a finite time interval at which there are no collisions in [1, 2], respectively.

If two equal positive charges are fixed at one coordinate axis at equal distances from the other coordinate axis along which two or three negative equal charges move, then there exists an equilibrium configuration of the negative charges and periodic solutions of their Coulomb equation of motion [3]. The center Lyapunov, Moser, and Weinstein theorems are applied by us in [3] to prove this fact. The first two of them demand an exclusion of resonances, which restrict values of the charges and yield the solutions in terms of convergent series. Periodic exact solutions are also found in planar Coulomb systems of $n-1$, $n > 2$ equal negative charges and one and three positive charges in [4, 5].

In this paper, we consider the Coulomb planar system of $n > 1$ point equal negative charges in a constant magnetic field and a field of n equal fixed positive charges located at a line (j th charge is placed at r_j) and construct

periodic solution of their equations of motion such that each negative charge moves close to its own positive fixed charge. The solution is given as a convergent series. We apply a generalization of the remarkable technique of Siegel invented by him for solving the three-body problem of the celestial mechanics [6, 7]. This result is formulated in the theorem in the end of the paper.

Earlier in [8, 9] we considered the space and planar systems of two equal negative and two fixed positive charges and proved with the help of the center Lyapunov theorem the existence of their periodic dynamics close to an equilibrium. We expect that there is a bifurcation in this system as in the case of gravitation two-center problem [10]. The results for the gravitation n -center problem, which can be used for dynamics in a system of a negative charge in the field of n fixed positive charges, are obtained in [11, 12].

The potential energy of our Coulomb system of $n > 1$ point equal negative charges e_0 and n fixed positive charges e_0 is given by

$$U(x_{(n)}) = e_0^2 \left[2^{-1} \sum_{1 \leq j \neq k \leq n} \frac{1}{|x_j - x_k|} - \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n} \frac{1}{|x_j - r_k|} \right], \quad (1)$$

$$x_j = (x_j^1, x_j^2), r_j = (r_j^1, 0) \in \mathbb{R}^2,$$

where $|x|$ is the Euclidean norm of x and $x_{(n)} = (x_1, \dots, x_n) \in \mathbb{R}^{2n}$. The equation of motion of the charges in a magnetic field h directed perpendicular to the plane is given by

$$m \frac{d^2 x_j}{dt^2} = -\frac{\partial U(x_{(n)})}{\partial x_j} + e_0 \frac{dx_j}{dt} \times h, \quad j = 1, \dots, n, \quad (2)$$

where $(v \times h)^1 = h^0 v^2$ and $(v \times h)^2 = -h^0 v^1$, $h^0 \in \mathbb{R}$. In the explicit form, it is written as

$$\begin{aligned} \frac{d^2 x_j}{dt^2} = & \frac{e_0^2}{m} \left[\sum_{1 \leq k \leq n, k \neq j} \frac{x_j - x_k}{|x_j - x_k|^3} - \sum_{1 \leq k \leq n} \frac{x_j - r_k}{|x_j - r_k|^3} \right] \\ & + \frac{e_0}{m} \frac{dx_j}{dt} \times h. \end{aligned} \quad (3)$$

Let us introduce the new (difference) variables x'_j .

$$x_j(t) = r_j + g x'_j(t), \quad g^3 = \frac{e_0^2}{m}. \quad (4)$$

If one omits the primes, the Coulomb equation for them is given by

$$\begin{aligned} \frac{d^2 x_j}{dt^2} = & -\frac{x_j}{|x_j|^3} + g^2 \left(\sum_{1 \leq k \leq n, k \neq j} \frac{r_j - r_k + g(x_j - x_k)}{|r_j - r_k + g(x_j - x_k)|^3} \right. \\ & \left. - \sum_{1 \leq k \leq n, k \neq j} \frac{r_j - r_k + g x_j}{|r_j - r_k + g x_j|^3} \right) + \frac{e_0}{m} \frac{dx_j}{dt} \times h. \end{aligned} \quad (5)$$

If $x_j = x_j^1 + i x_j^2$, $\tilde{x}_j = \tilde{x}_j^1 + i \tilde{x}_j^2$, $x_j^1 - i x_j^2 = x_j^* = \tilde{x}_j$, then this equation of motion is represented in the following complex form:

$$\begin{aligned} \frac{d^2 x_j}{dt^2} = & -x_j^{-1/2} \tilde{x}_j^{-3/2} \\ & + g^2 \sum_{1 \leq k \leq n, k \neq j} \left[(r_j - r_k + g(x_j - x_k))^{-1/2} \right. \\ & \cdot (r_j - r_k + g(\tilde{x}_j - \tilde{x}_k))^{-3/2} - (r_j - r_k + g x_j)^{-1/2} \\ & \cdot (r_j - r_k + g \tilde{x}_j)^{-3/2} \left. \right] - i h_0 \frac{dx_j}{dt}, \quad h_0 = \frac{h^0 e_0}{m}, \\ \frac{d^2 \tilde{x}_j}{dt^2} = & -\tilde{x}_j^{-1/2} x_j^{-3/2} \\ & + g^2 \sum_{1 \leq k \leq n, k \neq j} \left[(r_j - r_k + g(\tilde{x}_j - \tilde{x}_k))^{-1/2} \right. \\ & \cdot (r_j - r_k + g(x_j - x_k))^{-3/2} - (r_j - r_k + g \tilde{x}_j)^{-1/2} \\ & \cdot (r_j - r_k + g x_j)^{-3/2} \left. \right] + i h_0 \frac{d\tilde{x}_j}{dt}. \end{aligned} \quad (6)$$

That is,

$$\begin{aligned} \frac{d^2 x_j}{dt^2} = & -x_j^{-1/2} \tilde{x}_j^{-3/2} + f_j, \\ \frac{d^2 \tilde{x}_j}{dt^2} = & -\tilde{x}_j^{-1/2} x_j^{-3/2} + \tilde{f}_j. \end{aligned} \quad (7)$$

This equation can be considered if $x_j^1 - i x_j^2 = x_j^* \neq \tilde{x}_j$. It is solved in this paper, which is organized as follows. In the second section, we introduce power expansions in two complex (Siegel) variables for the coordinates in (7) and write down and solve an equation for their coefficients. In the third section, we obtain their majorant bounds that are used for the proof of convergence of the power expansions. Our main result is formulated in Theorem 2 concerning periodic solutions of (3). In the Appendix, we prove the majorant inequalities (32)-(33) that are together with the majorant inequalities (34)-(36) basic for the majorant technique.

2. Algebraic Form of the Equation of Motion

To solve (7), we introduce two Siegel complex variables, ξ and η , such that

$$\begin{aligned} \frac{d\xi}{dt} &= \alpha \xi, \\ \frac{d\eta}{dt} &= -\alpha \eta, \end{aligned} \quad (8)$$

$$\alpha = \frac{i}{4} (\xi \eta)^{-3},$$

$$x_j = \xi^4 u \left(1 + \sum_{k,l} a_{j,kl} \zeta_{k,l} \right) = u \xi^4 (1 + A_j), \quad (9)$$

$$k - |2l| \geq 0, \quad k > 0,$$

$$\tilde{x}_j = \eta^4 u \left(1 + \sum_{k,l} a_{j,kl} \zeta_{k,-l} \right) = u \eta^4 (1 + B_j), \quad (10)$$

$$\zeta_{k,l} = \zeta_{kl} = \xi^{k+2l} \eta^{k-2l},$$

where $a_{j,kl} = a_{j,k,l} \in \mathbb{R}$, $a_{j,k,l} = 0$ for $k < 3$ or $k < 6$ for a special choice of b and

$$u = (1 - b \zeta_3)^{1/3}, \quad \zeta_k = \zeta_{k0}, \quad b \in \mathbb{R}. \quad (11)$$

The choice of u is motivated by the expression of $\zeta_{k,l}$. It helps to derive the Siegel-type inequality (31) (see also the last sentence of the fourth section).

Note that $x_j^* = \tilde{x}_j$ if $\xi^* = \eta$. The last condition implies that the equations in (8) have the periodic solutions

$$\begin{aligned} \xi &= e^{\alpha t} \xi_0, \\ \eta &= e^{-\alpha t} \eta_0, \end{aligned} \quad (12)$$

$$\alpha = \frac{i}{4} (\xi_0 \eta_0)^{-3}$$

since $\xi\eta$ is a conserved quantity. In other words, we seek the periodic solutions of the equation of motion (7) with the representation (9)-(10). In order to achieve that goal, we have to calculate the first and second derivatives of the coordinates in (9)-(10), substitute them together with the coordinates into (7), obtain the equation for $a_{j,k,l}$, $a_{j,k,-l}$, and prove that it has a solution that provides convergence of the series in (9)-(10) for appropriate (sufficiently small) ξ, η .

Let us calculate the time derivatives of the complex coordinates in (9)-(10) satisfying (7). We set for simplicity $h^0 \geq 0$.

$$\begin{aligned}\frac{dx_j}{dt} &= 4\alpha\xi^4 u \left(1 + \sum_{k,l} (l+1) a_{j,kl} \zeta_{kl} \right), \\ \frac{d^2 x_j}{dt^2} &= (4\alpha)^2 \xi^4 u \left(1 + \sum_{k,l} (l+1)^2 a_{j,kl} \zeta_{kl} \right), \\ \frac{d\tilde{x}_j}{dt} &= -(4\alpha) \eta^4 u \left(1 + \sum_{k,l} (l+1) a_{j,kl} \zeta_{k,-l} \right), \\ \frac{d^2 \tilde{x}_j}{dt^2} &= (4\alpha)^2 \eta^4 u \left(1 + \sum_{k,l} (l+1)^2 a_{j,kl} \zeta_{k,-l} \right).\end{aligned}\quad (13)$$

As a result,

$$\begin{aligned}-i\xi^2 \eta^6 u^2 \frac{dx_j}{dt} &= \zeta_3 u^3 \left(1 + \sum_{k,l} (l+1) a_{j,kl} \zeta_{kl} \right), \\ \xi^2 \eta^6 u^2 \frac{d^2 x_j}{dt^2} &= -u^3 \left(1 + \sum_{k,l} (l+1)^2 a_{j,kl} \zeta_{kl} \right), \\ \xi^2 \eta^6 u^2 x_j^{-1/2} \tilde{x}_j^{-3/2} &= (1 + A_j)^{-1/2} (1 + B_j)^{-3/2} \\ &= \left(1 - \frac{1}{2} A_j + \dots \right) \left(1 - \frac{3}{2} B_j + \dots \right) \\ &= 1 - \frac{1}{2} A_j - \frac{3}{2} B_j + \dots\end{aligned}\quad (14)$$

The last three equalities and (7) yield

$$\begin{aligned}u^3 \left[\sum_{k,l} (l+1)^2 a_{j,kl} \zeta_{kl} + h_0 \zeta_3 \sum_{k,l} (l+1) a_{j,kl} \zeta_{kl} \right] + \frac{1}{2} A_j \\ + \frac{3}{2} B_j = D_j + F_j,\end{aligned}\quad (15)$$

where

$$\begin{aligned}D_j &= (1 + A_j)^{-1/2} (1 + B_j)^{-3/2} - 1 + \frac{1}{2} A_j + \frac{3}{2} B_j, \\ F_j &= -g^2 \xi^2 \eta^6 u^2 \sum_{1 \leq k \leq n, k \neq j} \left[\left[(r_j - r_k + g\xi^4 u (A_j - A_k))^{-1/2} \right. \right. \\ &\quad \cdot \left. \left. (r_j - r_k + g\eta^4 u (B_j - B_k))^{-3/2} - (r_j - r_k)^{-2} \right] \right]\end{aligned}$$

$$\begin{aligned}&- \left[(r_j - r_k + g\xi^4 u + g\xi^4 u A_j)^{-1/2} \right. \\ &\quad \cdot \left. (r_j - r_k + g\eta^4 u + g\eta^4 u B_j)^{-3/2} - (r_j - r_k)^{-2} \right] \Big\} \\ &- h_0 \zeta_3 u^3 + b \zeta_3,\end{aligned}\quad (16)$$

where $\xi^2 \eta^6 = \zeta_{4,-1}$. Note that the power expansion in the right-hand side without two last terms begins from $k > 5$. For $h^0 \neq 0$, we shall put $b = h_0$, implying that the last two terms give $b^2 \zeta_6$. (15) is rewritten as follows:

$$\sum_{k,l} (l+1)^2 a_{j,kl} \zeta_{kl} + \frac{1}{2} A_j + \frac{3}{2} B_j = D_j + F_j + C_j, \quad (17)$$

where

$$\begin{aligned}C_j &= \zeta_3 \sum_{k,l} [b(l+1)^2 - h_0(l+1)] a_{j,kl} \zeta_{kl} \\ &\quad + b h_0 \zeta_6 \sum_{k,l} (l+1) a_{j,kl} \zeta_{kl}.\end{aligned}\quad (18)$$

(17) is valid, since

$$\begin{aligned}\xi^2 \eta^6 &= \zeta_{4,-1}, \\ \xi^{4k_1} \eta^{4k_2} &= \zeta_{2(k_1+k_2), k_1-k_2}, \\ \zeta_{k,l} \zeta_{k',l'} &= \zeta_{k+k', l+l'}, \\ k_1, k_2 &\geq 0.\end{aligned}\quad (19)$$

By $F_{k,l}$ and F^+ , we will denote the coefficient in the power expansion of F and the power expansion corresponding to the coefficient $|F_{k,l}|$, respectively.

The following representation is true:

$$\begin{aligned}D_j + F_j + C_j &= \sum_{k,l} \rho_{j,kl} \zeta_{kl}, \\ \tilde{D}_j + \tilde{F}_j + \tilde{C}_j &= \sum_{k,l} \rho_{j,k,-l} \zeta_{kl},\end{aligned}\quad (20)$$

$$\rho_{j,kl} = \rho_{j,k,l} \in \mathbb{R}.$$

We derive from (17) and (20) the needed equation for $a_{j,kl}$, $a_{j,k,-l}$ (we take into account $\tilde{\zeta}_{k,l} = \zeta_{k,-l}$ when applying the tilde to (17)).

$$\begin{aligned}\left[(l+1)^2 + \frac{1}{2} \right] a_{j,kl} + \frac{3}{2} a_{j,k,-l} &= \rho_{j,kl}, \quad a_{j,k0} = \frac{1}{3} a_{j,k0}, \\ \left[(l+1)^2 + \frac{1}{2} \right] a_{j,k,-l} + \frac{3}{2} a_{j,kl} &= \rho_{j,k,-l},\end{aligned}\quad (21)$$

which can be solved easily for $l \neq \pm 1$. Note that $\rho_{j,kl}, \rho_{j,k,-l}$ depend on $a_{s,r,l}$, $1 \leq s \leq n$, $r < k$ since the expansion for D_j begins from the second powers of A_j, B_j , since it does not contain the term with $k = 0$ and there are coefficients $\xi^2 \eta^6, \zeta_6$,

ζ_3 in the right-hand side of the equality in (17). The obtained complex two-dimensional linear equation is solved as

$$a_{j,kl} = \frac{[(l-1)^2 + 1/2] \rho_{j,kl} - (3/2) \rho_{j,k,-l}}{l^2 (l^2 - 1)}, \quad (22)$$

$$l \neq 0, \pm 1, \quad 3a_{j,k0} = \rho_{j,k,0}.$$

From this equality that generates a solvable recursion relation for $a_{j,kl}$, it follows that

$$|(l+1)^s a_{j,kl}| \leq 2^{-1} c_0 (|\rho_{j,kl}| + |\rho_{j,k,-l}|), \quad (23)$$

$$l \neq \pm 1, \quad s = 0, 1, 2.$$

The equation for $a_{j,k,\pm 1}$ follows from the equations

$$\begin{aligned} (l+1)^2 a_{j,k+3,l} + \frac{1}{2} a_{j,k+3,l} + \frac{3}{2} a_{j,k+3,-l} \\ - [b(l+1)^2 - h_0(l+1)] a_{j,k,l} \\ - bh_0(l+1) a_{j,k-3,l} = \rho'_{j,k+3,l}, \\ (-l+1)^2 a_{j,k+3,-l} + \frac{3}{2} a_{j,k+3,l} + \frac{1}{2} a_{j,k+3,-l} \\ - [b(-l+1)^2 - h_0(-l+1)] a_{j,k,-l} \\ - bh_0(-l+1) a_{j,k-3,-l} = \rho'_{j,k+3,-l}, \end{aligned} \quad (24)$$

where

$$\rho'_{j,k,l} = (D_j + F_j)_{k,l}, \quad (25)$$

which results from (17). Then

$$\begin{aligned} \frac{9}{2} a_{j,k+3,1} + \frac{3}{2} a_{j,k+3,-1} - (4b - 2h_0) a_{j,k,1} - 2bh_0 a_{j,k-3,1} \\ = \rho'_{j,k+3,1}, \\ \frac{3}{2} a_{j,k+3,1} + \frac{1}{2} a_{j,k+3,-1} = \rho'_{j,k+3,-1}. \end{aligned} \quad (26)$$

Let us multiply the second equation by 3 and subtract the result from the first one. That is,

$$\begin{aligned} -(4b - 2h_0) a_{j,k,1} = 2bh_0 a_{j,k-3,1} + \rho'_{j,k+3,1} - 3\rho'_{j,k+3,-1}, \\ \frac{3}{2} a_{j,k,1} + \frac{1}{2} a_{j,k,-1} = \rho'_{j,k,-1}. \end{aligned} \quad (27)$$

These two equalities give the bounds

$$\begin{aligned} |a_{j,k,1}| &\leq |4b - 2h_0|^{-1} \\ &\cdot (2bh_0 |a_{j,k-3,1}| + |\rho'_{j,k+3,1}| + 3|\rho'_{j,k+3,-1}|), \\ |a_{j,k,-1}| &\leq 3|a_{j,k,1}| + 2|\rho'_{j,k,-1}|. \end{aligned} \quad (28)$$

3. Majorant Bounds

In this section, we derive majorant bounds for the terms in the right-hand side of (17), which permit proving that the series in (9)-(10) converges absolutely for $|\xi|^2 < q$, $|\eta|^2 < q$. We shall do it with the help of the advanced Cauchy majorant technique that is close to the one applied by Siegel for the solution of the generalized Hill problem in the celestial mechanics [7]. It is sufficient to prove the absolute convergence for $\xi^2 = \eta^2 = \zeta$, $|\zeta|^2 < q$, which implies that $\zeta_{k,l} = \zeta^k$, since $|\zeta_{k,l}| = |\xi|^{k+2l} |\eta|^{k-2l} < q^k$ if $|\xi|^2 < q$ and $|\eta|^2 < q$. The analog of the first condition was applied by Siegel for the solution of the Hill problem in the celestial mechanics [6].

Let

$$Z_j = \sum_{k,l} (|l+1|)^s |a_{j,kl}| \zeta^k, \quad s = 0, 1, 2, \quad Z = \sum_{j=1}^n Z_j, \quad (29)$$

where $k - |2l| \geq 0$; $s = 0, k > 2$ ($s = 2, k > 5$) correspond to the zero (nonzero, $b = h_0$) magnetic field. Let also $s = 1$ correspond to the case where $b = 0, h \neq 0$. Then the following majorant bounds are true:

$$\begin{aligned} A_j &\ll Z_j, \\ B_j &\ll Z_j, \\ \xi^2 &= \eta^2 = \zeta, \end{aligned} \quad (30)$$

where $f \ll g$ means that, in the power expansion for g , the coefficients are positive and exceed absolute values of the coefficients in the power expansion for f ; that is, $g - f \gg 0$.

Our aim is to derive with the help of (23) and (28) the basic Siegel-type inequality [7]:

$$\begin{aligned} Z &\ll c' \frac{2p'z^3 + z^4 + zZ + z^{-3}Z^2}{1 - c'(Z + z)}, \\ z &= \nu\zeta, \quad \nu = u^{-3} (1 + b\zeta^3), \quad c' \geq 1, \quad p' \geq 0. \end{aligned} \quad (31)$$

p' turns out to be zero for $b = h_0 \neq 0$. We shall apply the following inequalities for $x, y \in \mathbb{C}$ and X, Y given by power expansions (X^+, Y^+ will denote the power expansions whose coefficients coincide with absolute values of the coefficients of X, Y). Let $|x| = |y|$, $X \ll Q, Y \ll Q$; then

$$\begin{aligned} (x + X)^{-1/2} (y + Y)^{-3/2} - x^{-1/2} y^{-3/2} \\ \ll |x|^{-3} \frac{2Q}{1 - 2|x|^{-1}Q}, \end{aligned} \quad (32)$$

$$\begin{aligned} (x + X)^{-1/2} (y + Y)^{-3/2} \\ + x^{-1/2} y^{-3/2} \left(-1 + \frac{1}{2x}X + \frac{3}{2y}Y \right) \\ \ll |x|^{-4} \frac{4Q^2}{1 - 2|x|^{-1}Q}. \end{aligned} \quad (33)$$

Equations (32)-(33) are proven in the Appendix. We shall use also the following inequalities:

$$\frac{1}{1 - g_0 z^2 (Z + 1)} \ll \frac{1}{1 - (Z + g_0 z^2)} \quad (34)$$

$$\ll \frac{1}{1 - (Z + \sqrt{g_0} z)},$$

$$\frac{1}{1 - g_0 z^2 Z} \ll \frac{1}{1 - (Z + g_0 z^2)} \quad (35)$$

$$\ll \frac{1}{1 - (Z + \sqrt{g_0} z)},$$

$$z^k \ll \frac{1}{1 - z}, \quad k \in \mathbb{Z}^+,$$

$$\frac{1}{1 - X^+} \frac{1}{1 - Y^+} \ll \frac{1}{1 - X^+ - Y^+}, \quad (36)$$

$$\frac{1}{1 - c' (z + Z)} \ll \frac{1}{1 - c' (z + z^{-1} Z)}$$

$$\ll \frac{1}{1 - c' (z + z^{-3} Z)}, \quad c' \geq 1.$$

The first inequality in (34) follows from the fact that the monomials of $(g_0 z^2 (Z + 1))^n$ are determined by n monomials of $(Z + g_0 z^2)^{2n}$, $(Z + g_0 z^2)^{2n-1}, \dots, (Z + g_0 z^2)^n$ with greater coefficients. The second one follows from the fact that the monomials of $(Z + g_0 z^2)^n$ are determined by n monomials of $(Z + \sqrt{g_0} z)^{2n}$, $(Z + \sqrt{g_0} z)^{2n-1}, \dots, (Z + \sqrt{g_0} z)^n$ with greater coefficients ($C_n^k \leq C_{2n-k}^k$). Equation (35) follows from the fact that $(g_0 z^2 Z)^n$ is determined by the monomial of $(Z + g_0 z^2)^{2n}$ with the coefficient C_{2n}^{2n} . Equation (36) is easily derived. The last inequality follows from the first inequality in (34) ($z^{-1} Z$ and z are substituted instead of Z and z^2 , resp.).

Using (33) with $x = y = 1$, $X = A_j$, $Y = B_j$, we obtain

$$D_j^+ \ll \frac{4Z_j^2}{1 - 2Z_j},$$

$$\sum_{1 \leq j \leq n} D_j^+ \ll \frac{4Z^2}{1 - 2(Z + z)} \ll \frac{b_1 z^{-3} Z^2}{1 - 3(Z + z)}, \quad (37)$$

$$b_1 = 4,$$

$$\zeta^{-3} \sum_{1 \leq j \leq n} D_j^+ = z^{-3} v^3 \sum_{1 \leq j \leq n} D_j^+ \ll \frac{b_2 z^{-3} Z^2}{1 - (3b + 5)(Z + z)},$$

$$b_2 = 4(1 + b)^3, \quad b > 0.$$

Here we used (36) and

$$v^3 = \frac{(1 + b\zeta^3)^3}{(1 - b\zeta^3)^3} \ll \frac{(1 + b\zeta^3)^3}{1 - 3b\zeta^3} \ll \frac{(1 + b\zeta^3)^3}{1 - (3b + 1)\zeta} \quad (38)$$

$$\ll \frac{(1 + b)^3}{1 - (3b + 2)\zeta} \ll \frac{(1 + b)^3}{1 - (3b + 2)z}.$$

Let

$$r = \max_j \sum_{1 \leq k \leq n, k \neq j} |r_j - r_k|^{-2}, \quad (39)$$

$$r_0 = \min_{j,k} |r_j - r_k|.$$

Then, for the term $F_{1;j}$ in the first square bracket in the expression for F_j , we derive, putting $g_0 = 2r_0^{-1}g$ and using (32) with $x = r_j - r_k$, $y = x$, $X = g\zeta^4 u(A_j - A_k)$, $Y = g\eta^4 u(B_j - B_k)$ and (35)-(36),

$$F_{1;j}^+ \ll r\zeta^4 g^2 v^2 \frac{g_0 v \zeta^2 Z}{1 - g_0 v \zeta^2 Z}$$

$$\ll n^{-1} b_3 \frac{z^6 Z}{1 - g_0 z^2 Z},$$

$$b_3 = 2^{-2} (n - 1) n g_0^3,$$

$$\zeta^{-3} F_{1;j}^+ < n^{-1} b_3 \frac{z^3 Z}{1 - g_0 z^2 Z},$$

$$\sum_{1 \leq j \leq n} F_{1;j}^+ \ll \frac{b_3 z^6 Z}{1 - (Z + \sqrt{g_0} z)} \quad (40)$$

$$\ll \frac{b_3 z Z}{1 - (c_* + 1)(Z + z)},$$

$$\zeta^{-3} \sum_{1 \leq j \leq n} F_{1;j}^+ \ll \frac{b_3 z^3 Z}{1 - (Z + \sqrt{g_0} z)}$$

$$\ll \frac{b_3 z Z}{1 - (c_* + 1)(Z + z)},$$

where $c_* = 1 + \sqrt{g_0}$, $g_0 \leq 1$; $c_* = \sqrt{g_0}$, $g_0 > 1$. Here we used also $b > 0$.

$$u = \frac{(1 - b\zeta_3)}{(1 - b\zeta_3)^{2/3}} \ll \frac{1 + b\zeta^3}{1 - b\zeta^3} = v;$$

$$v \ll v^k, \quad k > 1; \quad (41)$$

$$1 \ll (1 - b\zeta_3)^{-1/3},$$

$$X^+, Y^+ \ll g\zeta^2 v (Z_j + Z_k).$$

For the term $F_{2;j}$ in the second square bracket in the expression for F_j , we obtain, using (32) with $x = r_j - r_k$, $y = x$, $X = g\zeta^4 u(A_j + 1)$, $Y = g\eta^4 u(B_j + 1)$ and (34) and (36),

$$F_{2;j}^+ \ll r g_0 g^2 \frac{z^6 (Z_j + 1)}{1 - g_0 z^2 (Z + 1)},$$

$$\sum_{1 \leq j \leq n} F_{2;j}^+ \ll \frac{z^6 (n^{-1} b_3 Z + b_3)}{1 - (Z + \sqrt{g_0} z)}$$

$$\ll b_4 \frac{zZ + z^4}{1 - (1 + c_*)(Z + z)},$$

$$b_4 = 2^{-2} (n^{-1}b_3 + b_3). \quad (42)$$

In order to exclude the contribution of two terms $\zeta_6 = \zeta_{6,0}$, $\zeta_{6,-2} = \xi^2 \eta^{10}$ in the bound derived from (28), we have to obtain a more accurate majorant inequality for F_j^+ starting from the equality

$$F_{2;j} = F_{3;j} - g^2 \xi^2 \eta^6 u^2 \sum_{1 \leq k \leq n, k \neq j} (r_j - r_k)^{-2} \cdot \left[\frac{\xi^4 u g}{2(r_j - r_k)} (A_j + 1) + \frac{3\eta^4 u g}{2(r_j - r_k)} (B_j + 1) \right]. \quad (43)$$

We will need also to majorize the following function:

$$F_{4;j} = g^2 \xi^2 \eta^6 u^2 \sum_{1 \leq k \leq n, k \neq j} (r_j - r_k)^{-2} \cdot \left[\frac{\xi^4 g}{2(r_j - r_k)} (uA_j + u - 1) + \frac{3\eta^4 g}{2(r_j - r_k)} (uB_j + u - 1) \right]. \quad (44)$$

That is,

$$F_{2;j} = F_{3;j} + F_{4;j} + \alpha_1 \zeta_{6,0} + \alpha_2 \zeta_{6,-2} \quad (45)$$

with appropriate α_j . Using (33) with $x = r_j - r_k$, $y = x$, $X = \xi^4 u(A_j + 1)$, $Y = \eta^4 u(B_j + 1)$ and (34) and (36), we obtain

$$F_{3;j}^+ \ll r v^2 \zeta^4 g^2 \frac{(g_0 v \zeta^2 (Z_j + 1))^2}{1 - g_0 v \zeta^2 (Z_j + 1)},$$

$$\zeta^{-3} F_{3;j}^+ \ll r g^2 z \frac{(g_0 z^2 (Z_j + 1))^2}{1 - g_0 z^2 (Z_j + 1)}, \quad z = v \zeta,$$

$$\zeta^{-3} \sum_{1 \leq j \leq n} F_{3;j}^+ \ll \frac{2^{-1} b_5 z^5 (Z + 1)^2}{1 - (Z + \sqrt{g_0} z)}$$

$$\ll \frac{b_5 (z^{-3} Z^2 + zZ + z^4)}{1 - (c_* + 1)(Z + z)},$$

$$b_5 = 2^{-1} n(n-1) g_0^4. \quad (46)$$

Let

$$r' = \max_j \sum_{1 \leq k \leq n, k \neq j} |r_j - r_k|^{-3}. \quad (47)$$

Then

$$F_{4;j}^+ \ll 2r' v^2 \zeta^6 g^3 (vZ_j + v - 1)$$

$$= 2r' g^3 v^2 \left(\zeta^6 vZ_j + \frac{2b\zeta^9}{1 - b\zeta^3} \right), \quad (48)$$

$$\zeta^{-3} \sum_{1 \leq j \leq n} F_{4;j}^+ \ll b_6 \frac{zZ + z^4}{1 - (1 + b + c_*)(Z + z)},$$

$$b_6 = 2^{-2} (n-1) g_0^3 (1 + 2b).$$

For the last two terms $F_{0;j}$ in the expression for F_j , we obtain also

$$F_{0;j}^+ \ll h_0 \zeta^3 (1 + b\zeta^3) + b\zeta^3 \ll h_0 z^3 (1 + bz^3) + bz^3. \quad (49)$$

For the nonzero magnetic field, we obtain

$$F_{0;j}^+ \ll b^2 \zeta^6 \ll b^2 z^6, \quad b = h_0,$$

$$\sum_{1 \leq j \leq n} F_{0;j}^+ \ll \frac{b_0 z^6}{1 - (Z + \sqrt{g_0} z)} \ll \frac{b_0 z^4}{1 - (c_* + 1)(Z + z)}, \quad (50)$$

$$b_0 = nh_0^2.$$

For the zero magnetic field, we derive

$$\sum_{1 \leq j \leq n} F_{0;j}^+ \ll \frac{b' z^3}{1 - (Z + \sqrt{g_0} z)} \ll \frac{b' z^3}{1 - (c_* + 1)(Z + z)}, \quad (51)$$

$$b' = nb.$$

We have also

$$C_j^+ \ll [\zeta^3 (b + h_0) + bh_0 \zeta^6] Z_j,$$

$$\sum_{1 \leq j \leq n} C_j^+ \ll \frac{b_7 zZ}{1 - (c_* + 1)(Z + z)}, \quad b_7 = b + h_0 + bh_0. \quad (52)$$

The same majorant bounds hold for $\tilde{D}_j, \tilde{F}_j, \tilde{C}_j$.

4. Main Result

In this section, we will prove (31), which permits proving the convergence of the series in (9)-(10), and formulate our main result in Theorem 2. We will take into account the fact that $a_{j,k,1} = 0$, $k < 3$. Equation (28) gives

$$\sum_k (|a_{j,k,1}| + |a_{j,k,-1}|) \zeta^k \ll \sum_k [|2b - h_0|^{-1} \cdot (4bh_0 |a_{j,k-3,1}| + 2(|\rho'_{j,k+3,1}| + 3|\rho'_{j,k+3,-1}|)) + 2|\rho'_{j,k,-1}|] \zeta^k \ll 4|2b - h_0|^{-1} bh_0 \zeta^3 Z_j$$

$$+ 2 \sum_{k,l} |\rho'_{j,k,l}| \zeta^k + G_j, \quad (53)$$

$$G_j = 6(\zeta^3 |2b - h_0|)^{-1} \sum_{k,l} (|D_{k,l}| + |F_{1;j,k,l}| + |F_{3;j,k,l}| + |F_{4;j,k,l}|) \zeta^k.$$

G_j does not contain the terms with $\zeta_6 = \zeta_{6,0}$, $\zeta_{6,-2} = \xi^2 \eta^{10}$ in the expression for $F_{2,j}$. $F_{0,j}$ does not contribute to this inequality as well. Besides,

$$\sum_k |a_{j,k,0}| \zeta^k \ll 3^{-1} \sum_k |\rho_{j,k,0}| \zeta^k \ll 3^{-1} \sum_{k,l} |\rho_{j,k,l}| \zeta^k. \quad (54)$$

Taking into account (23), one obtains ($\xi^2 = \eta^2 = \zeta$)

$$Z_j \ll (c_0 + 3^{-1}) \sum_{k,l} |\rho_{j,k,l}| \zeta^k + \nu \left(|2b - h_0|^{-1} \frac{4bh_0 z Z_j}{1 - (Z + z)} + \sum_{k,l} |\rho'_{j,k,l}| \zeta^k + G_j \right), \quad (55)$$

where $\nu = 4(\nu = 1)$ if the magnetic field is not zero (zero).

$$\begin{aligned} \sum_{k,l} |\rho_{j,k,l}| \zeta^k &\ll D_j^+ + F_j^+ + C_j^+, \\ \sum_{k,l} |\rho'_{j,k,l}| \zeta^k &\ll D_j^+ + F_j^+, \\ F_j^+ &\ll F_{1,j}^+ + F_{2,j}^+ + F_{0,j}^+, \\ G_j &= 6 \left(\zeta^3 |2b - h_0| \right)^{-1} (D_j^+ + F_{1,j}^+ + F_{3,j}^+ + F_{4,j}^+). \end{aligned} \quad (56)$$

As a result,

$$\begin{aligned} Z_j &\ll \left[2\nu + c_0 + 3^{-1} + 4\nu \left(\zeta^3 |2b - h_0| \right)^{-1} \right] (D_j^+ + F_{1,j}^+) \\ &\quad + (2\nu + c_0 + 3^{-1}) (F_{2,j}^+ + F_{0,j}^+) + (c_0 + 3^{-1}) C_j^+ \\ &\quad + |2b - h_0|^{-1} \frac{4\nu b h_0 z Z_j}{1 - (Z + z)} \\ &\quad + 6\nu \left(\zeta^3 |2b - h_0| \right)^{-1} (F_{3,j}^+ + F_{4,j}^+). \end{aligned} \quad (57)$$

Let us sum over j both sides of the above inequality for Z_j and utilize the majorant inequalities obtained in the previous section for all its terms. This gives

$$Z \ll \frac{c_2 (2p' z^3 + z^4 + zZ + z^{-3} Z^2)}{1 - c_1 (z + Z)}, \quad (58)$$

where $c_1 = 5 + 3b + \sqrt{g_0}$, $c_2 = b_3 + (2\nu + c_0 + 3^{-1})(b_8 + b_1 + b_3 + b_4) + 6\nu |2b - h_0|^{-1} (b_2 + b_3 + b_5 + b_6) + (c_0 + 3^{-1}) b_7$, $b_3 = |2b - h_0|^{-1} 4\nu b h_0$, $b_8 = 0$, $2p' = b' c_2^{-1} (b_8 = b_0, p' = 0)$ for the zero (nonzero, $b = h_0$) magnetic field.

As a result, we obtain (31) with

$$\begin{aligned} c' &= \sqrt{g_0} + c_2 + b'', \\ (2\nu + 3^{-1}) b_1 &> 8\nu + 1, \\ c' &> 8\nu + 1 + b'' + \sqrt{g_0}, \end{aligned} \quad (59)$$

where $b'' = 0 (b'' = 3b)$ if the magnetic field is nonzero (zero). Here we took into account $b_3 > 3b$ for $b = h_0$ and $2\nu b_1 = 8$.

It is majorized by (see the inequality below (36))

$$Z \ll c' \frac{2p_* z^3 + z^4 + zZ + z^{-3} Z^2}{1 - c' (z^{-3} Z + z)}, \quad (60)$$

$$2p_* = b' c'^{-1}, \quad b' = nb.$$

Let $h_0 \neq 0$, $b = h_0$, and

$$Z' = z + z^{-3} Z. \quad (61)$$

Then the last inequality for $p = 0$ is majorized by

$$\begin{aligned} Z' &\ll c \frac{2z + Z'^2}{4 - cZ'}, \\ c &= 2 + 4c', \\ c &> 4(8\nu + 1) + 2 + 4\sqrt{g_0}. \end{aligned} \quad (62)$$

Indeed, it implies that

$$\begin{aligned} z^{-3} Z &\ll \frac{cz(z + z^{-3} Z) - 4z}{4 - c(z^{-3} Z + z)} + c \\ &\cdot \frac{2z + z^2 + 2z^{-2} Z + z^{-6} Z^2}{4 - c(z^{-3} Z + z)} \\ &= ((2c - 4)z + 2cz^2 + 3cz^{-2} Z + cz^{-6} Z^2) \\ &\cdot (4 - c(z^{-3} Z + z))^{-1}. \end{aligned} \quad (63)$$

Or

$$\begin{aligned} Z &\ll ((2c - 4)z^4 + 2cz^5 + 3czZ + cz^{-3} Z^2) \\ &\cdot (4 - c(z^{-3} Z + z))^{-1}. \end{aligned} \quad (64)$$

Applying (36), we map $2cz^5$ into $2cz^4$. In this way, we transform the last relation into (60) with greater c' .

Equation (62) leads to

$$Z' \ll W, \quad W = c \frac{2z + W^2}{4 - cW}, \quad (65)$$

$$cW^2 - 2W + cz = 0, \quad W = c^{-1} (1 - \sqrt{1 - c^2 z}).$$

Moreover,

$$\begin{aligned} (1 - cW)^2 &= 1 - c^2 z, \\ 2cW &\ll (1 - cW)^{-2} - 1 = \frac{c^2 z}{1 - c^2 z}. \end{aligned} \quad (66)$$

Hence,

$$W \ll 2^{-1} \frac{cZ}{1 - c^2 z}, \quad |z| < c^{-2}. \quad (67)$$

We also have

$$|W(z)| \leq W(|z|) < 2^{-1} \frac{c|z|}{1 - c^2|z|} \leq c^{-1}, \quad (68)$$

$$|z| \leq (2c^2)^{-1}.$$

The following bounds also hold:

$$|Z'(z)| \leq Z'(|z|) \leq W(|z|) \leq c^{-1}, \quad |z| \leq (2c^2)^{-1}. \quad (69)$$

Let

$$|z| = |\nu\zeta| = \left| \frac{1 + b\zeta^3}{1 - b\zeta^3} \zeta \right| \leq c^{-2}. \quad (70)$$

The last inequality is easily solved taking into account $b|\zeta| < 1$, $|\zeta|^k < |\zeta| < 1$, $c > 1$, $k > 1$.

$$|\zeta| < \frac{c^{-2}}{1 + b + bc^{-2}} = q. \quad (71)$$

It follows from the stronger middle inequality (for $\zeta > 0$ it is obvious) that

$$\begin{aligned} |(1 + b\zeta^3)\zeta| &\leq |\zeta| + b|\zeta|^4 \leq |\zeta| + b|\zeta| \\ &< c^{-2}(1 - b|\zeta|) < c^{-2}(1 - b|\zeta|^3) \\ &\leq c^{-2}|1 - b\zeta^3|. \end{aligned} \quad (72)$$

The condition $|z| \leq (2c^2)^{-1}$ holds if $2|\zeta| < q$. Equation (71) is satisfied if

$$\begin{aligned} |\xi| &< \sqrt{q}, \\ |\eta| &< \sqrt{q}. \end{aligned} \quad (73)$$

From these bounds, one obtains

$$\begin{aligned} |Z_j(z)| &\leq Z_j(|z|) < Z(|z|) \leq |z|^4 + |z|^3 Z'(|z|) \\ &\leq 2c^{-8}. \end{aligned} \quad (74)$$

For the difference variables of negative charges x'_j from (9), we derive the bound

$$|x'_j| \leq |\xi|^4 (1 + b|\zeta|^3)^{1/3} (1 + Z(|z|)). \quad (75)$$

As a result, for the variables of negative charges x_j , we deduce the bound

$$\begin{aligned} |r_j - x_j| &< gq^2 (1 + bq^3)^{1/3} (1 + 2c^{-8}) \\ &< 2gc^{-4} (1 + c^{-2}) < 4gc^{-4} \\ &< 2r_0 g_0 (134 + 4\sqrt{g_0})^{-4} < (134)^{-2} \frac{2r_0}{16}, \end{aligned} \quad (76)$$

since

$$\begin{aligned} (1 + bq^3)^{1/3} &< (1 + c^{-6})^{1/3} < 1 + c^{-2}, \\ c &> 134 + 4\sqrt{g_0}. \end{aligned} \quad (77)$$

Remark 1. If $h^0 < 0$, then one has to substitute $|b| = -h_0$ in all the bounds and use

$$u = \frac{1 - b\zeta_3}{(1 - b\zeta_3)^{2/3}} \ll \frac{1 + |b|\zeta^3}{1 - |b|\zeta^3} = v. \quad (78)$$

Let us consider the case of the zero magnetic field and $b \neq 0$. Then (60) is majorized by ($a \geq 0$)

$$\begin{aligned} Z' &\ll c \frac{2p + 2z + Z'^2}{4 - cZ'} \ll c \frac{2p + 2z + Z'^2}{4 - cZ'} + 4a \\ &\ll c \frac{2p - 4aZ' + 2z + Z'^2 + 16c^{-1}a}{4 - cZ'}, \quad p = \frac{2nb}{c}. \end{aligned} \quad (79)$$

Let

$$\begin{aligned} W' &= c \frac{2p - 4aW' + 2z + W'^2 + 16c^{-1}a}{4 - cW'}, \\ cW'^2 - 2(1 + ca)W' + cz + cp + 8a &= 0. \end{aligned} \quad (80)$$

The solution of the quadratic equation is given by

$$W' = c^{-1} \left(1 + ca - \sqrt{(1 + ca)^2 - c^2z - c^2p - 8ac} \right). \quad (81)$$

Or

$$\begin{aligned} W' &= c^{-1} (1 + ca - d) + W_0 = a' + W_0, \\ W_0(z) &= c^{-1} (d - \sqrt{d^2 - c^2z}) = dW(d^{-2}z), \end{aligned} \quad (82)$$

where

$$d^2 = (1 + ca)^2 - c^2p - 8ac > 0. \quad (83)$$

Let

$$\begin{aligned} (1 + ca)^2 &= c^2p + 8ac + 1, \\ a^2 &= 6c^{-1}a + p, \\ (a - 3c^{-1})^2 &= p + 9c^{-2}. \end{aligned} \quad (84)$$

This results in

$$\begin{aligned} a &= 3c^{-1} + \sqrt{p + 9c^{-2}}, \\ d &= 1, \\ a' &= a. \end{aligned} \quad (85)$$

This condition means that W' is given by the convergent power expansion if $|z| < c^{-2}$ and $W' = a + W \gg 0$ and $Z' \ll W'$.

Note that if $c^2p < 1$, then we can put $a = 0$ and obtain $W' \gg 0$. But in our case $c^2p > 1$.

If (85) and $|z| \leq (2c^2)^{-1}$ hold, then

$$|Z'(z)| \leq Z'(|z|) \leq W'(|z|) \leq c^{-1} + a. \quad (86)$$

From these bounds, one obtains also

$$\begin{aligned} |Z_j(z)| &\leq Z_j(|z|) < Z(|z|) \leq |z|^4 + |z|^3 Z'(|z|) \\ &\leq c^{-6} (c^{-2} + c^{-1} + a), \\ a &< 6c^{-1} + \sqrt{p}. \end{aligned} \quad (87)$$

That is, if the condition $\sqrt{p} < c^5$, which trivially follows from $b < (2n)^{-1}$, holds, then

$$c^{-6} (c^{-2} + c^{-1} + a) < c^{-6} (8c^{-1} + \sqrt{p}) < 1. \quad (88)$$

This bound and (75), (85), and (87) give

$$\begin{aligned} |r_j - x_j| &< gq^2 (1 + bq^3)^{1/3} [1 + c^{-6} (c^{-2} + c^{-1} + a)] \\ &< 2gc^{-4} (1 + c^{-2}) < 4gc^{-4} \\ &< 2r_0 g_0 (38 + 4\sqrt{g_0})^{-4} < (38)^{-2} \frac{2r_0}{16}. \end{aligned} \quad (89)$$

Hence, we proved the following theorem.

Theorem 2. *Let the magnetic field be either zero and $b \neq 0$ or nonzero and $b = h_0$. Then the difference coordinates $x_j, \tilde{x}_j$ given by (9)-(10) are holomorphic functions in the hyperdisc $|\xi| < \sqrt{q}, |\eta| < \sqrt{q}$ and satisfy (7) if $\xi(t), \eta(t)$ satisfy (8). They are also periodic functions in t with the period $8\pi(\xi(0)\eta(0))^3$ if $\eta = \xi^*$. The coordinates x_j of the negative charges, expressed in terms of these difference coordinates with primes by (4) for $\eta = \xi^*$, satisfy the Coulomb equation of motion (3). Moreover, if $\sqrt{2}|\xi| \leq \sqrt{q}, \sqrt{2}|\eta| \leq \sqrt{q}$, and $\sqrt{p} < c^5$ for the zero magnetic field, then the following inequality is true:*

$$|r_j - x_j| < \frac{r_0}{8\nu'^2}, \quad (90)$$

where $\nu' = 134(38)$ if the magnetic field is nonzero (zero).

Note that (31) with $\nu = 1, \zeta = z$ holds also for the case where $b = 0, h_0 \neq 0$ with $2p' = b'c_2^{-1}, b' = nh_0$, which implies that $p = 2nh_0c^{-1}$ (this follows from (49) with $b = 0$). That is, Theorem 2 is true for the case and the condition $\sqrt{p} < c^5$ which restricts values of h_0, g_0 .

5. Conclusion

Theorem 2 determines periodic solution of the considered equation of motion such that each negative charge moves close to its own positive fixed charge and the distance between them exceeds $r_0 - r_0/4\nu'^2$. This excludes collisions between the charges.

If one considers arbitrary location of the equal positive charges on a complex plane, then it is necessary to consider complex $a_{j,k,l}$. In this case, the analog of (28) can be derived and all the used majorant bounds can be applied for the proof of the existence of periodic solution of the considered equation of motion.

We hope that the proposed result and technique will be useful for the quantum systems. Earlier classical Coulomb

systems, which are relevant to quantum atomic systems, were considered in [13, 14]. Our systems can be useful for a description of molecules with n nuclei.

Appendix

Here we prove (32)-(33). Let $|x| = |y|, X \ll Q, Y \ll Q$; then (32) follows from

$$\begin{aligned} (x+X)^{-1/2} (y+Y)^{-3/2} - x^{-1/2} y^{-3/2} \\ = x^{-1/2} y^{-3/2} \left[(1+x^{-1}X)^{-1/2} (1+y^{-1}Y)^{-3/2} - 1 \right] \\ \ll |x|^{-2} \left[(1-|x|^{-1}Q)^{-1/2} (1-|x|^{-1}Q)^{-3/2} - 1 \right] \\ = |x|^{-2} \left[(1-|x|^{-1}Q)^{-2} - 1 \right] \\ \ll |x|^{-2} \left[(1-2(|x|^{-1}Q))^{-1} - 1 \right] \\ = |x|^{-3} \frac{2Q}{1-2|x|^{-1}Q}. \end{aligned} \quad (A.1)$$

Equation (33) follows from

$$\begin{aligned} (x+X)^{-1/2} (y+Y)^{-3/2} + x^{-1/2} y^{-3/2} \left(-1 + \frac{1}{2x}X + \frac{3}{2y}Y \right) \\ = x^{-1/2} y^{-3/2} \left[(1+x^{-1}X)^{-1/2} (1+y^{-1}Y)^{-3/2} - 1 \right. \\ \left. + \frac{1}{2x}X + \frac{3}{2y}Y \right] \ll |x|^{-2} \\ \cdot \left[(1-|x|^{-1}Q)^{-1/2} (1-|x|^{-1}Q)^{-3/2} - 1 \right. \\ \left. - \frac{1}{2}|x|^{-1}Q - \frac{3}{2}|x|^{-1}Q \right] = |x|^{-2} \left[(1-|x|^{-1}Q)^{-2} \right. \\ \left. - 1 - 2|x|^{-1}Q \right] \ll |x|^{-2} \left[(1-2|x|^{-1}Q)^{-1} - 1 \right. \\ \left. - 2|x|^{-1}Q \right] = |x|^{-4} \frac{4Q^2}{1-2|x|^{-1}Q}. \end{aligned} \quad (A.2)$$

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

References

- [1] W. I. Skrypnik, "On the holomorphic solutions of Hamiltonian equations of motion of point charges," *Ukrainian Mathematical Journal*, vol. 63, no. 2, pp. 315–327, 2011.
- [2] W. I. Skrypnik, "On holomorphic solutions of the Darwin equations of motion of point charges," *Ukrainian Mathematical Journal*, vol. 65, no. 4, pp. 602–611, 2013.

- [3] W. I. Skrypnik, "Periodic and bounded solutions of the Coulomb equation of motion of two and three point charges with equilibrium in the line," *Ukrainian Mathematical Journal*, vol. 66, no. 5, pp. 758–774, 2014.
- [4] W. I. Skrypnik, "On exact solutions of Coulomb equation of motion of planar charges," *Journal of Geometry and Physics*, vol. 98, pp. 285–291, 2015.
- [5] W. I. Skrypnik, "On regular polygon solutions of Coulomb equation of motion of $n + 2$ charges n of which are planar," *Journal of Mathematical Physics*, vol. 57, no. 4, 2016.
- [6] C. L. Siegel and J. K. Moser, *Lectures on Celestial Mechanics*, Springer, New York, NY, USA, 1971.
- [7] C. L. Siegel, "Über eine periodische Lösung in ebenen Dreikörperproblem," *Mathematische Nachrichten*, vol. 4, pp. 28–35, 1951.
- [8] W. I. Skrypnik, "Coulomb dynamics near equilibrium of two equal negative charges in the field of two fixed identical positive charges," *Ukrainian Mathematical Journal*, vol. 68, no. 9, pp. 1466–1480, 2017.
- [9] W. I. Skrypnik, "Two-Dimensional Coulomb Dynamics of Two and Three Equal Negative Charges in the Field of Two Equal Fixed Positive Charges," *Ukrainian Mathematical Journal*, vol. 68, no. 11, pp. 1767–1781, 2017.
- [10] M. Seri, "The problem of two fixed centers: bifurcation diagram for positive energies," *Journal of Mathematical Physics*, vol. 56, no. 1, 012902, 14 pages, 2015.
- [11] A. Knauf, "The n -centre problem of celestial mechanics for large energies," *Journal of the European Mathematical Society*, vol. 4, no. 1, pp. 1–114, 2002.
- [12] A. Knauf and I. A. Taimanov, "On the integrability of the n -centre problem," *Mathematische Annalen*, vol. 331, no. 3, pp. 631–649, 2005.
- [13] I. Davies, A. Truman, and D. Williams, "Classical periodic solution of the equal-mass $2n$ -body problem, $2n$ -ion problem and the n -electron atom problem," *Physics Letters A*, vol. 99, no. 1, pp. 15–18, 1983.
- [14] K. Richter, G. Tanner, and D. Wintgen, "Classical mechanics of two-electron atoms," *Physical Review A: Atomic, Molecular and Optical Physics*, vol. 48, no. 6, article 4182, 1993.

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