Research Article

Existence of Nontrivial Solutions for Generalized Quasilinear Schrödinger Equations with Critical Growth

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We study the following generalized quasilinear Schrödinger equations with critical growth $-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = \lambda f(x, u) + g(u)|G(u)|^{2^*-2}G(u)$, $x \in \mathbb{R}^N$, where $\lambda > 0$, $N \geq 3$, $g(s) : \mathbb{R} \rightarrow \mathbb{R}^+$ is a C^1 even function, $g(0) = 1$, and $g'(s) \geq 0$ for all $s \geq 0$, where $G(u) := \int_0^u g(t)dt$. Under some suitable conditions, we prove that the equation has a nontrivial solution by variational method.

1. Introduction and Preliminaries

Consider the following generalized quasilinear Schrödinger equations with critical growth:

$$\begin{aligned} & -\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u \\ & = \lambda f(x, u) + g(u)|G(u)|^{2^*-2}G(u), \quad x \in \mathbb{R}^N, \end{aligned} \quad (1)$$

where $\lambda > 0$, $N \geq 3$, $g(s) : \mathbb{R} \rightarrow \mathbb{R}^+$ is a C^1 even function, $g(0) = 1$, and $g'(s) \geq 0$ for all $s \geq 0$.

The equations are related to the existence of solitary wave solutions for quasilinear Schrödinger equations

$$\begin{aligned} iz_t &= -\Delta z + W(x)z - k(x, |z|)z \\ & - \Delta(l(|z|^2))l'(|z|^2)z, \end{aligned} \quad (2)$$

where $z : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C}$, $W : \mathbb{R}^N \rightarrow \mathbb{R}$ is a given potential, $l : \mathbb{R} \rightarrow \mathbb{R}$, and $k : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ are suitable functions. The form of (2) has been derived as models of several physical phenomena corresponding to various types of $l(s)$. For instance, the case $l(s) = s$ models the time evolution of the condensate wave function in superfluid film [1, 2] and is called the superfluid film equation in fluid mechanics

by Kurihara [1]. In the case $l(s) = (1 + s)^{1/2}$, problem (2) models the self-channeling of a high-power ultra short laser in matter, the propagation of a high-irradiance laser in a plasma creates an optical index depending nonlinearly on the light intensity, and this leads to interesting new nonlinear wave equations; see [3–6]. For more physical motivations and more references dealing with applications, we can refer to [7–14] and references therein.

Set $z(t, x) = \exp(-iEt)u(x)$, where $E \in \mathbb{R}$ and u is a real function. Then (2) can be reduced to the corresponding equation of elliptic type (see [15]):

$$\begin{aligned} & -\Delta u + V(x)u - \Delta l(u^2)l'(u^2)u = f(x, u), \\ & x \in \mathbb{R}^N, \end{aligned} \quad (3)$$

where $f(x, u) = k(x, |u|)u$. If we take

$$g^2(u) = 1 + \frac{[l(u^2)]'}{2}, \quad (4)$$

then (1) turns into (3) (see [16]).

Moreover, problem (3) also arises in biological models and propagation of laser beams when $g(u)$ is a positive constant (see [17, 18]). In (3), if we set $l(u) = u$, that is,

$g^2(u) = 1 + 2u^2$, then we get the superfluid film equation in plasma physics:

$$-\Delta u + V(x)u - \Delta(u^2)u = f(x, u), \quad x \in \mathbb{R}^N; \quad (5)$$

if we set $l(u) = (1 + u)^{1/2}$, that is, $g^2(u) = 1 + u^2/2(1 + u^2)$, then we get the equation

$$\begin{aligned} & -\Delta u + V(x)u - \left[\Delta(1 + u^2)^{1/2} \right] \frac{u}{2(1 + u^2)^{1/2}} \\ & = f(x, u), \quad x \in \mathbb{R}^N, \end{aligned} \quad (6)$$

which models the self-channeling of a high-power ultrashort laser in matter.

In the past, the research on the existence of solitary wave solutions of Schrödinger equations (2) is for some given special function $l(s)$. In this paper, we will use a unified new variable replacement to study (2), constructed by Shen and Wang in [16]. Define the energy functional associated with (1) by

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2} \int_{\mathbb{R}^N} g^2(u) |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) u^2 dx \\ &\quad - \lambda \int_{\mathbb{R}^N} F(x, u) dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |G(u)|^{2^*} dx, \end{aligned} \quad (7)$$

where $F(x, u) := \int_0^u f(x, t) dt$. However, I_λ is not well defined in $H^1(\mathbb{R}^N)$ because of the term $\int_{\mathbb{R}^N} g^2(u) |\nabla u|^2 dx$. To overcome this difficulty, we make a change of variable constructed by Shen and Wang in [16]: $v := G(u) := \int_0^u g(t) dt$. Then we obtain

$$\begin{aligned} J_\lambda(v) &= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) G^{-1}(v)^2 dx \\ &\quad - \lambda \int_{\mathbb{R}^N} F(x, G^{-1}(v)) dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |v|^{2^*} dx. \end{aligned} \quad (8)$$

If u is a nontrivial solution of (1), then

$$\begin{aligned} \langle I'_\lambda(u), \varphi \rangle &= \int_{\mathbb{R}^N} \left[g^2(u) \nabla u \nabla \varphi + g(u) g'(u) |\nabla u|^2 \varphi \right. \\ &\quad \left. + V(x) u \varphi - \lambda f(x, u) \varphi \right. \\ &\quad \left. - g(u) |G(u)|^{2^*-2} G(u) \varphi \right] dx = 0 \end{aligned} \quad (9)$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Let $\varphi = (1/g(u))\psi$. By [16] we know that (9) is equivalent to

$$\begin{aligned} \langle J'_\lambda(v), \psi \rangle &= \int_{\mathbb{R}^N} \left[\nabla v \nabla \psi + V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} \psi \right. \\ &\quad \left. - \lambda \frac{f(x, G^{-1}(v))}{g(G^{-1}(v))} \psi - |v|^{2^*-2} v \psi \right] dx = 0 \end{aligned} \quad (10)$$

for all $\psi \in C_0^\infty(\mathbb{R}^N)$. Therefore, in order to find the nontrivial solution of (1), it suffices to study the existence of the nontrivial solutions of the following equations:

$$\begin{aligned} & -\Delta v + V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} - \lambda \frac{f(x, G^{-1}(v))}{g(G^{-1}(v))} - |v|^{2^*-2} v \\ & = 0. \end{aligned} \quad (11)$$

Recently, the authors studied generalized quasilinear Schrödinger equations with subcritical growth [19, 20], critical growth [21], and supercritical growth [22].

In order to reduce the statements for main results, we list the assumptions as follows:

(V₁) $V(x) \geq V_0 := \inf_{x \in \mathbb{R}^N} V(x) > 0$ for all $x \in \mathbb{R}^N$.

(V₂) $\lim_{|x| \rightarrow \infty} V(x) = V_\infty < +\infty$ and $V(x) \leq V_\infty$ for all $x \in \mathbb{R}^N$.

(f₁) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there exists $2 < p < 2^*$ such that

$$|f(x, t)| \leq C(1 + g(t)|G(t)|^{p-1}) \quad (12)$$

for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$.

(f₂) $f(x, t) = o(|t|)$ uniformly in $x \in \mathbb{R}^N$ as $|t| \rightarrow 0$.

(f₃) $(f(x, G^{-1}(t))/g(G^{-1}(t)))t - 2F(x, G^{-1}(t)) \geq (f(x, G^{-1}(st))/g(G^{-1}(st)))st - 2F(x, G^{-1}(st))$ for all $t \in \mathbb{R}$ and $s \in [0, 1]$.

(f₄) $f(x, t)t > 0$ for all $(x, t) \in \mathbb{R}^N \times \mathbb{R} \setminus \{0\}$.

(f₅) $\lim_{|t| \rightarrow +\infty} (F(x, G^{-1}(t))/t^2) = +\infty$ uniformly in $x \in \mathbb{R}^N$.

Set $E = H^1(\mathbb{R}^N)$ with the norm

$$\|u\|_E = \left[\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx \right]^{1/2}. \quad (13)$$

It is easy to prove that J_λ is well defined on E and $J_\lambda \in C^1(E, \mathbb{R})$ under our assumptions and its Gateaux derivative is given by

$$\begin{aligned} \langle J'_\lambda(v), \varphi \rangle &= \int_{\mathbb{R}^N} \left[\nabla v \nabla \varphi + V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} \varphi \right. \\ &\quad \left. - \lambda \frac{f(x, G^{-1}(v))}{g(G^{-1}(v))} \varphi - |v|^{2^*-2} v \varphi \right] dx \end{aligned} \quad (14)$$

for all $v, \varphi \in E$.

Our main result of this paper is as follows.

Theorem 1. Suppose that (V₁), (V₂), and (f₁)–(f₅) are satisfied. Then if $N \geq 5$, (1) admits a nontrivial solution for all $\lambda > 0$; if $N = 3, 4$, (1) admits a nontrivial solution for large λ .

Remark 2. Condition (f₃) is weaker than the following condition (f₆).

(f_6) $f(x, G^{-1}(t))/g(G^{-1}(t))t$ is nonincreasing on $(-\infty, 0)$ and nondecreasing on $(0, +\infty)$.

Indeed, set $l(s) = s^2 t (f(x, G^{-1}(t))/g(G^{-1}(t))) - 2F(x, G^{-1}(st))$, $\forall s \in [0, 1]$. Then

$$\begin{aligned} l'(s) &= 2st \frac{f(x, G^{-1}(t))}{g(G^{-1}(t))} - 2 \frac{f(x, G^{-1}(st))}{g(G^{-1}(st))} t \\ &= 2st \frac{f(x, G^{-1}(t))}{g(G^{-1}(t))} - 2t \frac{f(x, G^{-1}(st))}{g(G^{-1}(st))} (st). \end{aligned} \quad (15)$$

If (f_6) holds, then

$$l'(s) \geq 2st \frac{f(x, G^{-1}(t))}{g(G^{-1}(t))} - 2t \frac{f(x, G^{-1}(t))}{g(G^{-1}(t))} (st) = 0 \quad (16)$$

whenever $t > 0$ or $t < 0$. Hence $l(s)$ is nondecreasing on $[0, 1]$, and hence $l(1) \geq l(s)$ for all $s \in [0, 1]$. Consequently, (f_6) implies that

$$\begin{aligned} & t \frac{f(x, G^{-1}(t))}{g(G^{-1}(t))} - 2F(x, G^{-1}(t)) \\ & \geq s^2 t \frac{f(x, G^{-1}(t))}{g(G^{-1}(t))} - 2F(x, G^{-1}(st)) \\ & = s^2 t |t| \frac{f(x, G^{-1}(t))}{g(G^{-1}(t)) |t|} - 2F(x, G^{-1}(st)) \\ & \geq s^2 t |t| \frac{f(x, G^{-1}(st))}{g(G^{-1}(st)) |st|} - 2F(x, G^{-1}(st)) \\ & = st \frac{f(x, G^{-1}(st))}{g(G^{-1}(st))} - 2F(x, G^{-1}(st)) \end{aligned} \quad (17)$$

for all $s \in [0, 1]$; that is, the condition (f_3) holds.

From Remark 2 we obtain Corollary 3.

Corollary 3. Suppose that (V_1) , (V_2) , (f_1) – (f_2) , (f_4) – (f_5) , and (f_6) are satisfied. Then if $N \geq 5$, (1) admits a nontrivial solution for all $\lambda > 0$; if $N = 3, 4$, (1) admits a nontrivial solution for large λ .

Remark 4. In [16], Shen and Wang studied the existence of nontrivial solutions for generalized quasilinear Schrödinger equations

$$\begin{aligned} & -\operatorname{div}(g^2(u) \nabla u) + g(u) g'(u) |\nabla u|^2 + V(x) u \\ & = h(u), \quad x \in \mathbb{R}^N, \end{aligned} \quad (18)$$

where h is a subcritical nonlinearity satisfying the following conditions:

- (h_0) $h(t) = 0$ if $t \leq 0$.
- (h_1) $h(t) = o(t)$ as $t \rightarrow 0^+$.

(h_2) There exists $2 < p < 2^*$ such that

$$|h(t)| \leq C(1 + g(t)|G(t)|^{p-1}) \quad (19)$$

for all $t > 0$.

(h_3) There exists $\mu > 2$ such that, for any $t > 0$, there holds

$$0 < \mu g(G^{-1}(t)) H(G^{-1}(t)) \leq h(G^{-1}(t)) t. \quad (20)$$

As mentioned above, if we set $g^2(u) = 1 + 2u^2$, then we get the superfluid film equation in plasma physics

$$-\Delta u + V(x)u - \Delta(u^2)u = h(u), \quad x \in \mathbb{R}^N, \quad (21)$$

whose nontrivial solutions were studied in [23]. But our problem (1) is elliptic problem involving the critical exponent, so our result extends the results of the work [16, 23] to a critical setting. Moreover, the assumptions about the nonlinearity in this paper are different from the assumptions about the nonlinearity in [16, 23].

Remark 5. In [24], Deng et al. studied problem (1) and their result based on more harsh conditions:

- $(f_1)^*$ $f(x, t) \geq 0$ is differentiable with respect to $t \in [0, +\infty)$ for all $x \in \mathbb{R}^N$ and continuous with respect to $x \in \mathbb{R}^N$ for all $t \in [0, +\infty)$. Moreover, $f(x, t) \equiv 0$ for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}^-$.
- $(f_3)^*$ There exists $\delta \in (0, 2^* - 2)$ such that, for any $t > 0$, there holds $(1 + \delta)f(x, t) \leq G(t)[f(x, t)/g(t)]'$, which implies that there exists $\mu \in (2, 2^*)$ such that $f(x, t)G(t) \geq \mu g(t)F(x, t)$ for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$.

In this paper, we just assume that f is a continuous function. Moreover, there are functionals $f(x, t)$ satisfying (f_3) but not satisfying the above Ambrosetti-Rabinowitz type condition (see Remark 1.2 in [25]). Hence, our result is different from the result there.

2. Proof of Theorem 1

To begin with, we give some lemmas.

Lemma 6. For the functions g , G , and G^{-1} , the following properties hold:

- (1) the functions $G(\cdot)$ and $G^{-1}(\cdot)$ are strictly increasing and odd;
- (2) $G(s) \leq g(s)s$ for all $s \geq 0$; $G(s) \geq g(s)s$ for all $s \leq 0$;
- (3) $g(G^{-1}(s)) \geq g(0) = 1$ for all $s \in \mathbb{R}$;
- (4) $G^{-1}(s)/s$ is decreasing on $(0, +\infty)$ and increasing on $(-\infty, 0)$;
- (5) $|G^{-1}(s)| \leq (1/g(0))|s| = |s|$ for all $s \in \mathbb{R}$;
- (6) $|G^{-1}(s)|/g(G^{-1}(s)) \leq (1/g^2(0))|s| = |s|$ for all $s \in \mathbb{R}$;
- (7) $G^{-1}(s)s/g(G^{-1}(s)) \leq |G^{-1}(s)|^2$ for all $s \in \mathbb{R}$;

(8) $\lim_{|s| \rightarrow 0} (G^{-1}(s)/s) = 1/g(0) = 1$ and

$$\lim_{|s| \rightarrow \infty} \frac{G^{-1}(s)}{s} = \begin{cases} \frac{1}{g(\infty)}, & \text{if } g \text{ is bounded,} \\ 0, & \text{if } g \text{ is unbounded.} \end{cases} \quad (22)$$

Proof. Properties (1)–(3) are obvious. By (2), we have

$$\left(\frac{G^{-1}(s)}{s} \right)' = \frac{s - G^{-1}(s) g(G^{-1}(s))}{g(G^{-1}(s)) s^2} \leq 0 \quad (23)$$

for all $s > 0$ and

$$\left(\frac{G^{-1}(s)}{s} \right)' = \frac{s - G^{-1}(s) g(G^{-1}(s))}{g(G^{-1}(s)) s^2} \geq 0 \quad (24)$$

for all $s < 0$. Consequently, we obtain (4). By mean value theorem and (3), one has

$$\begin{aligned} |G^{-1}(s)| &= |G^{-1}(s) - G^{-1}(0)| = \frac{1}{g(G^{-1}(\theta s))} |s| \\ &\leq \frac{1}{g(0)} |s| \end{aligned} \quad (25)$$

for all $s \in \mathbb{R}$, where $\theta \in (0, 1)$; that is, (5) is proved. Obviously, (6) is a consequence of (3) and (5). Moreover, (7) is a consequence of (2). Finally, using L'Hospital's rule, we know that (8) is satisfied. This completes the proof. $\square$

Denote

$$\begin{aligned} h_\lambda(x, s) &= V(x) s - V(x) \frac{G^{-1}(s)}{g(G^{-1}(s))} \\ &\quad + \lambda \frac{f(x, G^{-1}(s))}{g(G^{-1}(s))}. \end{aligned} \quad (26)$$

Then

$$\begin{aligned} H_\lambda(x, s) &:= \int_0^s h_\lambda(x, t) dt \\ &= \frac{1}{2} V(x) [s^2 - G^{-1}(s)^2] + \lambda F(x, G^{-1}(s)). \end{aligned} \quad (27)$$

Consequently,

$$\begin{aligned} J_\lambda(v) &= \frac{1}{2} \int_{\mathbb{R}^N} [|\nabla v|^2 + V(x) v^2] dx \\ &\quad - \int_{\mathbb{R}^N} H_\lambda(x, v) dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |v|^{2^*} dx. \end{aligned} \quad (28)$$

Lemma 7. *The functions $h_\lambda(x, s)$ and $H_\lambda(x, s)$ enjoy the following properties under (f_1) – (f_5) :*

- (1) $\lim_{|s| \rightarrow 0} (h_\lambda(x, s)/s) = 0$ and $\lim_{|s| \rightarrow 0} (H_\lambda(x, s)/s^2) = 0$ uniformly in $x \in \mathbb{R}^N$;
- (2) $\lim_{|s| \rightarrow \infty} (h_\lambda(x, s)/|s|^{2^*-1}) = 0$ and $\lim_{|s| \rightarrow \infty} (H_\lambda(x, s)/|s|^{2^*}) = 0$ uniformly in $x \in \mathbb{R}^N$;

(3) $th_\lambda(x, t) - 2H_\lambda(x, t) \geq sth_\lambda(x, st) - 2H_\lambda(x, st)$ for all $t \in \mathbb{R}$ and $s \in [0, 1]$;

(4) $H_\lambda(x, s) \geq 0$ for all $(x, s) \in \mathbb{R}^N \times \mathbb{R}$;

(5) $\lim_{|s| \rightarrow +\infty} (H_\lambda(x, s)/s^2) = +\infty$ uniformly in $x \in \mathbb{R}^N$.

Proof. By (f_1) – (f_2) , for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\left| \frac{f(x, G^{-1}(s))}{g(G^{-1}(s))} \right| \leq \varepsilon |s| + C_\varepsilon |s|^{p-1} \quad (29)$$

for all $(x, s) \in \mathbb{R}^N \times \mathbb{R}$. Set $G^{-1}(s) = t$. Then Lemma 6(8) implies that

$$\begin{aligned} \lim_{|s| \rightarrow 0} \frac{h_\lambda(x, s)}{s} &= V(x) \left[1 - \frac{1}{g^2(0)} \right] + \lambda \lim_{|t| \rightarrow 0} \frac{f(x, t)}{g(t) G(t)} \\ &= 0 \end{aligned} \quad (30)$$

uniformly in $x \in \mathbb{R}^N$. Moreover, by Lemma 6(6) one has

$$\begin{aligned} \lim_{|s| \rightarrow \infty} \frac{h_\lambda(x, s)}{|s|^{2^*-1}} &= -V(x) \lim_{|s| \rightarrow \infty} \frac{G^{-1}(s)}{sg(G^{-1}(s))} \frac{s}{|s|^{2^*-1}} \\ &\quad + \lambda \lim_{|t| \rightarrow \infty} \frac{f(x, t)}{g(t) |G(t)|^{2^*-1}} = 0 \end{aligned} \quad (31)$$

uniformly in $x \in \mathbb{R}^N$. Similarly, we have

$$\lim_{|s| \rightarrow 0} \frac{H_\lambda(x, s)}{s^2} = 0 \quad (32)$$

uniformly in $x \in \mathbb{R}^N$ and

$$\lim_{|s| \rightarrow \infty} \frac{H_\lambda(x, s)}{|s|^{2^*}} = 0 \quad (33)$$

uniformly in $x \in \mathbb{R}^N$. Hence, (1) and (2) hold.

In the following, we set $l(t) = G^{-1}(t)^2 - G^{-1}(t)t/g(G^{-1}(t))$, $\forall t \in \mathbb{R}$. If $t \geq 0$, by Lemma 6(2) and $g'(t) \geq 0$ for $t \geq 0$, we have

$$G(t) \left[\frac{1}{g^2(t)} (g(t) - g'(t)t) \right] \leq t \quad (34)$$

for $t \geq 0$, which implies that

$$G(t) \left(\frac{t}{g(t)} \right)' \frac{1}{g(t)} \leq \frac{t}{g(t)} \quad (35)$$

for all $t \geq 0$. Let $r = G(t)$. Then

$$G(t) \frac{d}{dr} \left(\frac{t}{g(t)} \right) \leq \frac{t}{g(t)} \quad (36)$$

and hence

$$r \left[\frac{G^{-1}(r)}{g(G^{-1}(r))} \right]' \leq \frac{G^{-1}(r)}{g(G^{-1}(r))} \quad (37)$$

for $r \geq 0$. Consequently,

$$\begin{aligned} l'(t) &= \frac{2G^{-1}(t)}{g(G^{-1}(t))} - \left[\frac{G^{-1}(t)}{g(G^{-1}(t))} \right]' t - \frac{G^{-1}(t)}{g(G^{-1}(t))} \\ &= \frac{G^{-1}(t)}{g(G^{-1}(t))} - \left[\frac{G^{-1}(t)}{g(G^{-1}(t))} \right]' t \geq 0 \end{aligned} \quad (38)$$

for all $t \geq 0$, that is, $l(t)$ is increasing with respect to $t \geq 0$. Hence $l(st) \leq l(t)$ for all $s \in [0, 1]$ and $t \geq 0$; that is,

$$G^{-1}(st)^2 - \frac{G^{-1}(st)st}{g(G^{-1}(st))} \leq G^{-1}(t)^2 - \frac{G^{-1}(t)t}{g(G^{-1}(t))} \quad (39)$$

for all $s \in [0, 1]$ and $t \geq 0$. Note that Lemma 6(1) implies that $l(t)$ is an even function. Therefore, if $t < 0$, we easily obtain that $l(st) \leq l(t)$ for all $s \in [0, 1]$ and $t < 0$. Consequently,

$$G^{-1}(st)^2 - \frac{G^{-1}(st)st}{g(G^{-1}(st))} \leq G^{-1}(t)^2 - \frac{G^{-1}(t)t}{g(G^{-1}(t))} \quad (40)$$

for all $s \in [0, 1]$ and $t \in \mathbb{R}$. Combining with (f_3) , we can conclude (3). Moreover, (f_4) and Lemma 6(5) imply that $H(x, s) \geq 0$ for all $(x, s) \in \mathbb{R}^N \times \mathbb{R}$. Clearly, (f_5) and Lemma 6(5) imply that (5) is satisfied. This completes the proof. $\square$

Lemma 8. Suppose that (V_1) , (V_2) , and (f_1) – (f_2) are satisfied. Then the energy functional J_λ satisfies the following conditions:

- (i) There exist $\beta, \rho > 0$ such that $J_\lambda(v) \geq \beta$ for $\|v\|_E = \rho$.
- (ii) There exists $e \in E$ with $\|e\|_E > \rho$ such that $J_\lambda(e) < 0$.

Proof. (i) Set $S_\rho := \{u \in E : \|u\|_E = \rho\}$. By (f_1) – (f_2) , Lemmas 6(6) and 7(1), and (2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|H_\lambda(x, s)| \leq \varepsilon(|s|^2 + |s|^{2^*}) + C_\varepsilon |s|^p \quad (41)$$

for all $(x, s) \in \mathbb{R}^N \times \mathbb{R}$. Consequently, for $v \in S_\rho$, we have

$$\begin{aligned} J_\lambda(v) &\geq \frac{1}{2}C_1 \|v\|_E^2 - C_2\varepsilon \|v\|_E^2 - C_3\varepsilon \|v\|_E^{2^*} \\ &\quad - C_4C_\varepsilon \|v\|_E^p \\ &\geq \frac{1}{2}C_1\rho^2 - C_2\varepsilon\rho^2 - C_3\varepsilon\rho^{2^*} - C_4C_\varepsilon\rho^p := \beta > 0 \end{aligned} \quad (42)$$

for small $\varepsilon > 0$ and $\rho > 0$.

(ii) Take $v^* \in E \setminus \{0\}$. Then

$$\begin{aligned} J_\lambda(tv^*) &\leq \frac{1}{2}C_5t^2 \|v^*\|_E^2 - \frac{1}{2^*}t^{2^*} \int_{\mathbb{R}^N} |v^*|^{2^*} dx \\ &\quad + \varepsilon t^2 \int_{\mathbb{R}^N} |v^*|^2 dx + \varepsilon t^{2^*} \int_{\mathbb{R}^N} |v^*|^{2^*} dx \\ &\quad + C_\varepsilon t^p \int_{\mathbb{R}^N} |v^*|^p dx < 0 \end{aligned} \quad (43)$$

for large $t > 0$ and small $\varepsilon > 0$. Consequently, we can take $e := t^*v^*$ for some large $t^* > 0$ such that (ii) holds. This completes the proof. $\square$

Lemma 9. Suppose that (V_1) , (V_2) , and (f_1) – (f_4) are satisfied. Then there exists a bounded Cerami sequence $\{v_n\} \subset E$ for J_λ with $J_\lambda(v_n) \rightarrow c_\lambda \geq \beta > 0$, where

$$c_\lambda := \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} J_\lambda(\gamma(t)), \quad (44)$$

$$\Gamma := \{\gamma \in C([0, 1], E) : \gamma(0) = 0, J_\lambda(\gamma(1)) < 0\},$$

β is the constant appearing in Lemma 8.

Proof. By Lemma 8 and the mountain pass theorem without (PS) condition (see Theorem 4.1 in [26]), there exists a Cerami sequence $\{v_n\} \subset E$ satisfying

$$J_\lambda(v_n) \rightarrow c_\lambda \geq \beta > 0, \quad (45)$$

$$(1 + \|v_n\|_E) \|J'_\lambda(v_n)\|_{E^*} \rightarrow 0,$$

where

$$c_\lambda := \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} J_\lambda(\gamma(t)), \quad (46)$$

$$\Gamma := \{\gamma \in C([0, 1], E) : \gamma(0) = 0, J_\lambda(\gamma(1)) < 0\},$$

β is the constant appearing in Lemma 8.

Let $t_n \in [0, 1]$ be such that $J_\lambda(t_n v_n) = \max_{t \in [0, 1]} J_\lambda(t v_n)$. Then $\{J_\lambda(t_n v_n)\}$ is bounded from above. Indeed, without loss of the generality, we may assume that $t_n \in (0, 1)$ for all $n \in \mathbb{N}$. Hence, by Lemma 7(3) we have

$$\begin{aligned} J_\lambda(t_n v_n) &= J_\lambda(t_n v_n) - \frac{1}{2} \langle J'_\lambda(t_n v_n), t_n v_n \rangle \\ &= \left(\frac{1}{2} - \frac{1}{2^*} \right) t_n^{2^*} \int_{\mathbb{R}^N} |v_n|^{2^*} dx \\ &\quad + \int_{\mathbb{R}^N} \left[\frac{1}{2} t_n v_n h_\lambda(x, t_n v_n) - H_\lambda(x, t_n v_n) \right] dx \\ &\leq \left(\frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} |v_n|^{2^*} dx \\ &\quad + \int_{\mathbb{R}^N} \left[\frac{1}{2} v_n h_\lambda(x, v_n) - H_\lambda(x, v_n) \right] dx \\ &= J_\lambda(v_n) - \frac{1}{2} \langle J'_\lambda(v_n), v_n \rangle = c_\lambda + o(1). \end{aligned} \quad (47)$$

This shows that $\{J_\lambda(t_n v_n)\}$ is bounded from above.

Now, we prove that $\{v_n\}$ is bounded in E . Otherwise, if $\|v_n\|_E$ is unbounded, then, up to a subsequence, we may assume that $\|v_n\|_E \rightarrow +\infty$. Set $w_n = v_n / \|v_n\|_E$. Then there exists $w \in E$ such that $w_n \rightharpoonup w$ in E . By $J_\lambda(v_n) \rightarrow c_\lambda$, we have

$$\begin{aligned} o(1) &+ \frac{1}{2} \max\{1, V_\infty\} \\ &\geq \frac{1}{2^*} \frac{\|v_n\|_{2^*}^{2^*}}{\|v_n\|_E^2} + \int_{\mathbb{R}^N} \frac{H_\lambda(x, v_n)}{\|v_n\|_E^2} dx. \end{aligned} \quad (48)$$

Set $\Omega = \{x \in \mathbb{R}^N : w(x) \neq 0\}$. If $\text{meas}(\Omega) > 0$, then by Lemma 7(4) and Fatou Lemma, one has

$$\begin{aligned} & o(1) + \frac{1}{2} \max\{1, V_\infty\} \\ & \geq \frac{1}{2^*} \frac{\|v_n\|_{2^*}^{2^*}}{\|v_n\|_E^2} + \int_{\mathbb{R}^N} \frac{H_\lambda(x, v_n)}{\|v_n\|_E^2} dx \\ & \geq \frac{1}{2^*} \int_{\Omega} w_n^2 |v_n|^{2^*-2} dx \longrightarrow +\infty \end{aligned} \quad (49)$$

as $n \rightarrow \infty$. This is a contradiction. Hence $|\Omega| = 0$, that is, $w = 0$ a.e. on $\mathbb{R}^N$. For any $B > 0$, by $\|v_n\|_E \rightarrow +\infty$ we have

$$\begin{aligned} J_\lambda(t_n v_n) & \geq J_\lambda\left(\frac{B}{\|v_n\|_E} v_n\right) = J_\lambda(Bw_n) \\ & \geq \frac{B^2}{2} \min\{1, V_0\} - \int_{\mathbb{R}^N} H_\lambda(x, Bw_n) dx \\ & \quad - \frac{B^{2^*}}{2^*} \int_{\mathbb{R}^N} |w_n|^{2^*} dx \end{aligned} \quad (50)$$

for n sufficiently large. By (29), Lemmas 6(6) and 7(1), and (2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|h_\lambda(x, s)| \leq \varepsilon(|s|^2 + |s|^{2^*}) + C_\varepsilon |s|^p \quad (51)$$

for all $(x, s) \in \mathbb{R}^N \times \mathbb{R}$. Consequently,

$$\begin{aligned} \int_{\mathbb{R}^N} |w_n|^{2^*} dx & \leq \frac{\max\{1, V_\infty\}}{\|v_n\|_E^{2^*-2}} \\ & \quad - \frac{1}{\|v_n\|_E^{2^*}} \int_{\mathbb{R}^N} h_\lambda(x, v_n) v_n dx + o(1) \\ & \longrightarrow 0 \end{aligned} \quad (52)$$

as $n \rightarrow \infty$ and so $\int_{\mathbb{R}^N} |w_n|^p dx \rightarrow 0$ as $n \rightarrow \infty$ by using interpolation inequality. Moreover, (41) implies that

$$\begin{aligned} \left| \int_{\mathbb{R}^N} H_\lambda(x, Bw_n) dx \right| & \leq \varepsilon B^2 \int_{\mathbb{R}^N} w_n^2 dx \\ & \quad + \varepsilon B^{2^*} \int_{\mathbb{R}^N} |w_n|^{2^*} dx \\ & \quad + C_\varepsilon B^p \int_{\mathbb{R}^N} |w_n|^p dx. \end{aligned} \quad (53)$$

By the arbitrariness of ε , we obtain $\int_{\mathbb{R}^N} H_\lambda(x, Bw_n) dx \rightarrow 0$ as $n \rightarrow \infty$. Hence

$$\liminf_{n \rightarrow \infty} J_\lambda(t_n v_n) \geq \frac{B^2}{2} \min\{1, V_0\}, \quad \forall B > 0. \quad (54)$$

This contradicts the fact that $\{J_\lambda(t_n v_n)\}$ is bounded from above. Consequently, $\{v_n\}$ is bounded in E . This completes the proof of Lemma 9. $\square$

Lemma 10. Suppose that (V_1) , (V_2) , and $(f_1)-(f_5)$ are satisfied. Then if $N \geq 5$, the minimax level c_λ satisfies $c_\lambda < (1/N)S^{N/2}$ for all $\lambda > 0$; if $N = 3, 4$, the minimax level c_λ satisfies $c_\lambda < (1/N)S^{N/2}$ for large λ , where S is the best constant of the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$.

Proof. From the minimax characterization of c_λ we see that it is sufficient to show that there exists $v_0 \in E \setminus \{0\}$ such that $\sup_{t \geq 0} J_\lambda(t v_0) < (1/N)S^{N/2}$.

We follow the strategy used in [24] but need to modify some process. Given $\varepsilon > 0$, we consider the function

$$w_\varepsilon(x) = \frac{[N(N-2)\varepsilon]^{(N-2)/4}}{(\varepsilon + |x|^2)^{(N-2)/2}}, \quad (55)$$

which satisfies the following equations:

$$\begin{aligned} -\Delta u &= u^{2^*-1}, \quad \text{in } \mathbb{R}^N, \\ u &\in D^{1,2}(\mathbb{R}^N), \\ u(x) &> 0, \\ &\quad \text{in } \mathbb{R}^N. \end{aligned} \quad (56)$$

Moreover, $w_\varepsilon(x)$ satisfies

$$|\nabla w_\varepsilon|_2^2 = |w_\varepsilon|_{2^*}^{2^*} = S^{N/2}. \quad (57)$$

Let $\varphi \in C_0^\infty(\mathbb{R}^N, [0, 1])$ be such that $\varphi(x) \equiv 1$ for $|x| \leq \rho_\varepsilon$ and $\varphi(x) \equiv 0$ for $|x| \geq 2\rho_\varepsilon$, where $\rho_\varepsilon := \varepsilon^\tau$ with $\tau \in (1/4, 1/2)$. Set $\psi_\varepsilon(x) = \varphi(x)w_\varepsilon(x)$. Then

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^2 dx &= S^{N/2} + O(\varepsilon^{(N-2)/2}), \\ \int_{\mathbb{R}^N} |\psi_\varepsilon|^{2^*} dx &= S^{N/2} + O(\varepsilon^{N/2}), \\ \int_{\mathbb{R}^N} |\psi_\varepsilon| dx &\leq C\varepsilon^{(N-2)/4}, \\ \int_{\mathbb{R}^N} |\psi_\varepsilon|^{2^*-1} dx &\leq C\varepsilon^{(N-2)/4}, \\ \int_{\mathbb{R}^N} |\nabla \psi_\varepsilon| dx &\leq C\varepsilon^{(N-2)/4}, \\ \int_{\mathbb{R}^N} |\psi_\varepsilon|^2 dx &= \begin{cases} C\varepsilon + O(\varepsilon^{(N-2)/2}), & \text{if } N \geq 5, \\ C\varepsilon |\ln \varepsilon| + O(\varepsilon), & \text{if } N = 4, \\ O(\varepsilon^{1/2}), & \text{if } N = 3. \end{cases} \end{aligned} \quad (58)$$

Since $J_\lambda(0) = 0$ and $\lim_{t \rightarrow \infty} J_\lambda(t\psi_\varepsilon) = -\infty$, there exists $t_\varepsilon > 0$ such that $J_\lambda(t_\varepsilon\psi_\varepsilon) = \max_{t \geq 0} J_\lambda(t\psi_\varepsilon)$. We claim that there exist two positive constants t_1, t_2 independent of ε such that

$$t_1 \leq t_\varepsilon \leq t_2 \quad (59)$$

for small $\varepsilon > 0$. Indeed, by $\langle J'_\lambda(t_\varepsilon \psi_\varepsilon), \psi_\varepsilon \rangle = 0$ we have

$$\begin{aligned} & \frac{\int_{\mathbb{R}^N} [|\nabla \psi_\varepsilon|^2 + V(x) \psi_\varepsilon^2] dx}{|\psi_\varepsilon|_{2^*}^{2^*}} - t_\varepsilon^{2^*-2} \\ & - \frac{\int_{\mathbb{R}^N} h_\lambda(x, t_\varepsilon \psi_\varepsilon) t_\varepsilon \psi_\varepsilon dx}{t_\varepsilon^2 |\psi_\varepsilon|_{2^*}^{2^*}} = 0. \end{aligned} \quad (60)$$

By (29), Lemmas 6(6) and 7(1), and (2), for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$|h_\lambda(x, s)| \leq \delta |s|^{2^*} + C_\delta |s|^2 \quad (61)$$

for all $(x, s) \in \mathbb{R}^N \times \mathbb{R}$. Consequently,

$$\begin{aligned} & \left| \frac{\int_{\mathbb{R}^N} h_\lambda(x, t_\varepsilon \psi_\varepsilon) t_\varepsilon \psi_\varepsilon dx}{t_\varepsilon^2 |\psi_\varepsilon|_{2^*}^{2^*}} \right| \\ & \leq \frac{\int_{\mathbb{R}^N} [\delta t_\varepsilon^{2^*} \psi_\varepsilon^{2^*} + C_\delta t_\varepsilon^2 \psi_\varepsilon^2] dx}{t_\varepsilon^2 |\psi_\varepsilon|_{2^*}^{2^*}} \\ & = \delta t_\varepsilon^{2^*-2} + C_\delta \frac{|\psi_\varepsilon|_2^2}{|\psi_\varepsilon|_{2^*}^{2^*}} \\ & = \delta t_\varepsilon^{2^*-2} + C_\delta [S^{N/2} + O(\varepsilon^{N/2})]^{-1} |\psi_\varepsilon|_2^2 \\ & \leq \delta t_\varepsilon^{2^*-2} + CS^{-N/2} |\psi_\varepsilon|_2^2 \\ & = \delta t_\varepsilon^{2^*-2} \\ & + CS^{-N/2} \begin{cases} C\varepsilon + O(\varepsilon^{(N-2)/2}), & \text{if } N \geq 5, \\ C\varepsilon |\ln \varepsilon| + O(\varepsilon), & \text{if } N = 4, \\ O(\varepsilon^{1/2}), & \text{if } N = 3 \end{cases} \\ & = \delta t_\varepsilon^{2^*-2} + o(1) \end{aligned} \quad (62)$$

as $\varepsilon \rightarrow 0$. Note that

$$\begin{aligned} \frac{\|\psi_\varepsilon\|_E^2}{|\psi_\varepsilon|_{2^*}^{2^*}} &= \frac{|\nabla \psi_\varepsilon|_2^2 + |\psi_\varepsilon|_2^2}{|\psi_\varepsilon|_{2^*}^{2^*}} = \frac{1}{S^{N/2} + O(\varepsilon^{N/2})} \\ & \cdot \begin{cases} S^{N/2} + O(\varepsilon^{(N-2)/2}) + C\varepsilon + O(\varepsilon^{(N-2)/2}), & \text{if } N \geq 5, \\ S^{N/2} + O(\varepsilon^{(N-2)/2}) + C\varepsilon |\ln \varepsilon| + O(\varepsilon), & \text{if } N = 4, \\ S^{N/2} + O(\varepsilon^{(N-2)/2}) + O(\varepsilon^{1/2}), & \text{if } N = 3 \end{cases} \\ & \rightarrow 1 \end{aligned} \quad (63)$$

as $\varepsilon \rightarrow 0$. Hence by (60) one has

$$0 \geq \min\{1, V_0\} (1 + o(1)) - t_\varepsilon^{2^*-2} - \delta t_\varepsilon^{2^*-2} + o(1) \quad (64)$$

as $\varepsilon \rightarrow 0$, which implies that

$$t_\varepsilon \geq \left[\frac{\min\{1, V_0\}}{2(1+\delta)} \right]^{1/(2^*-2)} := t_1 > 0 \quad (65)$$

for $\varepsilon > 0$ small enough. On the other hand, (60) leads to

$$\begin{aligned} t_\varepsilon^{2^*-2} &\leq \max\{1, V_\infty\} \frac{\|\psi_\varepsilon\|_E^2}{|\psi_\varepsilon|_{2^*}^{2^*}} \\ &+ \left| \frac{\int_{\mathbb{R}^N} h_\lambda(x, t_\varepsilon \psi_\varepsilon) t_\varepsilon \psi_\varepsilon dx}{t_\varepsilon^2 |\psi_\varepsilon|_{2^*}^{2^*}} \right| \\ &\leq \max\{1, V_\infty\} (1 + o(1)) + \delta t_\varepsilon^{2^*-2} + o(1) \end{aligned} \quad (66)$$

as $\varepsilon \rightarrow 0$, which implies that

$$t_\varepsilon \leq \left[\frac{2 \max\{1, V_\infty\}}{1-\delta} \right]^{1/(2^*-2)} := t_2 < +\infty \quad (67)$$

for $\delta > 0$ and $\varepsilon > 0$ small enough.

Since $Q(t) := t^2/2 - t^{2^*}/2^*$ has only maximum at $t = 1$, one has

$$\begin{aligned} J_\lambda(t_\varepsilon \psi_\varepsilon) &= \frac{1}{2} t_\varepsilon^2 \int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^2 dx + \frac{1}{2} t_\varepsilon^2 \int_{\mathbb{R}^N} V(x) \psi_\varepsilon^2 dx \\ &- \int_{\mathbb{R}^N} H_\lambda(x, t_\varepsilon \psi_\varepsilon) dx \\ &- \frac{1}{2^*} t_\varepsilon^{2^*} \int_{\mathbb{R}^N} \psi_\varepsilon^{2^*} dx \\ &= \left(\frac{t_\varepsilon^2}{2} - \frac{t_\varepsilon^{2^*}}{2^*} \right) S^{N/2} + O(\varepsilon^{(N-2)/2}) \\ &+ \frac{1}{2} t_\varepsilon^2 \int_{\mathbb{R}^N} V(x) \psi_\varepsilon^2 dx \\ &- \int_{\mathbb{R}^N} H_\lambda(x, t_\varepsilon \psi_\varepsilon) dx \\ &\leq \frac{1}{N} S^{N/2} + O(\varepsilon^{(N-2)/2}) \\ &+ \frac{1}{2} t_\varepsilon^2 V_\infty \int_{\mathbb{R}^N} \psi_\varepsilon^2 dx \\ &- \int_{\mathbb{R}^N} H_\lambda(x, t_\varepsilon \psi_\varepsilon) dx. \end{aligned} \quad (68)$$

Notice that, for $x \in B_{\rho_\varepsilon}$, we have

$$\begin{aligned} t_\varepsilon \psi_\varepsilon &= t_\varepsilon w_\varepsilon = t_\varepsilon \frac{[N(N-2)\varepsilon]^{(N-2)/4}}{(\varepsilon + |x|^2)^{(N-2)/2}} \\ &\geq C t_\varepsilon \frac{[N(N-2)]^{(N-2)/4} \varepsilon^{(N-2)/4}}{\varepsilon^{\tau(N-2)}} \\ &\geq C t_1 [N(N-2)]^{(N-2)/4} \varepsilon^{(N-2)(1/4-\tau)} \rightarrow +\infty \end{aligned} \quad (69)$$

as $\varepsilon \rightarrow 0$, which combining with Lemma 7(4) and (5) implies that for any $M > 0$

$$\int_{\mathbb{R}^N} H_\lambda(x, t_\varepsilon \psi_\varepsilon) dx \geq M t_\varepsilon^2 \int_{B_{\rho_\varepsilon}} \psi_\varepsilon^2 dx \quad (70)$$

for $\varepsilon > 0$ small enough. Note that

$$\begin{aligned} & \int_{B_{\rho_\varepsilon}} \psi_\varepsilon^2 dx \\ &= [N(N-2)]^{(N-2)/2} N \omega_N \varepsilon \int_0^{\rho_\varepsilon/\sqrt{\varepsilon}} \frac{s^{N-1}}{(1+s^2)^{N-2}} ds, \\ & \int_0^{+\infty} \frac{s^{N-1}}{(1+s^2)^{N-2}} ds \geq \int_0^1 \frac{s^{N-1}}{(1+s^2)^{N-2}} ds \geq \frac{1}{N} \cdot \frac{1}{2^{N-2}} \\ &:= C > 0. \end{aligned} \quad (71)$$

Consequently,

$$\int_{\mathbb{R}^N} H_\lambda(x, t_\varepsilon \psi_\varepsilon) dx \geq MC\varepsilon \quad (72)$$

for $\varepsilon > 0$ small enough. Hence by (68)

$$\begin{aligned} J_\lambda(t_\varepsilon \psi_\varepsilon) &\leq \frac{1}{N} S^{N/2} + O(\varepsilon^{(N-2)/2}) \\ &\quad + \frac{1}{2} t_\varepsilon^2 V_\infty \int_{\mathbb{R}^N} \psi_\varepsilon^2 dx \\ &\quad - \int_{\mathbb{R}^N} H_\lambda(x, t_\varepsilon \psi_\varepsilon) dx \\ &\leq \frac{1}{N} S^{N/2} + O(\varepsilon^{(N-2)/2}) - MC\varepsilon \\ &\quad + C \begin{cases} \varepsilon + O(\varepsilon^{(N-2)/2}), & \text{if } N \geq 5, \\ \varepsilon |\ln \varepsilon| + O(\varepsilon), & \text{if } N = 4, \\ O(\varepsilon^{1/2}), & \text{if } N = 3. \end{cases} \end{aligned} \quad (73)$$

From this, we see that $J_\lambda(t_\varepsilon \psi_\varepsilon) < (1/N)S^{N/2}$ for $\varepsilon > 0$ small enough and M big enough if $N \geq 5$. Consequently, $c_\lambda < (1/N)S^{N/2}$ for all $\lambda > 0$ if $N \geq 5$.

In the following, we consider the case $N = 3, 4$. Indeed, if the conclusion is false, then there exists a sequence $\{\lambda_n\}$ with $\lambda_n \rightarrow +\infty$ such that $c_{\lambda_n} \geq (1/N)S^{N/2}$. Take $v \in E \setminus \{0\}$. Then by the proof of Lemma 8, there exists a unique $t_{\lambda_n} > 0$ such that $\max_{t>0} J_{\lambda_n}(tv) = J_{\lambda_n}(t_{\lambda_n} v)$. Hence

$$\begin{aligned} & t_{\lambda_n}^2 \int_{\mathbb{R}^N} |\nabla v|^2 dx + \int_{\mathbb{R}^N} V(x) \frac{G^{-1}(t_{\lambda_n} v)}{g(G^{-1}(t_{\lambda_n} v))} t_{\lambda_n} v dx \\ &= t_{\lambda_n}^{2^*} \int_{\mathbb{R}^N} |v|^{2^*} dx \\ &\quad + \lambda_n \int_{\mathbb{R}^N} \frac{f(x, G^{-1}(t_{\lambda_n} v))}{g(G^{-1}(t_{\lambda_n} v))} t_{\lambda_n} v dx. \end{aligned} \quad (74)$$

By Lemma 6(6) and (f_4) we get

$$\max\{1, V_\infty\} \|v\|_E^2 \geq t_{\lambda_n}^{2^*-2} \int_{\mathbb{R}^N} |v|^{2^*} dx, \quad (75)$$

which implies that $\{t_{\lambda_n}\}$ is bounded. Hence, up to a subsequence, there exists $t_0 \geq 0$ such that $t_{\lambda_n} \rightarrow t_0$ as $n \rightarrow \infty$. If $t_0 > 0$, then by (f_4) and Fatou lemma we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left[t_{\lambda_n}^{2^*} \int_{\mathbb{R}^N} |v|^{2^*} dx \right. \\ & \quad \left. + \lambda_n \int_{\mathbb{R}^N} \frac{f(x, G^{-1}(t_{\lambda_n} v))}{g(G^{-1}(t_{\lambda_n} v))} t_{\lambda_n} v dx \right] = +\infty. \end{aligned} \quad (76)$$

But, on the other hand, by Lemma 6(6) one has

$$\begin{aligned} & t_{\lambda_n}^{2^*} \int_{\mathbb{R}^N} |v|^{2^*} dx + \lambda_n \int_{\mathbb{R}^N} \frac{f(x, G^{-1}(t_{\lambda_n} v))}{g(G^{-1}(t_{\lambda_n} v))} t_{\lambda_n} v dx \\ & \leq \max\{1, V_\infty\} t_{\lambda_n}^2 \|v\|_E^2 \rightarrow \max\{1, V_\infty\} t_0^2 \|v\|_E^2, \end{aligned} \quad (77)$$

a contradiction. Hence $t_0 = 0$ and by Lemma 7(4) we know that

$$\begin{aligned} & \max_{t>0} J_{\lambda_n}(tv) = J_{\lambda_n}(t_{\lambda_n} v) \\ & \leq \frac{1}{2} \max\{1, V_\infty\} t_{\lambda_n}^2 \|v\|_E^2 \\ & \quad - \frac{1}{2^*} t_{\lambda_n}^{2^*} \int_{\mathbb{R}^N} |v|^{2^*} dx \rightarrow 0 \end{aligned} \quad (78)$$

as $n \rightarrow \infty$. Consequently,

$$\begin{aligned} 0 &< \frac{1}{N} S^{N/2} \leq c_{\lambda_n} \leq \inf_{u \in E \setminus \{0\}} \max_{t>0} J_{\lambda_n}(tu) \leq \max_{t>0} J_{\lambda_n}(tv) \\ &\rightarrow 0, \end{aligned} \quad (79)$$

a contradiction. This completes the proof. $\square$

Proof of Theorem 1. Since $\{v_n\} \subset E$ is a bounded Cerami sequence for J_λ at the level $c_\lambda > 0$, there exists $v \in E$ such that

$$\begin{aligned} & v_n \rightharpoonup v \quad \text{in } E, \\ & v_n \rightarrow v \quad \text{in } L_{\text{loc}}^q(\mathbb{R}^N) \text{ for } 1 \leq q < 2^*, \\ & v_n(x) \rightarrow v(x) \quad \text{a.e. on } \mathbb{R}^N. \end{aligned} \quad (80)$$

Using a standard argument, we know that $J'_\lambda(v) = 0$, that is, v is a weak solution of (11). Indeed, for any $\psi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} o(1) &= \langle J'_\lambda(v_n), \psi \rangle \\ &= \int_{\mathbb{R}^N} \nabla v_n \nabla \psi dx + \int_{\mathbb{R}^N} V(x) v_n \psi dx \\ &\quad - \int_{\mathbb{R}^N} |v_n|^{2^*-2} v_n \psi dx - \int_{\mathbb{R}^N} h_\lambda(x, v_n) \psi dx. \end{aligned} \quad (81)$$

Since $v_n \rightharpoonup v$ in E , one has

$$\begin{aligned} \int_{\mathbb{R}^N} \nabla v_n \nabla \psi \, dx &\longrightarrow \int_{\mathbb{R}^N} \nabla v \nabla \psi \, dx, \\ \int_{\mathbb{R}^N} V(x) v_n \psi \, dx &\longrightarrow \int_{\mathbb{R}^N} V(x) v \psi \, dx, \\ \int_{\mathbb{R}^N} |v_n|^{2^*-2} v_n \psi \, dx &\longrightarrow \int_{\mathbb{R}^N} |v|^{2^*-2} v \psi \, dx, \\ \int_{\mathbb{R}^N} h_\lambda(x, v_n) \psi \, dx &\longrightarrow \int_{\mathbb{R}^N} h_\lambda(x, v) \psi \, dx. \end{aligned} \quad (82)$$

Consequently,

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} \nabla v \nabla \psi \, dx + \int_{\mathbb{R}^N} V(x) v \psi \, dx \\ &\quad - \int_{\mathbb{R}^N} |v|^{2^*-2} v \psi \, dx - \int_{\mathbb{R}^N} h_\lambda(x, v) \psi \, dx \end{aligned} \quad (83)$$

for all $\psi \in C_0^\infty(\mathbb{R}^N)$. For any $\varphi \in E$, there exists a sequence $\{\psi_n\} \subset C_0^\infty(\mathbb{R}^N)$ such that $\psi_n \rightarrow \varphi$ in E . Hence

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} \nabla v \nabla \psi_n \, dx + \int_{\mathbb{R}^N} V(x) v \psi_n \, dx \\ &\quad - \int_{\mathbb{R}^N} |v|^{2^*-2} v \psi_n \, dx - \int_{\mathbb{R}^N} h_\lambda(x, v) \psi_n \, dx. \end{aligned} \quad (84)$$

Let $n \rightarrow \infty$, we get

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} \nabla v \nabla \varphi \, dx + \int_{\mathbb{R}^N} V(x) v \varphi \, dx \\ &\quad - \int_{\mathbb{R}^N} |v|^{2^*-2} v \varphi \, dx - \int_{\mathbb{R}^N} h_\lambda(x, v) \varphi \, dx; \end{aligned} \quad (85)$$

that is, $\langle J'_\lambda(v), \varphi \rangle = 0$ for all $\varphi \in E$. Hence $J'_\lambda(v) = 0$; that is, v is a weak solution of (11).

In the following, we prove that v is nontrivial. With the aid of Lemma 10, the proof follows essentially the proof of Theorem 1.1 in [16]. For completeness, we present the proof as follows. If the conclusion is false, we may assume $v = 0$. We divide the proof into four steps.

Step 1. We prove that $\{v_n\} \subset E$ is also a Cerami sequence for the functional $J_\lambda^\infty : E \rightarrow \mathbb{R}$, where

$$\begin{aligned} J_\lambda^\infty(v_n) &= \frac{1}{2} \int_{\mathbb{R}^N} [|\nabla v_n|^2 + V_\infty v_n^2] \, dx \\ &\quad - \int_{\mathbb{R}^N} H_\lambda(x, v_n) \, dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |v_n|^{2^*} \, dx. \end{aligned} \quad (86)$$

By (V_2) and $v_n \rightharpoonup 0$ in E , one has

$$J_\lambda(v_n) - J_\lambda^\infty(v_n) = \frac{1}{2} \int_{\mathbb{R}^N} [V(x) - V_\infty] v_n^2 \, dx \longrightarrow 0 \quad (87)$$

as $n \rightarrow \infty$. Similarly, we have

$$\begin{aligned} &\|J'_\lambda(v_n) - (J_\lambda^\infty)'(v_n)\|_{E^*} \\ &= \sup_{\|\varphi\|_E \leq 1} |\langle J'_\lambda(v_n) - (J_\lambda^\infty)'(v_n), \varphi \rangle| \\ &= \sup_{\|\varphi\|_E \leq 1} \left| \int_{\mathbb{R}^N} [V(x) - V_\infty] v_n \varphi \, dx \right| \longrightarrow 0 \end{aligned} \quad (88)$$

as $n \rightarrow \infty$. Consequently, $\{v_n\}$ is also a Cerami sequence of J_λ^∞ .

Step 2. There exist $\alpha, R > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\lim_{n \rightarrow \infty} \int_{B_R(y_n)} |v_n|^2 \, dx \geq \alpha > 0. \quad (89)$$

Indeed, by contradiction, then by Lemma 1.21 in [27], one has $v_n \rightarrow 0$ in $L^q(\mathbb{R}^N)$ for $2 < q < 2^*$. Notice that

$$\begin{aligned} o(1) &= \langle J'_\lambda(v_n), v_n \rangle \\ &= \int_{\mathbb{R}^N} [|\nabla v_n|^2 + V(x) v_n^2] \, dx \end{aligned} \quad (90)$$

$$- \int_{\mathbb{R}^N} h_\lambda(x, v_n) v_n \, dx - \int_{\mathbb{R}^N} |v_n|^{2^*} \, dx,$$

which combining with (51) leads to

$$\int_{\mathbb{R}^N} [|\nabla v_n|^2 + V(x) v_n^2] \, dx - \int_{\mathbb{R}^N} |v_n|^{2^*} \, dx \longrightarrow 0 \quad (91)$$

as $n \rightarrow \infty$. Consequently, there exists a constant $l \geq 0$ such that

$$\begin{aligned} \int_{\mathbb{R}^N} [|\nabla v_n|^2 + V(x) v_n^2] \, dx &\longrightarrow l, \\ \int_{\mathbb{R}^N} |v_n|^{2^*} \, dx &\longrightarrow l. \end{aligned} \quad (92)$$

Obviously, $l > 0$. Otherwise, $J_\lambda(v_n) \rightarrow 0$ as $n \rightarrow \infty$, which contradicts with $c_\lambda > 0$. Hence by the definition of S , we have

$$\begin{aligned} S &\leq \frac{\int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx}{\left(\int_{\mathbb{R}^N} |v_n|^{2^*} \, dx \right)^{2/2^*}} \\ &\leq \frac{\int_{\mathbb{R}^N} [|\nabla v_n|^2 + V(x) v_n^2] \, dx}{\left(\int_{\mathbb{R}^N} |v_n|^{2^*} \, dx \right)^{2/2^*}} \longrightarrow \frac{l}{l^{2/2^*}} = l^{2/N}; \end{aligned} \quad (93)$$

that is, $l \geq S^{N/2}$. Therefore, (41) implies that

$$\begin{aligned} c_\lambda + o(1) &= J_\lambda(v_n) \\ &= \frac{1}{2} \int_{\mathbb{R}^N} [|\nabla v_n|^2 + V(x) v_n^2] \, dx \\ &\quad - \int_{\mathbb{R}^N} H_\lambda(x, v_n) \, dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |v_n|^{2^*} \, dx \\ &\longrightarrow \left(\frac{1}{2} - \frac{1}{2^*} \right) l = \frac{1}{N} l \geq \frac{1}{N} S^{N/2}, \end{aligned} \quad (94)$$

as $n \rightarrow \infty$, which implies that $c_\lambda \geq (1/N)S^{N/2}$, a contradiction.

Step 3. After a translation of $\{v_n\}$ called $\{\tilde{v}_n\}$, then $\tilde{v}_n$ converges weakly to a nonzero critical point of J_λ^∞ .

Set $\tilde{v}_n(x) = v_n(x + \gamma_n)$. Since $\{v_n\} \subset E$ is a Cerami sequence of J_λ^∞ and $\|\tilde{v}_n\|_E = \|v_n\|_E$, arguing as in the case of $\{v_n\}$, we may assume $\tilde{v}_n \rightharpoonup \tilde{v}$ in E and $(J_\lambda^\infty)'(\tilde{v}) = 0$. So by Step 2 we know $\tilde{v} \neq 0$. By Lemma 7(3) and Fatou Lemma, one has

$$\begin{aligned} 2c_\lambda &= \liminf_{n \rightarrow \infty} [2J_\lambda^\infty(\tilde{v}_n) - \langle (J_\lambda^\infty)'(\tilde{v}_n), \tilde{v}_n \rangle] \\ &\geq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} [h_\lambda(x, \tilde{v}_n) \tilde{v}_n - 2H_\lambda(x, \tilde{v}_n)] dx \\ &\quad + \left(1 - \frac{2}{2^*}\right) \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\tilde{v}_n|^{2^*} dx \\ &\geq \int_{\mathbb{R}^N} [h_\lambda(x, \tilde{v}) \tilde{v} - 2H_\lambda(x, \tilde{v})] dx \\ &\quad + \left(1 - \frac{2}{2^*}\right) \int_{\mathbb{R}^N} |\tilde{v}|^{2^*} dx \\ &= 2J_\lambda^\infty(\tilde{v}) - \langle (J_\lambda^\infty)'(\tilde{v}), \tilde{v} \rangle = 2J_\lambda^\infty(\tilde{v}), \end{aligned} \quad (95)$$

which implies that $J_\lambda^\infty(\tilde{v}) \leq c_\lambda$.

Step 4. We use $\tilde{v}$ to construct a path which allows us to obtain a contradiction with the definition of mountain pass level c_λ .

Define the mountain pass level $c_\lambda^\infty := \inf_{\gamma \in \Gamma_\infty} \sup_{t \in [0,1]} J_\lambda^\infty(\gamma(t)) > 0$, where $\Gamma_\infty := \{\gamma \in C([0,1], E) : \gamma(0) = 0, J_\lambda^\infty(\gamma(1)) < 0\}$. It follows the arguments used in [28, 29], we can construct a path $\gamma : [0,1] \rightarrow E$ such that

$$\begin{aligned} \gamma(0) &= 0, \\ J_\lambda^\infty(\gamma(1)) &< 0, \\ \tilde{v} &\in \gamma([0,1]), \\ \gamma(t)(x) &> 0, \quad \forall x \in \mathbb{R}^N, t \in [0,1], \\ \max_{t \in [0,1]} J_\lambda^\infty(\gamma(t)) &= J_\lambda^\infty(\tilde{v}). \end{aligned} \quad (96)$$

Then $c_\lambda^\infty \leq \max_{t \in [0,1]} J_\lambda^\infty(\gamma(t)) = J_\lambda^\infty(\tilde{v})$. If $V(x) \equiv V_\infty$, we have already proved Theorem 1. If $V(x) \leq V_\infty$ but $V(x) \not\equiv V_\infty$, we take the path γ given by above, and by $\gamma \in \Gamma_\infty \subset \Gamma$, we have

$$\begin{aligned} c_\lambda &\leq \max_{t \in [0,1]} J_\lambda(\gamma(t)) = J_\lambda(\gamma(\bar{t})) < J_\lambda^\infty(\gamma(\bar{t})) \\ &\leq \max_{t \in [0,1]} J_\lambda^\infty(\gamma(t)) = J_\lambda^\infty(\tilde{v}) \leq c_\lambda, \end{aligned} \quad (97)$$

a contradiction. Consequently, $v \neq 0$. This completes the proof of Theorem 1. $\square$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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