Hindawi
Advances in Mathematical Physics
Volume 2018, Article ID 3916814, 6 pages
https://doi.org/10.1155/2018/3916814

Research Article

Conditional Lie-Bäcklund Symmetry Reductions and Exact Solutions of a Class of Reaction-Diffusion Equations

Xinyang Wang[1] and Junquan Song[2]

[1] *College of Civil Aviation, Nanjing University of Aeronautics and Astronautics, Nanjing 211106, China*
[2] *Department of Applied Mathematics, Zhejiang University of Technology, Hangzhou 310023, China*

Correspondence should be addressed to Junquan Song; mathjqsong@163.com

Received 25 January 2018; Revised 1 May 2018; Accepted 17 May 2018; Published 2 July 2018

Academic Editor: Giampaolo Cristadoro

The method of conditional Lie-Bäcklund symmetry is applied to solve a class of reaction-diffusion equations $u_t + u_{xx} + Q(x)u_x^2 + P(x)u + R(x) = 0$, which have wide range of applications in physics, engineering, chemistry, biology, and financial mathematics theory. The resulting equations are either solved exactly or reduced to some finite-dimensional dynamical systems. The exact solutions obtained in concrete examples possess the extended forms of the separation of variables.

1. Introduction

In this paper, we analyze a class of reaction-diffusion equations (RDEs)

$$u_t + u_{xx} + Q(x)u_x^2 + P(x)u + R(x) = 0, \tag{1}$$

which admit certain conditional Lie-Bäcklund symmetries (CLBSs). Equation (1) can be simplified from the following RDEs:

$$\begin{aligned} u_t + F(x)u_{xx} + Q(x)u_x^2 + G(x)u_x + P(x)u + R(x) \\ = 0, \end{aligned} \tag{2}$$

which have wide range of applications in physics, engineering, chemistry, biology, and financial mathematics theory [1–5]. Here the coefficient functions depend upon the variable x which typically represents the value of the underlying asset, such as the price of a stock upon which an option is placed. There exist several special cases of (2), say, the Black-Scholes-Merton equation, the Longstaff equation [6], the Vasicek equation [7], and the Cox-Ingersoll-Ross equation [8], in which $Q(x)$ is zero. In order to keep the computations as simple as and consistent with this requirement, authors in [4] transform (2) into (1) by using an equivalence point transformation, namely, by means of

$$u(t,x) = U(t,X) + g(x), \quad X = h(x), \tag{3}$$

with

$$\begin{aligned} h'(x)^2 &= \frac{1}{F(x)}, \\ g'(x) &= \frac{F'(x) - 2G(x)}{4Q(x)}, \end{aligned} \tag{4}$$

Equation (2) can be transformed into

$$\begin{aligned} &U_t + U_{xx} + \frac{Q(x)}{F(x)}U_x^2 + P(x)U + R(x) + g(x)P(x) \\ &\quad - \frac{G(x)^2}{4Q(x)} + \frac{F'(x)^2}{16Q(x)}, \\ &- \frac{F(x)G'(x)}{2Q(x)} + \frac{F(x)G(x)Q'(x)}{2Q(x)^2} \\ &\quad - \frac{F(x)F'(x)Q'(x)}{4Q(x)^2} + \frac{F(x)F''(x)}{4Q(x)} = 0, \end{aligned} \tag{5}$$

where the primes denote differentiation with respect to x. Then, with the inverse transformation $x = h^{-1}(X)$, this equation can be performed as (1) on reversion to lower case variables and redefinition of the coefficient functions as appropriate. When $Q(x) = constant$ and $P(x) = 0$, (1) becomes

$$u_t + u_{xx} + ku_x^2 + R(x) = 0, \tag{6}$$

which can simplify the linear form of

$$v_t + v_{xx} + f(x)\, v_x = 0, \tag{7}$$

under the transformation $u = \log(v)/k + \int f(x)dx/2k + \gamma t$ with $4kR(x) = -(2f' + f^2 + 4k\gamma)$.

It is known that symmetry reductions and exact solutions play important roles in the study of RDEs. The conditional Lie-Bäcklund symmetry (CLBS) method introduced by Zhdanov [9] and Fokas and Liu [10, 11] firstly has been proved to be very powerful to classify equations or specify the functions appeared in the equations and construct the corresponding group invariant solutions. Furthermore, authors have shown that CLBS is closely related to the invariant subspace; namely, exact solutions defined on invariant subspaces for equations or their variant forms can be obtained by using the CLBS method [12–24].

Motivated by the form of (1), we set the following second-order nonlinear CLBSs:

$$\eta = u_{xx} + a_1(x)\, u_x^2 + a_2(x)\, u_x + a_3(x)\, u + a_4(x), \tag{8}$$

which are very powerful to specify the functions appeared in (1) and construct the corresponding exact solutions. The remainder of this paper is organized as follows. In Section 2, some equations of the form (1) admitting CLBSs generated by (8) are obtained. CLBS reductions and exact solutions of two concrete examples are used to illustrate the results. Section 3 is devoted to conclusions and discussions.

2. Equations Admitting CLBSs and Two Examples

To consider further, we need the following proposition derived in [9, 10].

Proposition 1. *RDEs (1) admit CLBSs (8) if and only if $\eta' E = 0$ whenever u satisfies (1) and $\eta = 0$, where the prime denotes the Gateaux derivative, that is, $\eta' E = (\mathrm{d}/\mathrm{d}\epsilon)|_{\epsilon=0}\eta[u + \epsilon E]$ and $E = -u_{xx} - Q(x)u_x^2 - P(x)u - R(x)$.*

A direct computation from the above proposition yields

$$\begin{aligned}
\eta' E = {} & 2a_1^2(a_1 - Q)\, u_x^4 + \left[2a_1'(Q - 2a_1)\right. \\
& \left. + 2a_1\left(Q' + 2a_1a_2 - 2Qa_2\right)\right] u_x^3 + \left[2a_2'(Q - a_1)\right. \\
& + a_1\left(-P - 4Qa_4 + 2a_2^2 + 4a_1a_4\right) \\
& \left. + a_2\left(3Q' - 2Qa_2 - 4a_1'\right) + a_3Q - Q'' + a_1''\right] u_x^2 \\
& + 4a_1a_3(a_1 - Q)\, uu_x^2 + \left(4a_1a_2a_3 - 2a_1P' + 2Qa_3'\right. \\
& \left. - 4Qa_2a_3 - 4a_1'a_3 + 4Q'a_3\right) uu_x + \left[2a_3' - 2P' + a_2''\right. \\
& - 2a_1R' + 2Qa_4' - 2a_2a_2' \\
& \left. + 4a_4\left(-Qa_2 + a_1a_2 - a_1' + Q'\right)\right] u_x + 2a_3^2(a_1 - Q) \\
& \cdot u^2 + \left(-a_2P' - 2a_2'a_3 + a_3'' - P'' - 4Qa_3a_4\right. \\
& \left. + 4a_1a_3a_4\right) u + a_4\left(P - 2Qa_4 - 2a_2' + 2a_1a_4\right) + a_4'' \\
& - a_3R - a_2R' - R'' = 0.
\end{aligned} \tag{9}$$

To vanish all the coefficients of (9), we have the following overdetermined system:

$$\begin{aligned}
& 2a_1^2(a_1 - Q) = 0, \\
& 2a_1'(Q - 2a_1) + 2a_1\left(Q' + 2a_1a_2 - 2Qa_2\right) = 0, \\
& 2a_2'(Q - a_1) + a_1\left(-P - 4Qa_4 + 2a_2^2 + 4a_1a_4\right) \\
& \quad + a_2\left(3Q' - 2Qa_2 - 4a_1'\right) + a_3Q - Q'' + a_1'' = 0, \\
& 4a_1a_3(a_1 - Q) = 0, \\
& 4a_1a_2a_3 - 2a_1P' + 2Qa_3' - 4Qa_2a_3 - 4a_1'a_3 \\
& \quad + 4Q'a_3 = 0, \\
& 2a_3' - 2P' + a_2'' - 2a_1R' + 2Qa_4' - 2a_2a_2' \\
& \quad + 4a_4\left(-Qa_2 + a_1a_2 - a_1' + Q'\right) = 0, \\
& 2a_3^2(a_1 - Q) = 0, \\
& -a_2P' - 2a_2'a_3 + a_3'' - P'' - 4Qa_3a_4 + 4a_1a_3a_4 = 0, \\
& a_4\left(P - 2Qa_4 - 2a_2' + 2a_1a_4\right) + a_4'' - a_3R - a_2R' \\
& \quad - R'' = 0.
\end{aligned} \tag{10}$$

Solving this system, we can obtain the unknown functions in (1) and the corresponding CLBSs (8). From the first and seventh equations of system (10), it is apparent that the solutions can be divided into two cases including $a_1(x) = Q(x)$ and $a_1(x) = a_3(x) = 0, a_1(x) \neq Q(x)$.

Case 1 ($a_1(x) = Q(x) = 1$). Substituting $a_1(x), Q(x)$ into the third equation of the above system (10), we can obtain $a_3(x) = P(x)$. Furthermore we can derive $a_2(x) = \mu_1/P^{1/2}$, $P(x) \neq 0$, or $P(x) = 0$ from the eighth equation of (10) by substituting all these.

When $P(x) = 0$, we can obtain $R(x) = (1/2)a_2' + a_4 - (1/2)a_2^2 + \mu_1$ from the sixth equation of (10). Then substituting $R(x)$ into (10) again, we obtain $a_2 = 0$ or $a_4 = (1/4)a_2^2 +$

$(1/2)a_2' + a_2'^2/4a_2^2 - a_2''/2a_2 + \mu_2/a_2^2$, resulting in these two cases:

(i) $u_t + u_{xx} + u_x^2 + a_2' - (1/4)a_2^2 + a_2'^2/4a_2^2 - a_2''/2a_2 + \mu_2/a_2^2 + \mu_1 = 0$,

$$\eta = u_{xx} + u_x^2 + a_2 u_x + \frac{1}{4}a_2^2 + \frac{1}{2}a_2' + \frac{a_2'^2}{4a_2^2} - \frac{a_2''}{2a_2} + \frac{\mu_2}{a_2^2}, \quad a_2 \neq 0; \tag{11}$$

(ii) $u_t + u_{xx} + u_x^2 + R(x) = 0$,

$$\eta = u_{xx} + u_x^2 + R(x) - \mu_1. \tag{12}$$

When $P(x) \neq 0$, substituting $a_2(x) = \mu_1/P^{1/2}$ into the sixth equation of system (10), it arrives at $R(x) = -\mu_1^2/2P - \mu_1 P'/4P^{3/2} + a_4 + \mu_2$. Then this system is reduced to

$$\begin{aligned} &4\mu_1 P''' P^2 - P''\left(18\mu_1 P' P + 4\mu_1^2 P^{3/2}\right) \\ &\quad + \mu_1 P'\left(4P^3 - 8\mu_1^2 P + 16P^2 a_4\right) + 10\mu_1^2 P^{1/2} P'^2 \\ &\quad + 15\mu_1 P'^3 + 8\mu_1^2 P^{7/2} - 16\mu_2 P^{9/2} - 16\mu_1 P^3 a_4' \\ &= 0. \end{aligned} \tag{13}$$

However, there are two unknown functions $P(x)$, $a_4(x)$ to be determined with only one determining equation, resulting in

$$\begin{aligned} a_4 = \frac{1}{16P^{7/2}} \int \Bigg[&-8P' P^{3/2}\mu_1^2 \\ &+ \left(10P'^2 P - 4P'' P^2 + 8P^4\right)\mu_1 + 4P' P^{7/2} \\ &+ 15P'^3 P^{1/2} + 4P''' P^{5/2} - 18P' P'' P^{3/2} \\ &- \frac{16P^5\mu_2}{\mu_1}\Bigg] \mathrm{d}x + \mu_3 P, \quad \mu_1 \neq 0. \end{aligned} \tag{14}$$

Noting that when $\mu_1 = 0$, it leads to $\eta = u_{xx} + u_x^2 + P(x)u + R(x) = 0$, which reveals $u_t = 0$ and implies that $u(x,t)$ is independent of t which we are not interested. Therefore, it is impossible to obtain the general solutions. However we can solve it explicitly for several special cases.

If $P(x) = k$, we derive $a_4 = (\sqrt{k}/2)(\mu_1 - 2\mu_2 k/\mu_1)x + k\mu_3$. Then let $\mu_1 = s\sqrt{k}$, $\mu_2 = -s(a/k - s/2)$, $\mu_3 = as/k^2 + \beta/k$; the solution becomes

(i) $u_t + u_{xx} + u_x^2 + ku + ax + \beta = 0$,

$$\eta = u_{xx} + u_x^2 + s u_x + ku + ax + \beta + \frac{sa}{k}. \tag{15}$$

If $P(x) = e^{\lambda x}$, then $a_4 = -(2\mu_2/\lambda\mu_1)e^{(3/2)\lambda x} + ((1/4)\lambda x + \mu_3)e^{\lambda x} - (\mu_1/\lambda)e^{(1/2)\lambda x} - (\mu_1\lambda/4)e^{-(1/2)\lambda x} + (\mu_1^2/4)e^{-\lambda x} - \lambda^2/16$. For $\lambda \neq 0$, consequently $\eta = 0$ can not be easily solved. If $P(x) = x^\lambda$ ($\lambda \neq 0, \pm 2$), then $a_4 = (\lambda/4)x^\lambda[\ln(x) + \mu_3] + (\mu_1^2/4)x^{-\lambda x} - (2\mu_2/(\lambda+2)\mu_1)x^{(3/2)\lambda+1} - (\mu_1/(\lambda-2))x^{(1/2)\lambda+1} - (\mu_1\lambda/4)x^{-(1/2)\lambda-1} - (\lambda^2+4\lambda)/16x^2$ leading to the fact that $\eta = 0$ also can not be easily solved.

If $P(x) = x^2$, then $a_4 = -(\mu_2/2\mu_1)x^4 + 2(\mu_1+1)x^2\ln(x) + (\mu_1+1)(\mu_1-3)/4x^2 + \mu_3 x^2$. To make η solvable, we let $\mu_1 = -1$, $\mu_2 = 2a$, $\mu_3 = \beta$; then the solution becomes

(i) $u_t + u_{xx} + u_x^2 + x^2 u + (x^4 + 2)a + \beta x^2 = 0$,

$$\eta = u_{xx} + u_x^2 - \frac{1}{x}u_x + x^2 u + \left(ax^2 + \beta\right)x^2. \tag{16}$$

If $P(x) = x^{-2}$, then $a_4 = (\mu_1^2/4)x^2 + 3\mu_1/4 - (1/2 + \mu_2/\mu_1)(\ln(x)/x^2) + \mu_3/x^2$. To make η solvable, we let $\mu_1 = a$, $\mu_2 = -(1/2)a$, $\mu_3 = \beta$; then the solution becomes

(i) $u_t + u_{xx} + u_x^2 + x^{-2}u - a(ax^2 - 3)/4 + \beta/x^2 = 0$,

$$\eta = u_{xx} + u_x^2 + axu_x + x^{-2}u + \frac{a\left(ax^2+3\right)}{4} + \frac{\beta}{x^2}. \tag{17}$$

Case 2 ($a_1(x) = Q(x) \neq$ costant). By similar calculation, we can get $a_2(x) = \mu_2 a_1/a_1'$, $a_3(x) = \mu_1 a_1'^2/a_1^2$, and $P(x) = -\mu_2 + \mu_1 a_1'^2/a_1^2$, then system (10) is transformed into the following equations:

$$\begin{aligned} &\mu_2 a_1 a_1''' a_1' - 2\mu_2 a_1 a_1''^2 + \mu_2 a_1'' a_1'^2 + 2a_1 R' a_1'^3 \\ &\quad - 2a_1 a_4' a_1'^3 + 2\mu_2^2 a_1 a_1'^2 - 2\mu_2^2 a_1^2 a_1'' = 0, \\ &2\mu_2 a_1^3 a_1'' a_4 - \mu_1 a_1'^4 R + \mu_1 a_1'^4 a_4 - R'' a_1^2 a_1'^2 + a_4'' a_1^2 a_1'^2 \\ &\quad - 3\mu_2 a_1^2 a_1'^2 a_4 - \mu_2 a_1^3 a_1' R' = 0. \end{aligned} \tag{18}$$

This system is so complicated that we can not even obtain general formations. Consequently, special solutions of the above system are given as follows:

(i) $u_t + u_{xx} + x^{-2}u_x^2 + ku + (3/4)kx^2 + a = 0$,

$$\eta = u_{xx} + x^{-2}u_x^2 + \frac{k}{2}xu_x + \frac{1}{16}k^2x^4 + \frac{3}{4}kx^2; \tag{19}$$

(ii) $u_t + u_{xx} + x^2 u_x^2 + ku + \dfrac{k}{4x^2} - (k^2\ln[x])/4 + a = 0$,

$$\eta = u_{xx} + x^2 u_x^2 - \frac{k}{2}xu_x + \frac{k}{4x^2} + \frac{k^2}{16}; \tag{20}$$

(iii) $u_t + u_{xx} + \csc^2[\lambda x]u_x^2 + 2\lambda^2 u + a = 0$,

$$\eta = u_{xx} + \csc^2[\lambda x]\, u_x^2 + \lambda\tan[\lambda x]\, u_x, \quad \lambda \neq 0; \tag{21}$$

(iv) $u_t + u_{xx} + \sec^2[\lambda x]u_x^2 + 2\lambda^2 u + a = 0$,

$$\eta = u_{xx} + \sec^2[\lambda x]\, u_x^2 - \lambda\cot[\lambda x]\, u_x, \quad \lambda \neq 0; \tag{22}$$

(v) $u_t + u_{xx} + e^{\lambda x}u_x^2 - \lambda^2 u + a = 0$,

$$\eta = u_{xx} + e^{\lambda x}u_x^2 + \lambda u_x, \quad \lambda \neq 0. \tag{23}$$

Case 1 ($a_1(x) = a_3(x) = 0, P(x) = \text{costant}(= k)$). In this case, after a lengthy calculation, we have

$$\begin{aligned} &2Qa_2' - Q'' + 3a_2Q' - 2Qa_2^2 = 0, \\ &-4Qa_2a_4 + 4Q'a_4 + 2Qa_4' - 2a_2'a_2 + a_2'' = 0, \\ &-2Qa_4^2 - R'' + a_4'' + ka_4 - a_2R' - 2a_2'a_4 = 0. \end{aligned} \tag{24}$$

Special solutions of the above system are given as follows:

(i) $u_t + u_{xx} + x^{-2}u_x^2 + ku + a + \beta x^2 = 0,$

$$\eta = u_{xx} - x^{-1}u_x; \tag{25}$$

(ii) $u_t + u_{xx} + x^2u_x^2 - 3/4x^4\ln^2[x] - 1/x^4\ln[x] + 5/4x^4 - \gamma^2/4\ln^2[x] + \gamma/x^2\ln^2[x] + a\ln^2[x] + \beta = 0,$

$$\begin{aligned} \eta = u_{xx} &+ \frac{\ln[x] - 1}{x\ln[x]}u_x \\ &+ \frac{2\ln^2[x] - \ln[x] - 2 + 2\gamma x^2}{2x^4\ln^2[x]}; \end{aligned} \tag{26}$$

(iii) $u_t+u_{xx}+x^\lambda u_x^2+(1/16)\lambda(3\lambda+4)x^{-2-\lambda}+ax^{-(1/2)\lambda+1}+\beta = 0,$

$$\eta = u_{xx} + \frac{\lambda}{2x}u_x + \frac{1}{8}\left(2\lambda + \lambda^2\right)x^{-2-\lambda}, \quad \lambda \neq 0, \pm 2; \tag{27}$$

(iv) $u_t+u_{xx}+x^\lambda u_x^2+((\lambda+3)(\lambda-1)/4)x^{-2-\lambda}+ax^{2-\lambda}+\beta = 0,$

$$\eta = u_{xx} + \frac{\lambda - 1}{x}u_x + (\lambda - 1)x^{-2-\lambda}, \quad \lambda \neq 0, \pm 2; \tag{28}$$

(v) $u_t + u_{xx} + u_x^2 + ku - (\gamma^2 - 2\gamma)/4x^2 - (1/2)\gamma k\ln[x] + ax^2 + \beta = 0,$

$$\eta = u_{xx} - \frac{1}{x}u_x + \frac{\gamma}{x^2}. \tag{29}$$

Case 2 ($a_1(x) = a_3(x) = 0, P(x) \neq \text{costant}$). By similar calculation, we can obtain $a_2(x) = -P''/P'$ and $Q(x) = (\mu_1P + \mu_2)/P'^2$, then system (10) is reduced to

$$\begin{aligned} &2P'\left(\mu_1P + \mu_2\right)a_4' + 4\left(\mu_1P'^2 - \mu_1PP'' - \mu_2P''\right)a_4 \\ &\quad + P'''P''P' - P''''P'^2 - 2P'^4 = 0, \\ &P'^2a_4'' - 2\left(\mu_1P + \mu_2\right)a_4^2 + \left(2P'''P' - 2P''^2 + PP'^2\right)a_4 \\ &\quad + P''P'R' - P'^2R'' = 0. \end{aligned} \tag{30}$$

We solve out the result with special cases listed in the following:

(i) $u_t + u_{xx} + u_x^2 + (ax + \beta)u - (1/12)a^2x^4 + (1/6)a(\beta - 3\gamma)x^3 + (1/2)\gamma(\beta - 2\gamma)x^2 + \theta_1x + \theta_2 = 0,$

$$\eta = u_{xx} + ax + \gamma; \tag{31}$$

(ii) $u_t+u_{xx}+u_x^2+kx^2u-(\gamma^2-2\gamma)/4x^2-(1/2)\gamma kx^2\ln[x]+ax^2+\beta = 0,$

$$\eta = u_{xx} - \frac{1}{x}u_x + \frac{\gamma}{x^2} + \frac{k}{2}x^2; \tag{32}$$

(iii) $u_t + u_{xx} + e^{\lambda x}u_x^2 + (a + \beta e^{-\lambda x})u + (\beta + a\beta/4\lambda^2)e^{-2\lambda x} - (\gamma^2/\lambda^2)e^{\lambda x} + (\gamma x/\lambda)(a + \beta e^{-\lambda x}) + \theta_1e^{-\lambda x} + \theta_2 = 0,$

$$\eta = u_{xx} + \lambda u_x + \frac{1}{2}\beta e^{-2\lambda x} + \gamma, \quad \lambda \neq 0; \tag{33}$$

(iv) $u_t + u_{xx} + e^{\lambda x}u_x^2 + (a + \beta e^{-(1/2)\lambda x})u + (3/2 + (-6\gamma + 2a)/3\lambda^2)\beta e^{-(3/2)\lambda x} - (\beta^2/3\lambda^2)e^{-2\lambda x} + ((2a\gamma - 4\gamma^2)/\lambda^2 + 2\gamma)e^{-\lambda x} + \theta_1e^{-(1/2)\lambda x} + \theta_2 = 0,$

$$\eta = u_{xx} + \frac{\lambda}{2}u_x + \beta e^{-(3/2)\lambda x} + \gamma e^{-\lambda x}, \quad \lambda \neq 0; \tag{34}$$

(v) $u_t + u_{xx} + \sec^2[\lambda x]u_x^2 + (a + \beta\sin[\lambda x])u + (2\gamma + (\beta^2 + 6\gamma^2 - 3a\gamma)/6\lambda^2)\cos^2[\lambda x] - (\beta^2/12\lambda^2)\cos^4[\lambda x] + (3/2 + (3\gamma - a)/6\lambda^2)\beta\sin[\lambda x]\cos^2[\lambda x] + \theta_1\sin[\lambda x] + \theta_2 = 0,$

$$\begin{aligned} \eta = u_{xx} &+ \lambda\tan[\lambda x]u_x + \beta\sin[\lambda x]\cos^2[\lambda x] \\ &+ \gamma\cos^2[\lambda x], \quad \lambda \neq 0; \end{aligned} \tag{35}$$

(vi) $u_t + u_{xx} + \csc^2(\lambda x)u_x^2 + (a + \beta\cos[\lambda x])u + (2\gamma + (\beta^2 + 6\gamma^2 - 3a\gamma)/6\lambda^2)\sin^2[\lambda x] - (\beta^2/12\lambda^2)\sin^4[\lambda x] + (3/2 + (3\gamma - a)/6\lambda^2)\beta\sin^2[\lambda x]\cos[\lambda x] + \theta_1\cos[\lambda x] + \theta_2 = 0,$

$$\begin{aligned} \eta = u_{xx} &- \lambda\cot[\lambda x]u_x + \beta\sin^2(\lambda x)\cos[\lambda x] \\ &+ \gamma\sin^2[\lambda x], \quad \lambda \neq 0. \end{aligned} \tag{36}$$

Thus we have obtained 21 classes of equations (11)-(36) with form (1) which admit certain second-order CLBSs. To reduce and solve equations by means of corresponding CLBSs, one solves $\eta = 0$ to obtain u as a function of x with x-independent integration constants and then substitutes this solution into (1) to determine the time evolution of these constants. Next, we only present two examples to illustrate this approach.

Example 1. Equation

$$u_t + u_{xx} + x^{-2}u_x^2 + ku + \frac{3}{4}kx^2 + a = 0 \tag{37}$$

admits the CLBS

$$\eta = u_{xx} + x^{-2}u_x^2 + \frac{k}{2}xu_x + \frac{k^2}{16}x^4 + \frac{3}{4}kx^2. \tag{38}$$

The corresponding solutions are given by

$$\begin{aligned} u(x,t) = &-\frac{1}{16}kx^4 + C_1(t)x + C_1^2(t)\ln\left[x - C_1(t)\right] \\ &+ C_2(t), \end{aligned} \tag{39}$$

where $C_1(t)$ and $C_2(t)$ satisfy the finite-dimensional dynamical system

$$C_1' = -\frac{k}{2}C_1,$$
$$C_2' = -kC_2 + \frac{k}{2}C_1^2 - a. \tag{40}$$

Exact solutions can be obtained as

$$C_1 = c_1 e^{-(k/2)t},$$
$$C_2 = \left(\frac{1}{2}c_1^2 kt - \frac{a}{k}e^{kt} + c_2\right)e^{-kt} \tag{41}$$

with two arbitrary constants c_1 and c_2.

Example 2. Equation

$$u_t + u_{xx} + x^{-2}u_x^2 + ku + a + \beta x^2 = 0 \tag{42}$$

admits the CLBS

$$\eta = u_{xx} - \frac{1}{x}u_x. \tag{43}$$

The corresponding solutions are given by

$$u(x,t) = C_1(t) + C_2(t)x^2, \tag{44}$$

where $C_1(t)$ and $C_2(t)$ satisfy the system

$$C_1' = -kC_1 - 2C_2 - 4C_2^2 - a,$$
$$C_2' = -kC_2 - \beta. \tag{45}$$

This dynamical system can be solved and exact solutions are

$$C_1 = \frac{1}{k^3}\left[4c_2^2k^2e^{-2kt} + \left(8c_2t\beta - 2c_2tk + c_1k\right)k^2e^{-kt} - ak^2 + 2\beta k - 4\beta^2\right],$$
$$C_2 = -\frac{\beta}{k} + c_2e^{-kt} \tag{46}$$

with two arbitrary constants c_1 and c_2. Here the solutions $u(x,t)$ preserve the forms of the separation of variables $u = \phi_1(t) + \phi_2(t)\psi(x)$, which are associated with the invariant subspace $\mathscr{L}\{1, x^2\}$ mentioned in [3] and the references therein.

3. Conclusions and Discussions

In this paper, we have discussed RDEs (1) by means of CLBS with characteristic (8). The key for this method is to determine presumably the form of the CLBS. For (1), we found that nonlinear CLBS (8) is very effective, which can yield some interesting symmetry reductions and exact solutions. Two examples are considered to illustrate this method in terms of the compatibility of CLBSs and the governing equations. Generally speaking, the obtained solutions cannot be derived within the framework of Lie's classical method and nonclassical method.

In addition, it must be pointed out that, for the corresponding equations with certain fractional derivative [25],

$$\frac{\partial^\alpha}{\partial t^\alpha}u + u_{xx} + Q(x)u_x^2 + P(x)u + R(x) = 0, \quad 0 < \alpha < 1, \tag{47}$$

we can do similar work including CLBS classification, reductions, and exact solutions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work is supported in part by the Foundation of the Department of Education of Zhejiang Province (Grants nos. Y201432067; Y201432097).

References

[1] J. D. Murray, *Journal of Mathematical Biology*, vol. 19, Springer-Verlag, New York, NY, USA, 1989.

[2] A. Okubo and S. A. Levin, *Diffusion and ecological problems: modern perspectives*, Springer-Verlag, New York, NY, USA, 2001.

[3] V. A. Galaktionov and S. R. Svirshchevskii, *Exact Solutions And Invariant Subspaces of Nonlinear Partial Differential Equations in Mechanics And Physics*, Chapman & Hall/CRC, Chapman & Hall/CRC, London, UK, 2007.

[4] N. M. Ivanova, C. Sophocleous, and P. G. Leach, "Group classification of a class of equations arising in financial mathematics," *Journal of Mathematical Analysis and Applications*, vol. 372, no. 1, pp. 273–286, 2010.

[5] M. L. Gandarias, M. Redondo, and M. S. Bruzón, "Some weak self-adjoint Hamilton-Jacobi-Bellman equations arising in financial mathematics," *Nonlinear Analysis: Real World Applications*, vol. 13, no. 1, pp. 340–347, 2012.

[6] F. A. Longstaff, "A nonlinear general equilibrium model of the term structure of interest rates," *Journal of Financial Economics*, vol. 23, no. 2, pp. 195–224, 1989.

[7] O. Vasicek, "An equilibrium characterization of the term structure," *Journal of Financial Economics*, vol. 5, no. 2, pp. 177–188, 1977.

[8] J. C. Cox, J. E. Ingersoll Jr., and S. A. Ross, "An intertemporal general equilibrium model of asset prices," *Econometrica*, vol. 53, no. 2, pp. 363–384, 1985.

[9] R. Z. Zhdanov, "Conditional Lie-Bäcklund symmetry and reduction of evolution equations," *Journal of Physics A: Mathematical and General*, vol. 28, no. 13, pp. 3841–3850, 1995.

[10] A. S. Fokas and Q. M. Liu, "Nonlinear interaction of traveling waves of nonintegrable equations," *Physical Review Letters*, vol. 72, no. 21, pp. 3293–3296, 1994.

[11] A. S. Fokas and Q. M. Liu, "Generalized conditional symmetries and exact solutions of non-integrable equations," *Theoretical and Mathematical Physics*, vol. 99, no. 2, pp. 263–277, 1994.

[12] C. Qu, "Group classification and generalized conditional symmetry reduction of the nonlinear diffusion-convection equation with a nonlinear source," *Studies in Applied Mathematics*, vol. 99, no. 2, pp. 107–136, 1997.

[13] C. Z. Qu, L. N. Ji, and L. Z. Wang, "Conditional lie Bäcklund symmetries and sign-invariants to quasi-linear diffusion equations," *Studies in Applied Mathematics*, vol. 119, no. 4, pp. 355–391, 2007.

[14] L. Ji and C. Qu, "Conditional Lie-Bäcklund symmetries and invariant subspaces to non-linear diffusion equations," *IMA Journal of Applied Mathematics*, vol. 76, no. 4, pp. 610–632, 2011.

[15] S. Shen, C. Qu, Y. Jin, and L. Ji, "Maximal dimension of invariant subspaces to systems of nonlinear evolution equations," *Chinese Annals of Mathematics, Series B*, vol. 33, no. 2, pp. 161–178, 2012.

[16] J. Song, S. Shen, Y. Jin, and J. Zhang, "New maximal dimension of invariant subspaces to coupled systems with two-component equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 18, no. 11, pp. 2984–2992, 2013.

[17] W.-X. Ma, "A refined invariant subspace method and applications to evolution equations," *Science China Mathematics*, vol. 55, no. 9, pp. 1769–1778, 2012.

[18] W.-X. Ma and Y. Liu, "Invariant subspaces and exact solutions of a class of dispersive evolution equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 10, pp. 3795–3801, 2012.

[19] L. N. Ji, "Conditional Lie-Bäcklund symmetries and invariant subspaces to nonlinear diffusion equations with source," *Physica A: Statistical Mechanics and its Applications*, vol. 391, no. 24, pp. 6320–6331, 2012.

[20] W. Feng and L. N. Ji, "Conditional Lie-Bäcklund symmetries and functionally separable solutions of the generalized inhomogeneous nonlinear diffusion equations," *Physica A: Statistical Mechanics and its Applications*, vol. 392, no. 4, pp. 618–627, 2013.

[21] Y. Ye, W.-X. Ma, S. Shen, and D. Zhang, "A class of third-order nonlinear evolution equations admitting invariant subspaces and associated reductions," *Journal of Nonlinear Mathematical Physics*, vol. 21, no. 1, pp. 132–148, 2014.

[22] R. Cherniha and V. Davydovych, "Nonlinear reaction-diffusion systems with a non-constant diffusivity: conditional symmetries in no-go case," *Applied Mathematics and Computation*, vol. 268, pp. 23–34, 2015.

[23] L. N. Ji, C. Z. Qu, and S. Shen, "Conditional Lie-Bäcklund symmetry of evolution system and application for reaction-diffusion system," *Studies in Applied Mathematics*, vol. 133, no. 1, pp. 118–149, 2014.

[24] L. Ji, "Conditional Lie-Bäcklund symmetries and differential constraints for inhomogeneous nonlinear diffusion equations due to linear determining equations," *Journal of Mathematical Analysis and Applications*, vol. 440, no. 1, pp. 286–299, 2016.

[25] J. Hu, Y. Ye, S. Shen, and J. Zhang, "Lie symmetry analysis of the time fractional KdV-type equation," *Applied Mathematics and Computation*, vol. 233, pp. 439–444, 2014.