Research Article

On Regularity of a Weak Solution to the Navier–Stokes Equations with the Generalized Navier Slip Boundary Conditions

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1. Introduction

1.1. Navier–Stokes' Initial-Boundary Value Problem. We assume that \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) with a smooth boundary and \( T \) is a given positive number. The motion of a viscous incompressible fluid with constant density (which is for simplicity assumed to be equal to one) in domain \( \Omega \) in the time interval \( (0, T) \) is described by the Navier–Stokes equations:

\[
\begin{align*}
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} &= -\nabla p + \text{div} [2\gamma \mathbb{D}(\mathbf{v})] + \mathbf{f}, \\
\text{div} \mathbf{v} &= 0
\end{align*}
\]

(1)

(2)

(in \( \Omega \times (0, T) \)) for the unknowns \( \mathbf{v} \equiv (v_1, v_2, v_3) \) and \( p \) (the velocity and the pressure). Symbol \( \gamma \) denotes the kinematic coefficient of viscosity (it is supposed to be a positive constant) and \( \mathbb{D}(\mathbf{v}) = (\nabla \mathbf{v})_{\text{sym}} := (1/2)[\nabla \mathbf{v} + (\nabla \mathbf{v})^T] \) is the so-called rate of deformation tensor. In this paper, we consider (1) and (2) with generalized Navier's slip boundary conditions:

\[
\begin{align*}
\mathbf{v} \cdot \mathbf{n} &= 0, \\
[2\gamma \mathbb{D}(\mathbf{v}) \cdot \mathbf{n}]_\tau + \mathbb{K} \cdot \mathbf{v} &= 0
\end{align*}
\]

(3)

(4)

(\( \mathbb{K} \) is a nonnegative 2nd-order tensor defined a.e. on \( \partial \Omega \) such that \( \mathbb{K}(\mathbf{x}) \cdot \mathbf{a} \) is tangential to \( \partial \Omega \) at point \( \mathbf{x} \in \partial \Omega \) if vector \( \mathbf{a} \) is tangential to \( \partial \Omega \) at point \( \mathbf{x} \). Condition (4) generalizes the “classical” Navier boundary condition \( [2\gamma \mathbb{D}(\mathbf{v}) \cdot \mathbf{n}]_\tau + \kappa \mathbf{v} = 0 \), where \( \kappa \geq 0 \) is the coefficient of friction between the fluid and the boundary. The replacement of \( \kappa \mathbf{v} \) by \( \mathbb{K} \cdot \mathbf{v} \) reflects the fact that the microscopic structure of \( \partial \Omega \) can vary from point to point, it need not produce the same resistance in all tangential directions, and it may therefore divert the flow to the side. In this paper, we assume that \( \mathbb{K} \) in (4) is a trace (on \( \partial \Omega \)) of a tensor-valued function from \( W^{1,2}(\Omega)^{3 \times 3}, \) which is also denoted by \( \mathbb{K} \). Problem (1)–(4) is completed by the initial condition

\[
\mathbf{v} |_{t=0} = \mathbf{v}_0 \quad \text{in} \, \Omega.
\]

(5)

1.2. Shortly on Regularity Criteria for Weak Solutions to System (1) and (2). Existence of a global regular solution and uniqueness of a weak solution are still the fundamental open questions in the theory of the Navier–Stokes equation in 3D. There exist a series of posteriori assumptions on weak solutions that exclude the development of possible...
singularities. (They are usually called the “criteria of regularity.”) The assumptions concern various quantities, like the velocity or some of its components (see, e.g., [1–4]), the gradient of velocity or some of its components (see, e.g., [3, 5]), the vorticity or only two of its components (see, e.g., [1, 6]), the direction of vorticity (see [7, 8]), and the pressure (see, e.g., [9–11]). The absence of a blow-up (i.e., the nonexistence of singularities) in a weak solution has also been proven under certain assumptions on the integrability of the positive part of the middle eigenvalue of the rate of deformation tensor $D(v)$ in [12].

Most of the known regularity criteria can be applied in the case when either $\Omega = \mathbb{R}^2$ (like those from [1, 3, 5]) or they exclude singularities in the interior of $\Omega$, but not the singularities on the boundary. (This concerns, e.g., the criteria from [2, 12].) As to criteria, valid up to the boundary, we can cite, for example, the papers [13] (where the so-called suitable weak solution is shown to be bounded locally near the boundary if it satisfies Serrin’s conditions near the boundary and the trace of the pressure is bounded on the boundary), [14] (where an analogy of the well-known Caffarelli–Kohn–Nirenberg criterion for the regularity of a suitable weak solution at the point $(x_0, t_0) \in \Omega \times (0, T)$, e.g., [15], is also proven for points on a flat part of the boundary), and [16, 17] (for some generalizations of the criterion from [14], however, also valid only on a flat part of the boundary). A generalization of the criterion from [14] for points $(x_0, t_0)$ on a “smooth” curved part of the boundary can be found in paper [18]. In paper [19], the authors show that if a weak solution satisfies Serrin’s integrability conditions in a neighborhood of a “smooth” part of the boundary then the solution is regular up to this part of the boundary. In all these papers, the authors used the no-slip boundary condition $v = 0$ on $\partial \Omega \times (0, T)$ (or on the relevant part of this set).

1.3. On the Results of This Paper. In Section 2 of this paper, we consider (1) and (2) with generalized Navier’s boundary conditions (3) and (4) and we prove results analogous to those from [12], however, extended so that they hold up to the boundary of $\Omega$. (See Theorem 1.)

Note that while the regularity criteria that consider some components of the velocity or the velocity gradient depend on the observer’s frame, the criterion that uses the eigenvalues of tensor $D(v)$ is frame indifferent. Also note that the study of regularity of a weak solution in the neighborhood of the boundary requires a special technique, which is subtler than the one applied in the interior and closely connected with the used boundary conditions. This can be, for example, documented by the fact that the same result as the one obtained in Section 2 and stated in Theorem 1, for system (1) and (2) with the no-slip boundary condition, is not known.

1.4. Notation. Vector functions and spaces of vector functions are denoted by boldface letters.

(i) The norms of scalar- or vector- or tensor-valued functions with components in $L^p(\Omega)$ (resp., $W^{k,l}(\Omega)$) are denoted by $\| \cdot \|_p$ (resp., $\| \cdot \|_{k,l}$. The norm in $L^p(\partial \Omega)$ is denoted by $\| \cdot \|_{\partial \Omega}$. Norms in other spaces on $\partial \Omega$ are denoted by analogy.

(ii) $L^2(\Omega)$ is the closure in $L^2(\Omega)$ of the linear space of all infinitely differentiable divergence-free vector functions with a compact support in $\Omega$. The orthogonal projection of $L^2(\Omega)$ onto $L^2(\Omega)$ is denoted by $P_{\nu}$. (iii) $W^{1,2}(\Omega) = W^{1,2}(\Omega) \cap L^2(\Omega)$. We denote by $W^{-1,2}(\Omega)$ the dual space to $W^{1,2}(\Omega)$ and by $(\cdot, \cdot)_{\Omega}$ the duality between elements of $W^{-1,2}(\Omega)$ and $W^{1,2}(\Omega)$.

(iv) $\| \cdot \|_{\partial \Omega}$ denotes the norm of a vector-valued or tensor-valued function with the components in $L^2(\partial \Omega)$.

1.5. A Weak Solution of Problem (1)–(5) and Theorem on Structure. For $v_0 \in L^2(\Omega)$ and $f \in L^2(0, T; L^2(\Omega))$, a function $v \in L^2(0, T; W^{1,2}(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ is called a weak solution of problem (1)–(5) if it satisfies

$$
\int_0^T \int_{\Omega} \left( -\partial_t \phi \cdot v + v \cdot \nabla \phi + 2 \nu \| \nabla v \| : \nabla \phi \right) \, dx \, dt
+ \int_0^T \int_{\partial \Omega} (\kappa \cdot v) \cdot \nu \, ds \, dt
= \int_0^T \langle f, \phi \rangle_{\Omega} \, dt + \int_0^T \int_{\partial \Omega} v_0 \cdot \phi(0, 0) \, ds
$$

for all infinitely differentiable divergence-free vector functions $\phi$ in $\Omega \times [0, T]$, such that $\phi \cdot n = 0$ on $\partial \Omega \times [0, T]$ and $\phi(., T) = 0.$ The existence of a weak solution of problem (1)–(3) and (5) with “classical” Navier’s boundary condition $2\nu \| \nabla (v) \cdot n \|_1 + \kappa v = 0$ follows, for example, from papers [20, 21]. (Note that the more general case of a time-varying domain $\Omega$ is considered in [21] and in the same methods, one can also extend the existing results from [20, 21] to problem (1)–(5), which includes generalized Navier boundary condition (4). Moreover, by analogy with the Navier–Stokes equations with the no-slip boundary condition $v = 0$ on $\partial \Omega \times (0, T)$, the weak solution can be constructed so that it satisfies the so-called strong energy inequality:

$$
\| v(t) \|_{L^2}^2 + 4 \nu \int_0^t \int_0^s \| \nabla (v(\theta)) \|_2^2 \, d\theta \, d\theta
+ 2 \nu \int_0^t \int_0^s \int_{\Omega} v(\theta) \cdot \kappa v(\theta) \, ds \, d\theta
\leq \| v(s) \|_{L^2}^2 + \int_s^t \langle f(\theta), v(\theta) \rangle_{\Omega} \, d\theta
$$

for a.a.s $s \in (0, T)$ and all $t \in (s, T)$.

In contrast to Navier–Stokes equations (1) and (2) with the no-slip boundary condition, whose theory is relatively well elaborated, the equations with generalized Navier’s boundary conditions (3) and (4) have not yet been given so much attention. This is why a series of important results, well known from the theory of equations (1), (2) with the no-slip boundary condition, have not been explicitly proven in literature for equations with boundary conditions (3), (4), although many of them can be obtained in a similar or
almost the same way. This concerns except others the local
in time existence of a strong solution (here, however, one can
cite the papers [20, 22], where the local in time existence
of a strong solution is proven in the case when \( \kappa = \kappa_1, \)
\( \kappa \geq 0 \), the uniqueness of the weak solution, and the so-
called theorem on structure. This theorem states that if the
specific volume force \( f \) is at least in \( L^2(0, T; L^2(\Omega)) \) and \( v \) is a
weak solution of the Navier–Stokes problem with the no-slip
boundary condition, satisfying the strong energy inequality,
then \( (0, T) = \bigcup_{j \in \mathbb{N}} (a_j, b_j) \cup \mathbb{G} \), where set \( \Gamma \) is at most
countable, the intervals \( (a_j, b_j) \) are pairwise disjoint, the 1D
Lebesgue measure of set \( \mathbb{G} \) is zero, and solution \( v \) coincides
with a strong solution in the interior of each of the time
intervals \( (a_j, b_j) \). (See, e.g., [23] for more details.) In this
paper, we also use the theorem on structure, but we apply it
to the Navier–Stokes problem with boundary conditions \( (3), \)
\( (4) \). (As is mentioned above, the validity of the theorem for
invariants of \( K \) is equal to zero, because
\( \partial_t \mathbf{D}(\mathbf{v}) = \mathbf{Q} \mathbf{K} \mathbf{v} \mathbf{Q} \mathbf{D}(\mathbf{v}) \), where \( \mathbf{K} \) is symmetric and
depends on \( \mathbf{v} \) and \( t \). Since the dynamic stress tensor
\( \mathbf{T}_d(\mathbf{v}) \equiv 2 \mathbf{v} \mathbf{D}(\mathbf{v}) \) in the Newtonian fluid, the eigenvalues
of \( \mathbf{D}(\mathbf{v}) \) coincide, up to the factor \( 2 \mathbf{v} \), with the principal dynamic
stresses. The eigenvalues are the roots of the characteristic
equation of tensor \( \mathbf{D}(\mathbf{v}) \), that is, the equation
\( F(\lambda) = \lambda^3 - E_1 \lambda^2 + E_2 \lambda - E_3 = 0 \), where \( E_1, E_2, E_3 \) are the principal
invariants of \( \mathbf{D}(\mathbf{v}) \). The invariant \( E_1 \) is equal to zero, because
\( \text{Tr} \mathbf{D}(\mathbf{v}) \equiv \lambda_1 + \lambda_2 + \lambda_3 = 0 \). Furthermore,
\[ E_2 = \lambda_2 \lambda_3 + \lambda_1 \lambda_3 + \lambda_1 \lambda_2 = -\frac{1}{2} \left( \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \right) \leq 0 \quad (8) \]
and \( E_3 = \xi_1 \xi_2 \xi_3 \). Put \( \xi^* = \sqrt{-1/3} \mathbf{E}_2 \). (\( \xi_k^* \) are the points
on the \( \xi \)-axis, where \( F(\xi) = 0 \).) Obviously, \( E_2 = 0 \) implies
\( \lambda_1 = \lambda_2 = \lambda_3 = 0 \). Thus, assume that \( E_2 < 0 \). Then \( \text{sgn}(\xi) \neq \text{sgn}(\mathbf{E}_2) \). The root \( \xi_2 \) lies between \( \xi^* = E_2/E_3 \) (the point
where the tangent line to the graph of \( F \) at the point \( (0, -E_3) \)
intersects the \( \xi \)-axis) and \( (3/2)E_3/E_2 \equiv (3/2)\xi^* \) (the point
where the line connecting the points \( (0, -E_3) \) and \((\xi^*, F(\xi^*)) \)
\( (E_3 > 0) \) or \( (-\xi^*, F(-\xi^*)) \) (if \( E_3 < 0 \) intersects the \( \xi \)-axis).
The positive part of \( \xi_2 \) satisfies \( 0 \leq \xi_2 \leq (3/2)\xi^* \). Define
function \( \mathcal{E} \) by the formula
\[ \mathcal{E}(x,t) = \begin{cases} 0 & \text{if } E_2(x,t) = 0, \\ \xi_2(x,t) \equiv \frac{E_3(x,t)}{E_2(x,t)} & \text{if } E_2(x,t) < 0. \end{cases} \quad (9) \]
Now, we observe that the statement of Theorem 1 is also valid
when condition (ii) is replaced by the condition
\begin{equation}
(iii) \quad \xi^* 
\end{equation}
Moreover, the statement of Theorem 1 is also valid
when condition (ii) is replaced by the condition
\begin{equation}
(ii) \quad \xi^* \in \mathbb{L}^1(0, T; \mathbb{L}^2(\Omega)) \quad \text{for some } r \in [1, \infty), s \in (3/2, \infty],
\end{equation}
satisfying \( 2/r + 3/s = 2 \).

Proof of Theorem 1. We assume that \( t_0 \) is in one of the intervals
\( (a_j, b_j) \) (see Section 1.5) and \( t_0 < t < b_j \). We may assume
without the loss of generality that \( b_j \) is the largest number \( \leq T \)
such that \( v \) is “smooth” on the time interval \( (t_0, b_j) \). Then
there are two possibilities: (a) the first singularity of solution
\( v \) (after the time instant \( t_0 \)) develops at any time \( t \in (t_0, T] \).
Assume, by contradiction, that the possibility (a) takes place. In this case,
\( b_j \) is called the epoch of irregularity.

There exists an associated pressure \( p \) so that \( v \) and \( p \)
satisfy (1), (2) a.e. in \( \Omega \times (a_j, b_j) \). Multiplying (1) by \( P_a \mathbf{Dv} \)
and integrating in \( \Omega \), we obtain
\[ \int_{\Omega} \partial_t v \cdot P_a \Delta v \, dx + \int_{\Omega} v \cdot \nabla v \cdot P_a \Delta v \, dx = \| P_a \Delta v \|_2^2. \quad (10) \]
The first integral on the left hand side can be treated as follows:
\[ \int_{\Omega} \partial_t v \cdot P_a \Delta v \, dx = \int_{\Omega} \partial_t v \cdot \Delta v \, dx 
\]
\[ = 2 \int_{\Omega} \partial_t v \cdot \div \mathbf{D}(v) \, dx 
\]
\[ = 2 \int_{\Omega} \partial_t v \cdot [\mathbf{D}(v) \cdot \mathbf{n}] \, dS 
\]
\[ - 2 \int_{\Omega} \partial_t \nabla v : \mathbf{D}(v) \, dx 
\]
\[ = - \frac{1}{v} \int_{\Omega} \partial_t v \cdot (v \cdot v) \, dS 
\]
\[ \frac{d}{dt} \int_{\Omega} [\mathbf{D}(v)]^2 \, dx 
\]
\[ = - \frac{d}{2v} \int_{\Omega} v \cdot \mathbf{K} \cdot v \, dS 
\]
\[ \frac{d}{dt} \| \mathbf{D}(v) \|_2^2. \quad (11) \]

Before we estimate the second integral on the left hand side
of (10), we recall some inequalities:
(α) the Friedrichs-type inequality \( \|u\|_2 \leq c_1 \|\nabla u\|_2 \) (see, e.g., [24, Exercise II.5.15]), satisfied for all functions \( u \in W^{1,2}(\Omega) \) such that \( u \cdot n = 0 \) on \( \partial \Omega 

(β) The inequality \( \|\nabla^2 u\|_2 \leq c_1 (\|\Delta u\|_2 + \|u\|_2) \), which holds for \( u \in W^{2,2}(\Omega) \) that satisfy Navier's boundary conditions (3), (4) (following from [20, Theorem 3.1]).

The Helmholtz decomposition of \( \Delta u \) is \( \Delta u = P_n \Delta u + \nabla \varphi \), where

\[
(\begin{align*}
(\text{a}) & \quad \Delta \varphi = 0 \quad \text{in} \ \Omega, \\
(\text{b}) & \quad \frac{\partial \varphi}{\partial n} = \Delta u \cdot n \quad \text{on} \ \partial \Omega.
\end{align*})
\]

The next lemma brings the crucial estimates of \( \|\nabla \varphi\|_2 \) and \( \|\varphi\|_{2,2} \).

**Lemma 3.** There exist \( c_3, c_4, c_5, c_6 > 0 \) such that if \( u \) is a divergence-free function from \( W^{2,2}(\Omega) \) that satisfies boundary conditions (3), (4) and \( \varphi \) is a solution of the Neumann problem (12) then

\[
\begin{align*}
\|\nabla \varphi\|_2 & \leq c_5 \|\nabla (\kappa \cdot u)\|_2 + c_4 \|u\|_{1,2}, \quad (13) \\
\|\varphi\|_{2,2} & \leq c_6 \|P_n \Delta u\|_2 + c_6 \|u\|_2. \quad (14)
\end{align*}
\]

**Proof.** The right hand side \( \Delta u \cdot n \) in the boundary condition (12) equals

\[
-\nabla \cdot \nabla \varphi = -\nabla \cdot (\nabla (\nabla \varphi)) = \nabla \cdot (\varphi \nabla \varphi) = \nabla \cdot (\nabla \varphi) = \nabla \cdot (\nabla \varphi).
\]

(The vector field \( \nabla (\nabla \varphi) \) is tangential because \( \nabla \varphi \) is normal. Hence the term \( \nabla (\nabla \varphi) \cdot n \) equals zero on \( \partial \Omega \).) The tangential component of \( \nabla \varphi \), that is, \( \nabla \varphi \), equals \( \nabla \cdot \nabla \varphi \). In order to express \( \nabla \cdot \nabla \varphi \), we apply the formula \( 2[D(u) \cdot n]_2 = \nabla u \cdot n - 2u \cdot \nabla n \) (see, e.g., [20]). Hence, using also the boundary condition (4), we obtain

\[
\begin{align*}
(\nabla \varphi) \cdot n & = \nabla \cdot (\nabla (\nabla \varphi) \cdot n) \\
& = \nabla \cdot (\nabla \varphi) = \nabla \cdot (\nabla \varphi) = \nabla \cdot (\nabla \varphi).
\end{align*}
\]

Thus, boundary condition (12)(b) takes the form

\[
\frac{\partial \varphi}{\partial n} = -\nabla \cdot (\nabla \varphi) \cdot n. \quad (17)
\]

In comparison to (12)(b), the right hand side of (17) contains only the first-order derivatives of \( u \). The classical theory of solution of the Neumann problem now implies that

\[
\begin{align*}
\|\nabla \varphi\|_2 & \leq C \|\nabla (\nabla \varphi) \cdot n\|_2 \leq C \|\nabla \varphi\|_{1,2}, \\
\|\varphi\|_{2,2} & \leq C \|\nabla \varphi\|_2 + \|\varphi\|_2. \quad (18)
\end{align*}
\]

(We use \( C \) as a generic constant.) The right hand side can be estimated by means of continuity of the linear operator, acting from the space \( L^2_\text{div}(\Omega) \) (which is the space function \( w \in L^2(\Omega) \), whose divergence in the sense of distributions is in \( L^2(\Omega) \), with the norm \( \|w\|_{L^2} + \|\nabla w\|_{L^2} \) to \( W^{1/2,2}(\partial \Omega) \), which assigns to "smooth" functions \( w \in L^2_\text{div}(\Omega) \) the normal component \( w \cdot n \). Thus, we obtain the estimate

\[
\begin{align*}
\|\nabla \varphi\|_2 & \leq C \|\nabla \varphi\|_2 \leq C \|\nabla \varphi\|_2, \quad (19)
\end{align*}
\]

(18)
Lemma 3 and the boundary conditions (3), (4), the integrals $I_1, I_2,$ and $I_3$ can be treated as follows:

\[ I_1 = \int_{\partial \Omega} (\mathbf{v} \cdot \nabla \mathbf{v})_n \cdot (\mathbf{n} \cdot \nabla \mathbf{v})_n \, \mathrm{d}S \]

\[
= \int_{\partial \Omega} [\nu (\partial_j \mathbf{v})_n] \cdot [n_k (\partial_k \mathbf{v}_m)_n] \, dS
\]

\[
= \int_{\partial \Omega} [\nu \partial_i (\partial_j \mathbf{v}_n) - \nu_j v_i (\partial_j \mathbf{v}_n)] \cdot [n_k (\partial_k \mathbf{v}_m)_n] \, dS
\]

\[
= - \int_{\partial \Omega} [\nu v_i (\partial_j \mathbf{v}_n)] \cdot [n_k (\partial_k \mathbf{v}_m)_n] \, dS
\]

\[
= - \int_{\partial \Omega} \{[\nu v_i (\partial_j \mathbf{v}_n)] \cdot [n_k (\partial_k \mathbf{v}_m)_n]\} \, \mathrm{d}x
\]

\[
\leq C \|\mathbf{v}\|_{\infty} \|\nabla \mathbf{v}\|_2^2 \leq C \|\mathbf{v}\|_{2,2}^2 \|\nabla \mathbf{v}\|_2^2
\]

\[
= C (\|P_0 \Delta \mathbf{v}\|_2 + \|\mathbf{v}\|_2) \|\nabla \mathbf{v}\|_2^2
\]

\[
\leq \delta \|P_0 \Delta \mathbf{v}\|_2^2 + C (\delta) \|\nabla \mathbf{v}\|_2^2 + C.
\]

\[ I_2 = \int_{\partial \Omega} (\mathbf{v} \cdot \nabla \mathbf{v})_r \cdot (\mathbf{n} \cdot \nabla \mathbf{v}) \, dS\]

\[
= \int_{\partial \Omega} (\mathbf{v} \cdot \nabla \mathbf{v})_r \cdot [\mathbf{n} \cdot (\mathbf{v} + (\nabla \mathbf{v})^T)] \, dS
\]

\[
= - \int_{\partial \Omega} (\mathbf{v} \cdot \nabla \mathbf{v})_r \cdot [\mathbf{n} \cdot (\nabla \mathbf{v})^T] \, dS
\]

\[
= \int_{\partial \Omega} (\mathbf{v} \cdot \nabla \mathbf{v})_r \cdot [2 \mathbb{D}(\mathbf{v}) \cdot \mathbf{n}]_r \, dS
\]

\[
= - \int_{\partial \Omega} (\mathbf{v} \cdot \nabla \mathbf{v})_r \cdot [\nabla (\mathbf{n} \cdot \mathbf{v}) - \mathbf{n} \cdot \nabla \mathbf{v}] \, dS.
\]

Since $(\mathbf{v} \cdot \nabla \mathbf{v})_r$ is tangential and $\mathbf{n} \cdot \mathbf{v} = 0$ on $\partial \Omega$, the scalar product $(\mathbf{v} \cdot \nabla \mathbf{v})_r \cdot \nabla (\mathbf{n} \cdot \mathbf{v})$ is equal to zero. Thus, if we also use boundary condition (4), the inequalities in $(\alpha)$ and $(\beta)$, and Lemma 3, we get

\[ |I_2| = \left| - \frac{1}{\nu} \int_{\partial \Omega} (\mathbf{v} \cdot \nabla \mathbf{v})_r \cdot (\kappa \cdot \mathbf{v}) \, dS \right|
\]

\[
+ \int_{\partial \Omega} (\mathbf{v} \cdot \nabla \mathbf{v})_r \cdot \nabla (\mathbf{n} \cdot \mathbf{v}) \, dS \leq C \int_{\partial \Omega} |\mathbf{v}|^2 |\nabla \mathbf{v}| \, (\|\kappa\|_2 + 1)
\]

\[
+ 1 \|\mathbf{S}\|_2 \leq C \|\mathbf{v}\|_{2,2}^2 \|\nabla \mathbf{v}\|_{4,2} (\|\kappa\|_4 + 1)
\]

\[
\leq C \|\mathbf{v}\|_{2,2}^2 \|\nabla \mathbf{v}\|_{2,2} (\|\kappa\|_2 + 1) \leq C \|\mathbf{v}\|_{2,2}^2 (\|P_0 \Delta \mathbf{v}\|_2 + \|\mathbf{v}\|_2)
\]

\[
+ \|\nabla \mathbf{v}\|_2 \leq \delta \|P_0 \Delta \mathbf{v}\|_2^2 + C (\delta) \|\nabla \mathbf{v}\|_2^4 + C,
\]

\[
I_3 = \int_{\Omega} \int_3 \mathbf{v}_j (\partial_j \mathbf{v}_i) (\partial_k \mathbf{v}_i) + v_j (\partial_k \mathbf{v}_i) (\partial_k \mathbf{v}_i) \, dS
\]

\[
= \int_{\Omega} \int_3 \mathbf{v}_j (\partial_j \mathbf{v}_i) (\partial_k \mathbf{v}_i) \, dS.
\]

If we denote $(i, j, k) = 1, 2, 3$ and $s_{ij} := (1/2)[(\partial_i \mathbf{v}_j + (\partial_j \mathbf{v}_i)]$ (the entries of tensor $\mathbb{D}$) and $s_{ij} := (1/2)[(\partial_i \mathbf{v}_j) - (\partial_j \mathbf{v}_i)]$ (the entries of the skew-symmetric part of $(\nabla \mathbf{v})$, we obtain

\[ I_3 = \int_{\Omega} \int_3 \left( d_{kj} + s_{kj} \right) \left( d_{ji} + s_{ji} \right) \left( d_{ki} + s_{ki} \right) \, dS
\]

\[
= \int_{\Omega} \int_3 \left[ d_{kj}d_{ij}d_{ik} + d_{kj}s_{ij} + d_{ji}s_{ki} + d_{ki}s_{kj} \right] \, dS.
\]

As $s_{ji} = -s_{ij}$, we have $d_{kj}s_{ij} + s_{ki}s_{kj} = d_{kj}s_{ij} + d_{kj}s_{kj} = 0$. Hence

\[ I_3 = \int_{\Omega} \int_3 \left[ d_{kj}d_{ij}d_{ik} + d_{ji}s_{ki} \right] \, dS
\]

\[
= \int_{\Omega} \int_3 d_{kj}d_{ij}d_{ik} \, dS - \frac{1}{4} \int_{\Omega} \int_3 d_{ji}d_{ki}d_{ij} \, dS.
\]

where $\omega_i$ are the components of $\omega := \text{curl} \mathbf{v}$. The estimates (27), (29) and the identity (32) yield

\[ \int_{\Omega} \mathbf{v} \cdot \nabla \mathbf{v} \cdot \Delta \mathbf{v} \, dS \leq 2\delta \|P_0 \Delta \mathbf{v}\|_2^2 + C (\delta) \|\nabla \mathbf{v}\|_2^4 + C
\]

\[
- \int_{\Omega} d_{kj}d_{ij}d_{ik} \, dS
\]

\[
+ \frac{1}{4} \int_{\Omega} d_{ji}d_{ki}d_{ij} \, dS.
\]

The integral on the left hand side of (33) can also be treated in another way:

\[ \int_{\Omega} \mathbf{v} \cdot \nabla \mathbf{v} \cdot \Delta \mathbf{v} \, dS = - \int_{\Omega} \mathbf{v} \cdot \nabla \mathbf{v} \cdot \text{curl}^2 \mathbf{v} \, dS
\]

\[ = - \int_{\Omega} \mathbf{v} \cdot \nabla \mathbf{v} \cdot (\mathbf{n} \times \text{curl} \mathbf{v}) \, dS
\]

\[ - \int_{\Omega} \text{curl} \mathbf{v} \cdot (\mathbf{n} \times \text{curl} \mathbf{v}) \, dS.
\]

The integrals on the right hand side can be estimated or modified as follows:

\[ \left| \int_{\Omega} \mathbf{v} \cdot \nabla \mathbf{v} \cdot (\mathbf{n} \times \text{curl} \mathbf{v}) \, dS \right|
\]

\[
= \int_{\Omega} \left( \mathbf{v} \cdot \nabla \mathbf{v} \right) \cdot \left[ (2 \mathbb{D} \cdot \mathbf{n}) + 2 \mathbf{v} \cdot \nabla \mathbf{v} \right] \, dS
\]

\[
\leq \frac{1}{\nu} \int_{\Omega} |\mathbf{v}|_2^2 |\nabla \mathbf{v}| \, (\|\kappa\|_2 + 1) \, dS
\]

\[
\leq C \|\mathbf{v}\|_{2,2}^2 \|\nabla \mathbf{v}\|_{2,2} (\|\kappa\|_4 + 1)
\]

\[
\leq \delta \|P_0 \Delta \mathbf{v}\|_2^2 + C (\delta) \|\nabla \mathbf{v}\|_4^2 + C.
\]
Integrating this inequality on the time interval \( [0, \tau] \), we get

\[
\frac{1}{4} \int_{\Omega} v \cdot \nabla \cdot \nabla \psi \, dx \leq \frac{\delta}{4} \| P_2 \nabla \psi \|_2^2 + C(\delta) \| \nabla \psi \|_2^4 + C \| \nabla \psi \|_2^2 \leq \frac{1}{4} \int_{\Omega} d_{ij} \psi_{ij} \, dx.
\]

Summing (33) and (37), we obtain

\[
\frac{5}{4} \int_{\Omega} v \cdot \nabla \cdot \nabla \psi \, dx \leq \frac{9}{4} \| P_2 \nabla \psi \|_2^2 + C(\delta) \| \nabla \psi \|_2^4 + C \| \nabla \psi \|_2^2 \leq \frac{1}{4} \int_{\Omega} d_{ij} \psi_{ij} \, dx.
\]

Dividing this inequality by \( 5/4 \), we get

\[
\frac{d}{dt} \| D(v) \|_2^2 + \frac{1}{2v} \frac{d}{dt} \| \mathbf{c} \cdot v \|_2^2 + \| P_2 \Delta \|_2^2 \leq -\frac{12}{5} \int_{\Omega} (-\xi_1) (\xi_2) + \xi_3 \, dx + C_1 \| \nabla \psi \|_2^2 \| D(v) \|_2^2 + C_6.
\]

Integrating this inequality on the time interval \( [t_0, t_1] \), we obtain

\[
\| D(v) \|_{L^2(\Omega, t_0, t_1)}^2 + \frac{y}{2} \| P_2 \Delta \|_{L^2(\Omega, t_0, t_1)}^2 \leq \eta \| D(v(t_0)) \|_2^2 + \xi_0 \int_{t_0}^{t_1} (-\xi_1) (\xi_2) + \xi_3 \, dx + C_1 \| \nabla \psi \|_2^2 \| D(v) \|_2^2 + C_6.
\]

Assume that \( t_1 = b_t \) and \( t_1 - t_0 < \xi \), where \( \xi \) is so small that \( \xi \in (0, T) \) such that \( 0 \leq t_1 < t_2 \leq T \) for any \( t_1, t_2 \in (0, T) \) such that \( 0 \leq t_1 < t_2 \leq T \). Then

\[
\| D(v) \|_{L^2(\Omega, t_0, t_1)}^2 + \frac{y}{2} \| P_2 \Delta \|_{L^2(\Omega, t_0, t_1)}^2 \leq 2\xi \| D(v(t_0)) \|_2^2 + C_1 \| \nabla \psi \|_2^2 + C_1 \| \nabla \psi \|_2^2 + C_6.
\]

From this, we observe that \( b_t \) cannot be an epoch of irregularity of the weak solution \( v \). The proof of Theorem 1 is completed. \( \square \)
Conflicts of Interest

The authors declare that they have no conflicts of interest.

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