Hindawi
Advances in Mathematical Physics
Volume 2018, Article ID 4743567, 11 pages
https://doi.org/10.1155/2018/4743567

Research Article

Approximate Symmetry Analysis and Approximate Conservation Laws of Perturbed KdV Equation

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Received 9 February 2018; Revised 11 August 2018; Accepted 27 August 2018; Published 9 September 2018

Academic Editor: Boris G. Konopelchenko

Approximate symmetries, which are admitted by the perturbed KdV equation, are obtained. The optimal system of one-dimensional subalgebra of symmetry algebra is obtained. The approximate invariants of the presented approximate symmetries and some new approximately invariant solutions to the equation are constructed. Moreover, the conservation laws have been constructed by using partial Lagrangian method.

1. Introduction

The partial differential equations (PDEs) with small parameter have been arisen in mathematics, physics, mechanics, etc. For these perturbed equations, the finding of analytic solutions, conservation laws, symmetries, etc. is essential in these related fields. Traditionally, various perturbation methods are used to solve such equations. In addition, emerging innovative methods such as the inverse scattering transformation method, homotopy perturbation method, Adomian decomposition method, and approximate symmetry method have been developed fast in the past few decades [1–4]. The approximate symmetry method, which is developed by Baikov, Gazizov, and Ibragimov [3, 4] in the 1980s, shows an effectiveness to obtain approximate solutions to a perturbed PDEs. The idea behind this development was the extension of Lie's theory in which a small perturbation of the original equation is encountered. The new method maintains the essential features of the standard Lie symmetry method and provides us with the most widely applicable technique to find the approximate solutions to a perturbed differential equations [5, 6].

The construction of conservation laws is one of the key applications of symmetries to physical problems [7, 8]. But for perturbed systems, one hindrance is how to construct conservation laws. Combined with the Noether theory and Lagrangian principle, the approximate symmetry method yields the approximate conservation laws for a perturbed PDEs. There are mainly two methods. One approach, based on generalizations of the Noether's theorem to perturbed equations, is given to get approximate conservation laws via the approximate Noether symmetries associated with a Lagrangian of the perturbed equations [5, 9]. This method depends on the existence of the Lagrangian functional for underlying differential equations. The other approach (also called partial Lagrangian method), which is presented by Johnpillai, Kara, and Mahomed [10, 11], demonstrates how one can construct approximate conservation laws of Eular-Type PDEs via approximate Noether-type symmetry operators associated with partial Lagrangian. Partial Lagrangian method can construct approximate conservation laws of perturbed equations via approximate operators that are not necessarily approximate symmetry operators of the underlying system of equations. This method is used for more general equations as even the equations do not admit essential Lagrangian.

In this paper, we consider perturbed KdV equation

$$u_t - 6uu_x + u_{xxx} + \varepsilon\alpha\left(u + 2xu_x\right) - \varepsilon\beta\left(2u + xu_x\right) = 0, \tag{1}$$

where ε is a small parameter, while α and β are arbitrary constants. Obviously, the equation has no Lagrangian because of its odd order. In [12], E. S. Benilov and B. A. Malomed

discussed the equation based on the inverse scattering transformation and showed its integrability. When $\beta = 2\alpha$ (1) has important applications in the description of nonlinear ion acoustic waves in an inhomogeneous plasma. For many other applications of (1), please refer to [2, 12, 13] and the references therein.

In this article, with the application of approximate symmetry method and partial Lagrangian method, we will investigate (1) and show its all first-order approximate symmetries. Furthermore, we will also construct several approximate solutions and approximate conservation laws of the equation. The rest of the paper is organized as follows. Section 2 gives some basic concepts and notations. Section 3 analyzes approximate symmetries of (1) by applying the approximate symmetry method, developed by Baikov, Gazizov, and Ibragimov. We compute the optimal system of the presented approximate symmetries. In Section 4 we generate the approximate invariants of presented approximate symmetries and construct corresponding approximately invariant solutions. Section 5, the final part, presents approximate conservation laws via partial Lagrangian method.

2. Notations and Definitions

We will use the following notations and definitions. Let G be a one-parameter approximate transformation group:

$$\tilde{z}^i \approx f(x, a, \varepsilon) \equiv f_0^i(z, a) + \varepsilon f_1^i(z, a), \quad i = 1, \ldots, N. \tag{2}$$

An approximate equation

$$F(z, \varepsilon) \equiv F_0(z) + \varepsilon F_1(z) \approx 0 \tag{3}$$

is said to be *approximately invariant* with respect to G or admits G if

$$F(\tilde{z}, \varepsilon) \approx F(f(z, a, \varepsilon), \varepsilon) = o(\varepsilon) \tag{4}$$

whenever $z = (z^1, \ldots, z^N)$ satisfies (3).

If $z = (x, u, u_{(1)}, \ldots, u_{(k)})$, where independent variables $x = (x^1, \ldots, x^n)$, dependent variables u and $u_{(k)}$ denote the collections of all kth-order partial derivatives, then (3) becomes an approximate differential equation of k, and G is an approximate symmetry group of the differential equation.

Theorem 1 (see [5]). *Equation (3) is approximately invariant under the approximate transformation group (2) with the generator*

$$X = X_0 + \varepsilon X_1 \equiv \xi_0^i(z) \frac{\partial}{\partial z^i} + \varepsilon \xi_1^i(z) \frac{\partial}{\partial z^i}, \tag{5}$$

if and only if

$$\left[X^{(k)} F(z, \varepsilon)\right]_{F \approx 0} = o(\varepsilon), \tag{6}$$

or

$$\left[X_0^{(k)} F_0(z) + \varepsilon \left(X_1^{(k)} F_0(z) + X_0^{(k)} F_1(z)\right)\right]_{(3)} = o(\varepsilon) \tag{7}$$

in which k is order of equation and $X^{(k)}$ is $k-th$ order prolongation of X. The operator (5) satisfying (7) is called an infinitesimal approximate symmetry or an approximate operator admitted by (3). Accordingly, (7) is termed the determining equation for approximate symmetries.

Remark 2. The determining equation (7) can be written as follows:

$$X_0^{(k)} F_0(z) = \lambda(z) F_0(z), \tag{8}$$

$$X_1^{(k)} F_0(z) + X_0^{(k)} F_1(z)\big) = \lambda(z) F_1(z). \tag{9}$$

The factor $\lambda(z)$ is determined by (8) and then substituted in (9). The latter equation must hold for all solutions of $F_0(z) = 0$. Comparing (8) with the determining equation of exact symmetries, we obtain the following statement.

Theorem 3 (see [5]). *If (3) admits an approximate transformation group with the generator $X = X_0 + \varepsilon X_1$, where $X_0 \neq 0$, then the operator*

$$X_0 = \xi_0^i \frac{\partial}{\partial z^i} \tag{10}$$

is an exact symmetry of

$$F_0(z) = 0. \tag{11}$$

Remark 4. It is manifested from (8) and (9) that if X_0 is an exact symmetry of (11), then $X = \varepsilon X_0$ is an approximate symmetry of (3).

Definition 5 (see [5]). Equations (11) and (3) are termed an unperturbed equation and a perturbed equation, respectively. Under the conditions of Theorem 3, the operator X_0 is called a stable symmetry of the unperturbed equation (11). The corresponding approximate symmetry generator $X = X_0 + \varepsilon X_1$ for the perturbed equation (10) is called a deformation of the infinitesimal symmetry X_0 of (11) caused by the perturbation $\varepsilon F_1(x)$. In particular, if the most general symmetry Lie algebra of Eq.(11) is stable, we say that the perturbed equation (3) inherits the symmetries of the unperturbed equation.

3. Approximate Symmetry Analysis

3.1. Exact Symmetries. Let us write the approximate group generator in the form

$$\begin{aligned} X &= X_0 + \varepsilon X_1 \\ &= (\xi_0 + \varepsilon \xi_1) \frac{\partial}{\partial x} + (\tau_0 + \varepsilon \tau_1) \frac{\partial}{\partial y} + (\eta_0 + \varepsilon \eta_1) \frac{\partial}{\partial u}, \end{aligned} \tag{12}$$

where ξ_i, τ_i, and η_i $(i = 0, 1)$ are unknown functions of x, t, and u.

Solving the determining equation

$$X_0^{(k)} F_0(z)\big|_{F_0(z)=0} = 0, \tag{13}$$

for the exact symmetries of the unperturbed equation $u_t - 6uu_x + u_{xxx} = 0$, we can get

$$\begin{aligned} \eta_{0t} &= 0, \\ \eta_{0x} &= 0, \\ \tau_{0x} &= 0, \\ \tau_{0u} &= 0, \\ \eta_{0u} + 2\xi_{0x} &= 0, \\ \xi_{0xx} &= 0, \\ \xi_{0u} &= 0, \\ 6\eta_0 + \xi_{0t} + 12u\xi_{0x} &= 0, \\ 3\xi_{0x} - \tau_{0t} &= 0. \end{aligned} \tag{14}$$

Then, we obtain $\xi_0 = c_1x - 6c_2t + c_3, \tau_0 = 3c_1t + c_2, \eta_0 = -2c_1u + c_2$, where c_1, c_2, c_3, c_4 are arbitrary constants. Hence,

$$\begin{aligned} X_0 = {} & (c_1x - 6c_2t + c_3)\frac{\partial}{\partial x} + (3c_1t + c_4)\frac{\partial}{\partial t} \\ & + (-2c_1u + c_2)\frac{\partial}{\partial u}. \end{aligned} \tag{15}$$

In other words, $u_t - 6uu_x + u_{xxx} = 0$ admits the four-dimensional Lie algebra with the basis

$$\begin{aligned} X_0^1 &= x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}, \\ X_0^2 &= -6t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \\ X_0^3 &= \frac{\partial}{\partial x}, \\ X_0^4 &= \frac{\partial}{\partial u}. \end{aligned} \tag{16}$$

3.2. Approximate Symmetries and Optimal System

3.2.1. Approximate Symmetries: The Cases α and β Are Arbitrary. First we need to determine the auxiliary function H by virtue of (15), by

$$H = \frac{1}{\varepsilon}\left[X_0^{(k)}\left(F_0(z) + \varepsilon F_1(z)\right)\Big|_{F_0(z)+\varepsilon F_1(z)=0}\right]. \tag{17}$$

Substituting the expression (15) of the generator X_0 into above equation we obtain the auxiliary function:

$$\begin{aligned} H = {} & c_2\alpha - 2c_2\beta + (3c_1\alpha - 6c_1\beta)u \\ & + (2c_3\alpha - 12c_2\alpha t + 6c_1\alpha x - c_3\beta + 6c_2\beta t - 3c_1\beta x) \\ & \cdot u_x. \end{aligned} \tag{18}$$

Now, calculate the operators X_1 by solving the inhomogeneous determining equation for deformations:

$$X_1^{(k)}F_0(z)\Big|_{F_0(z)=0} + H = 0. \tag{19}$$

Above determining equation yields

$$\begin{aligned} & c_2\alpha - 2c_2\beta - 6c_1\alpha u - 6c_1\beta u + \eta_{1t} = 0, \\ & 2c_1\alpha - 2\eta_{1x} = 0, \\ & \tau_{1x} = 0, \\ & \eta_{1u} = 0, \\ & \eta_{1u} + 2\xi_{1x} = 0, \\ & \xi_{1xx} = 0, \\ & \tau_{1u} = 0, \\ & 3\xi_{1x} - \tau_{1t} = 0, \\ & 2c_3\alpha - c_3\beta + 6c_1\alpha x - 3c_1\beta x - 12c_2\alpha t + 6c_2\beta t - 6\eta_1 \\ & \quad - \xi_{1t} - 12u\xi_{1x} = 0. \end{aligned} \tag{20}$$

Solving this system, we obtain

$$\begin{aligned} X_1 = {} & \left[-3(c_1\alpha + c_1\beta)tx + (2c_3\alpha - c_3\beta - 6d_1)t \right. \\ & \left. - 3(c_2\alpha + c_2\beta)t^2 + d_2\right]\frac{\partial}{\partial x} \\ & + \left[-\frac{9(c_1\alpha + c_1\beta)t^2}{2} + d_3\right]\frac{\partial}{\partial t} + \left[\frac{3c_1\alpha x}{2}\right. \\ & \left. + (2c_2\beta + 6c_1\alpha u - c_2\alpha + 6c_1\beta u)t + d_1\right]\frac{\partial}{\partial u}. \end{aligned} \tag{21}$$

Then, we obtain the following approximate symmetries of (1)

$$\begin{aligned} X = {} & \left[c_1x - 6c_2t + c_3 + \varepsilon\left[-3(c_1\alpha + c_1\beta)tx \right.\right. \\ & \left. + (2c_3\alpha - c_3\beta - 6d_1)t - 3(c_2\alpha + c_2\beta)t^2 \right. \\ & \left.\left. + d_2\right]\right]\frac{\partial}{\partial x} + \left[3c_1t + c_2 + \varepsilon\left[-\frac{9(c_1\alpha + c_1\beta)t^2}{2}\right.\right. \\ & \left.\left. + d_3\right]\right]\frac{\partial}{\partial t} + \left[-2c_1u + c2 + \varepsilon\left[\frac{3c_1\alpha x}{2}\right.\right. \\ & \left.\left. + (2c_2\beta + 6c_1\alpha u - c_2\alpha + 6c_1\beta u)t + d_1\right]\right]\frac{\partial}{\partial u}, \end{aligned} \tag{22}$$

and we have

$$\begin{aligned} v_1 = {} & x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} + \varepsilon\left[-3(\alpha + \beta)tx\frac{\partial}{\partial x}\right. \\ & \left. - \frac{9}{2}(\alpha + \beta)t^2\frac{\partial}{\partial t} + \left(\frac{3\alpha x}{2} + 6\alpha ut + 6\beta ut\right)\frac{\partial}{\partial u}\right], \\ v_2 = {} & -6t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + \varepsilon\left[-3(\alpha + \beta)t^2\frac{\partial}{\partial x}\right. \\ & \left. + (2\beta - \alpha)t\frac{\partial}{\partial u}\right], \\ v_3 = {} & \frac{\partial}{\partial x} + \varepsilon\left[(2\alpha - \beta)t\frac{\partial}{\partial x}\right], \end{aligned}$$

$$
\begin{aligned}
v_4 &= \frac{\partial}{\partial u},\\
v_5 &= \varepsilon v_2,\\
v_6 &= \varepsilon v_3,\\
v_7 &= \varepsilon v_4.
\end{aligned}
\tag{23}
$$

Tables 1, 2, and 3 of commutator, evaluated in the first-order of precision, show that above operators span a seven-dimensional approximate Lie algebra L_7.

3.2.2. Approximate Symmetries: The Case $\alpha = 2\beta$. When $\alpha = 2\beta$, the equation admits seven-dimensional approximate Lie algebra L_7 as follows:

$$
\begin{aligned}
v_1 &= x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} + \varepsilon\left[-9\alpha t x\frac{\partial}{\partial x} - \frac{27}{2}\alpha t^2\frac{\partial}{\partial t}\right.\\
&\quad \left. + \left(\frac{3\alpha x}{2} + 6\alpha u t + 6\beta u t\right)\frac{\partial}{\partial u}\right],\\
v_2 &= -6t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + \varepsilon\left[-9\alpha t^2\frac{\partial}{\partial x}\right],\\
v_3 &= \frac{\partial}{\partial x} + \varepsilon\left[3\alpha t\frac{\partial}{\partial x}\right],\\
v_4 &= \frac{\partial}{\partial u},\\
v_5 &= \varepsilon v_2,\\
v_6 &= \varepsilon v_3,\\
v_7 &= \varepsilon v_4.
\end{aligned}
\tag{24}
$$

3.2.3. Approximate Symmetries and Optimal System: The Case $\alpha = -\beta$. When $\alpha = -\beta$, the equation admits seven-dimensional approximate Lie algebra L_7 as follows:

$$
\begin{aligned}
v_1 &= x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} + \varepsilon\left[\frac{3\alpha x}{2}\frac{\partial}{\partial u}\right],\\
v_2 &= -6t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + \varepsilon\left[-3\alpha t\frac{\partial}{\partial u}\right],\\
v_3 &= \frac{\partial}{\partial x} + \varepsilon\left[3\alpha t\frac{\partial}{\partial x}\right],\\
v_4 &= \frac{\partial}{\partial u},\\
v_5 &= \varepsilon v_2,\\
v_6 &= \varepsilon v_3,\\
v_7 &= \varepsilon v_4.
\end{aligned}
\tag{25}
$$

It is worth nothing that the seven-dimensional approximate Lie algebra $L_7 = g$ is solvable and its finite sequence of ideals is as follows:

$$
\begin{aligned}
0 \subset \langle v_6\rangle \subset \langle v_5, v_6\rangle \subset \langle v_3, v_5, v_6\rangle \subset \langle v_2, v_3, v_5, v_6\rangle\\
\subset \langle v_2, v_3, v_4, v_5, v_6\rangle \subset \langle v_2, v_3, v_4, v_5, v_6, v_7\rangle \subset g
\end{aligned}
\tag{26}
$$

In the following, we will construct the optimal system of above Lie algebra L_7. The method used here for obtaining the one-dimensional optimal system of subalgebras is that given in [7]. This approach is taking a general element from the Lie algebra and reducing it to its simplest equivalent form by applying carefully chosen adjoint transformations that are defined as follows.

Definition 6 (see [7]). Let G be a Lie group. An optimal system of s-parameter subgroups is a list of conjugacy inequivalent s-parameter subalgebras with the property that any other subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of s-parameter subalgebras forms an optimal system if every s-parameter subalgebra of g is equivalent to a unique member of the list under some element of the adjoint representation: $\widetilde{h} = Ad(g(h))$, $g \in G$.

Theorem 7 (see [7]). *Let H and $\widetilde{H}$ be connected s-dimensional Lie subgroups of the Lie group G with corresponding Lie subalgebras h and $\widetilde{h}$ of the Lie algebra g of G. Then $H = gHg^{-1}$ are conjugate subgroups if and only if $\widetilde{h} = Ad(r(h))$ are conjugate subalgebras.*

By Theorem 7, the problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. To compute the adjoint representation, we use the Lie series

$$
\begin{aligned}
Ad\left(\exp\left(\mu v_i\right)\right) v_j = v_j - \mu\left[v_i, v_j\right] + \frac{\mu^2}{2}\left[v_i, \left[v_i, v_j\right]\right]\\
- \cdots = \exp\left(ad\left(-\mu v_i\right)\right) v_j,
\end{aligned}
\tag{27}
$$

where $[v_i, v_j]$ is the commutator for the Lie algebra and μ is a parameter,and $i, j = 1, \ldots, 7$. In this manner, we construct Table 4 with the (i, j)th entry indicating $Ad(\exp(\mu v_i))v_j$.

Theorem 8. *An optimal system of one-dimensional approximate symmetry algebra (case $\alpha = -\beta$) of equation (1) is provided by $v_7, v_6, v_1 + bv_6, v_3 + bv_7, v_2 + bv_5 + cv_7, v_5 + bv_3 + cv_7, v_4 + bv_2 + cv_5 + dv_7$.*

Proof. Considering the approximate symmetry algebra g of (1), whose adjoint representation was determined in the Table 4, our task is to simplify as many of the coefficients a_i as possible through judicious applications of adjoint maps to V_i, so that V_i is equivalent V_i' under the adjoint representation.

Given a nonzero vector

$$
V_1 = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 + a_5 v_5 + a_6 v_6 + a_7 v_7. \tag{28}
$$

Table 1: Approximate commutators of approximate symmetry algebra of (1) ($a = 2\alpha - \beta$, $b = 2\beta - \alpha$, $c = \alpha + \beta$, $d = 3\alpha/2$).

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1	0	$2v_2$	$-dv_5 - v_3$	$-3v_4 + 3\varepsilon c v_1$	$2v_5$	$-v_6$	$-3v_7$
v_2	$-2v_2$	0	0	$6v_3 - bv_5$	0	0	$6v_6$
v_3	$dv_5 + v_3$	0	0	$-av_6$	0	0	0
v_4	$3v_4 - 3\varepsilon c v_1$	$-6v_3 + bv_5$	av_6	0	$-6v_6$	0	0
v_5	$-2v_5$	0	0	$6v_6$	0	0	0
v_6	v_6	0	0	0	0	0	0
v_7	$3v_7$	$-6v_6$	0	0	0	0	0

Table 2: Approximate commutators of approximate symmetry algebra of (1) ($\alpha = 2\beta$, $a = 3\alpha$).

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1	0	$2v_2$	$\frac{-a}{2}v_5 - v_3$	$-3v_4 + 3\varepsilon a v_1$	$2v_5$	$-v_6$	$-3v_7$
v_2	$-2v_2$	0	0	$-6v_3 + \frac{5}{2}v_5$	0	0	$6v_6$
v_3	$v_3 + \frac{5}{2}v_5$	0	0	$-av_6$	0	0	0
v_4	$3v_4 - 3\varepsilon a v_1$	$6v_3 - \frac{5}{2}v_5$	av_6	0	$-6v_6$	0	0
v_5	$-2v_5$	0	0	$6v_6$	0	0	0
v_6	v_6	0	0	0	0	0	0
v_7	$3v_7$	$-6v_6$	0	0	0	0	0

Table 3: Approximate commutators of approximate symmetry algebra of (1) ($\alpha = -\beta$, $a = 3\alpha$).

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1	0	$2v_2$	$\frac{-a}{2}v_5 - v_3$	$-3v_4$	$2v_5$	$-v_6$	$-3v_7$
v_2	$-2v_2$	0	0	$6v_3 + av_5$	0	0	$6v_6$
v_3	$\frac{a}{2}v_5 + v_3$	0	0	$-av_6$	0	0	0
v_4	$3v_4$	$-6v_3 - av_5$	av_6	0	$-6v_6$	0	0
v_5	$-2v_5$	0	0	$6v_6$	0	0	0
v_6	v_6	0	0	0	0	0	0
v_7	$3v_7$	$-6v_6$	0	0	0	0	0

Table 4: Adjoint representation of approximate symmetry of (1).

Ad	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1	v_1	$e^{-2\mu}v_2$	$e^{-\mu}v_3 + \frac{a}{2}e^{-\mu}v_5 - \frac{a}{2}v_5$	$e^{3\mu}v_4$	$e^{-2\mu}v_5$	$e^{-\mu}v_6$	$e^{3\mu}v_7$
v_2	$v_1 + 2\mu v_2$	v_2	v_3	$v_4 - \mu(6v_3 + av_5)$	v_5	v_6	$v_7 - 6\mu v_6$
v_3	$v_1 - \mu\left(\frac{av_5}{2} + v_3\right)$	v_2	v_3	$v_4 + \mu(av_6)$	v_5	v_6	v_7
v_4	$v_1 - 3\mu v_4$	$v_2 + \mu(6v_3 + av_5)$	$v_3 - a\mu v_6$	v_4	$v_5 + 6\mu v_6$	v_6	v_7
v_5	$v_1 + 2\mu v_5$	v_2	v_3	$v_4 - 6\mu v_6$	v_5	v_6	v_7
v_6	$v_1 - \mu v_6$	v_2	v_3	v_4	v_5	v_6	v_7
v_7	$v_1 - 3\mu v_7$	$v_2 + 6\mu v_7$	v_3	v_4	v_5	v_6	v_7

First, suppose that $a_1 \neq 0$. Scaling V_1 if necessary, we can assume that $a_1 = 1$. As for the 7th column of the Table 4, we have

$$\begin{aligned} V_1' &= Ad\left(\exp\left(\left(\frac{a_4}{3}v_4\right)\right)\right) \circ Ad\left(\exp\left(\left(\frac{-a_2}{2}v_2\right)\right)\right) \\ &\quad \circ Ad\left(\exp\left(\left(\frac{a_7}{3}v_7\right)\right)\right) \\ &\quad \circ Ad\left(\exp\left(\left(a_3 + 2a_2a_4\right)v_3\right)\right) \\ &\quad \circ Ad\left(\exp\left(\left(\frac{-a_5}{2} + \frac{aa_3}{4} + \frac{aa_2a_4}{12}\right)v_5\right)\right)V_1 \\ &= v_1 + \left(a_6 - \frac{aa_3a_4}{3} + 2a_4a_5 + 3a_2a_7\right)v_6. \end{aligned} \tag{29}$$

The remaining approximate one-dimensional subalgebras are spanned by vectors of the above form with $a_1 = 0$. If $a_4 \neq 0$, we have

$$V_2 = a_2 v_2 + a_3 v_3 + v_4 + a_5 v_5 + a_6 v_6 + a_7 v_7. \tag{30}$$

Next, we act on V_2 to cancel the coefficients of v_3 and v_6 as follows:

$$\begin{aligned} V_2' &= Ad\left(\exp\left(\left(\frac{-a_3}{6a_2} v_4\right)\right)\right) \\ &\quad \circ Ad\left(\exp\left(\left(\frac{-a_5}{6a_2} + \frac{aa_3^2}{36a_2} + \frac{a_6}{6}\right) v_5\right)\right)\right) V_2. \\ &= a_2 v_2 + v_4 + \left(a_5 - \frac{aa_3}{6} v_5 + a_7 v_7. \end{aligned} \tag{31}$$

If $a_1, a_4 = 0$ and $a_2 \neq 0$, the nonzero vector

$$V_3 = v_2 + a_3 v_3 + a_5 v_5 + a_6 v_6 + a_7 v_7 \tag{32}$$

is equivalent to

$$\begin{aligned} V_3' &= Ad\left(\exp\left(\left(\frac{-a_3}{6} v_4\right)\right)\right) \\ &\quad \circ Ad\left(\exp\left(\left(\frac{-a_6}{6} + \frac{-aa_3^2}{36} + \frac{a_6 a_5}{6}\right) v_7\right)\right)\right) V_3 \\ &= v_2 + \left(a_5 - \frac{aa_3}{6}\right) v_5 + a_7 v_7. \end{aligned} \tag{33}$$

If $a_1, a_2, a_4 = 0$ and $a_5 \neq 0$, we scale to make $a_5 = 1$ and then

$$V_4 = a_3 v_3 + v_5 + a_6 v_6 + a_7 v_7 \tag{34}$$

is equivalent to V_4' under the adjoint representation

$$V_4' = Ad\left(\exp\left(\left(\frac{a_6}{6a_7} v_2\right)\right)\right) V_4 = a_3 v_3 + v_5 + a_7 v_7. \tag{35}$$

If $a_1, a_2, a_4, a_5 = 0$ and $a_3 \neq 0$, in the same way as before, the nonzero vector

$$V_5 = v_3 + a_6 v_6 + a_7 v_7 \tag{36}$$

can be simplified as

$$V_5' = Ad\left(\exp\left(\left(\frac{a_6}{6a_7} v_2\right)\right)\right) V_2 = v_3 + a_7 v_7. \tag{37}$$

If $a_1, a_2, a_3, a_4, a_5 = 0$ and $a_7 \neq 0$, we can act

$$V_6 = a_6 v_6 + v_7 \tag{38}$$

by $Ad(\exp(((a_6/6)v_2)))$, to cancel the coefficient of v_6, leading to

$$V_6' = Ad\left(\exp\left(\left(\frac{a_6}{6} v_2\right)\right)\right) V_6 = v_7. \tag{39}$$

The last remaining case occurs when $a_1, a_2, a_3, a_4, a_5, a_7 = 0$ and $a_6 \neq 0$, for which our earlier simplifications were unnecessary, because the only remaining vectors are the multiples of a_6, on which the adjoint representation acts trivially. □

4. Approximately Invariant Solutions

In this section we use two different techniques to construct new approximate solutions of (1) when $\alpha = -\beta$.

4.1. Approximately Invariant Solutions I. In the beginning of this section we compute an approximately invariant solution based on the $X = v_2$. The approximate invariants for X are determined by

$$\begin{aligned} X(J) &= \left(-6t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + \varepsilon\left(-3\alpha t\frac{\partial}{\partial u}\right)\right)(J_0 + \varepsilon J_1) \\ &= o(\varepsilon). \end{aligned} \tag{40}$$

Equivalently

$$\begin{gathered} \left(-6t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right)(J_0) = 0, \\ \left(-3\alpha t\frac{\partial}{\partial u}\right)(J_1) + \left(-6t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right)(J_0) = 0. \end{gathered} \tag{41}$$

The first equation has two functionally independent solutions $J_0 = t$ and $J_0 = x/6t + u$. The simplest solutions of the second equation are, respectively, $J_1 = 0$ and $J_1 = -\alpha x/2 + t$. Therefore, we have two independent invariants to $x/6t + u + \varepsilon(-\alpha x/2 + t)$ and $\varphi(t)$ with respect to X.

Letting $x/6t + u + \varepsilon(3\alpha t u) = \varphi(t)$, we obtain $u = \varphi(t) - x/6t - \varepsilon(-\alpha x/2 + t)$ for the approximately invariant solutions.

Therefore, we can obtain

$$\begin{aligned} u(x,t) \approx -\frac{x}{6t} + \Gamma_0(t) + \varepsilon\left(-t - \frac{\alpha x}{2} + \Gamma_1(t)\right. \\ + \Gamma_0(t)\Gamma_1'(t). \end{aligned} \tag{42}$$

Putting (42) into (1), we obtain

$$\begin{aligned} \Gamma_0(t) &= \frac{C_1}{T} \\ \Gamma_1(t) &= \frac{C_2}{t^3} + t + \frac{C_2}{t}, \end{aligned} \tag{43}$$

where C_1 and C_2 are arbitrary constants. Hence, we obtained the approximately invariant solution to (1)

$$u(x,t) \approx -\frac{x}{6t} + \frac{C_1}{t} + \varepsilon\left(\frac{C_2}{t} - \frac{\alpha x}{2}\right). \tag{44}$$

In this manner, we compute approximate invariants with respect to the generators of Lie algebra and optimal system, as shown in Table 5.

4.2. Approximately Invariant Solution II. Now, we apply a different technique to find approximately invariant solutions for (1). We will begin with one exact solution

$$u = w(\zeta) = \frac{-1}{2}\frac{1}{\cosh^2\zeta} \tag{45}$$

Table 5: Approximate invariants with respect to the generators of Lie algebra and optimal system.

Operator	Approximate Invariants
v_1	$\frac{t}{x^3}, x^2u - \varepsilon\left(\frac{\alpha x^3}{2}\right)$
v_2	$t, \frac{x}{6t} + u + \varepsilon\left(\frac{-\alpha x}{2} + t\right)$
v_3	t, u
v_4	x, u
v_5	$t, \frac{x}{6t} + u$
v_6	t, u
v_7	x, u
$v_1 + bv_6$	$\frac{t}{x^3} + \varepsilon(bx^{-4}t), x^2u - \varepsilon\left(\frac{\alpha x^3}{2} + \frac{2bx}{3u}\right)$
$v_3 + bv_7$	$t - \varepsilon(bx), u$
$v_4 + bv_2 + cv_5 + dv_7$	$\frac{t^2}{2} + \frac{x}{6b} + \varepsilon\left(\left(\frac{c}{b} - d\right)t\right), \frac{b}{u} - t + \varepsilon\left(\frac{3\alpha t^2}{2} + dt\right)$
$v_2 + bv_5 + cv_7$	$t - \varepsilon\left(\frac{u}{c}\right), \frac{x}{6t} + u + \varepsilon(3\alpha tu - \frac{cx^2}{72t^3})$
$v_5 + bv_3 + cv_7$	$t - \varepsilon\left(\frac{cx}{b}\right), u$

of the unperturbed equation $u_t - 6uu_x + u_{xxx} = 0$. Here, $\zeta = (x - t)/2$ is an invariant of the group with the generator

$$X^0 = \frac{\partial}{\partial t} + \frac{\partial}{\partial x}. \tag{46}$$

The function w given by (45) is invariant under the operator (46).

Using the generators $X^1 = \partial/\partial x$, admitted by $u_t - 6uu_x + u_{xxx} = 0$, we will take the approximate symmetry

$$X = X^0 + \varepsilon X^1 = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial x} \tag{47}$$

and use it looking for the approximately invariant solution of (1) in the form

$$u = w + \varepsilon v(t, x). \tag{48}$$

Then the invariant equation test $X(u - w - \varepsilon v)|_{(48)} = o(\varepsilon)$ can be written as

$$X^0(u - w) + \varepsilon\left[X^1(u - w) - X^0(v)\right]\Big)_{u=w} = 0. \tag{49}$$

Note that X^0 does not contain the differentiation in u; therefore (49) becomes

$$\left[X^1(u - w) - X^0(v)\right]_{u=w} = 0 \tag{50}$$

whence we obtain the following differential for $v(t, x)$:

$$v_t + v_x = -(w)_x. \tag{51}$$

Because $w = w(\zeta)$, so (51) can be integrated by the variables

$$\zeta = \frac{x - t}{2}, \qquad \rho = t. \tag{52}$$

Then, denoting by w' the derivative of w with respect to ζ, we have

$$(w)_t = -\frac{1}{2}(w)', \qquad (w)_x = \frac{1}{2}(w)' \tag{53}$$

and (51) becomes

$$v_\rho = -\frac{1}{2}(w)'. \tag{54}$$

The integration yields

$$v = -\frac{1}{2}\rho(w)' + \psi(x). \tag{55}$$

Returning to the variables t, x, we have

$$v = -t(w)_x + \psi(x). \tag{56}$$

Inserting this v in (48) and substituting it into (1) we obtain $\psi = ((-1 - 3\beta)/6)x$.

Thus, the approximate symmetry (47) provides the following approximately invariant solution (approximate travelling wave):

$$u(x, t) = \frac{-1}{2} \cdot \frac{1}{\cosh^2((x - t)/2)} + \varepsilon\left[-\frac{t}{2}\frac{1}{\cosh^2((x - t)/2)} \cdot \tanh\left(\frac{x - t}{2}\right) + \frac{-1 - 3\beta}{6}x\right]. \tag{57}$$

5. Approximate Conservation Laws

Approximate Lie symmetry can be used to construct the approximate conservation laws, but in this section, we will use partial Lagrangians to construct approximate conservation laws of (1); this is a more concise method.

Definition 9 (see [10]). An operator Y is a kth-order approximate Lie-Bäcklund symmetry:

$$Y = Y_0 + \varepsilon Y_1 + \cdots + \varepsilon^k Y_k, \tag{58}$$

where

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^i \frac{\partial}{\partial u^\alpha}, \quad i = 1, 2, \ldots, n \tag{59}$$

and

$$Y_b = \xi_b^i \frac{\partial}{\partial x^i} + \eta_b^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_{b,i}^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{b,i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \cdots, \quad b = 0, \ldots, k, \tag{60}$$

where $\xi_b^i, \eta_b^\alpha \in A$ and the additional coefficients are

$$\begin{aligned} \zeta_{b,i}^\alpha &= D_i\left(W_b^\alpha\right) + \xi_b^j u_{ij}^\alpha, \\ \zeta_{b,i_1 i_2}^\alpha &= D_{i_1} D_{i_2}\left(W_b^\alpha\right) + \xi_b^j u_{j i_1 i_2}^\alpha, \quad i, j = 1, \ldots, n \end{aligned} \tag{61}$$

and W_b^α is the Lie characteristic function defined by

$$W_b^\alpha = \eta_b^\alpha - \xi_b^j u_j^\alpha. \tag{62}$$

Definition 10 (see [10]). The approximate Noether operator associated with an approximate Lie-Bäcklund symmetry operator Y is given by

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u^\alpha} + \sum_{s \geq 1} D_{i_1} \ldots D_{i_s}\left(W^\alpha\right) \frac{\delta}{\delta u_{i i_1 i_2 \ldots i_s}^\alpha}, \quad i = 1, \ldots, n, \tag{63}$$

where

$$N^i = N_0^i + \varepsilon N_1^i + \cdots + \varepsilon^k N_k^i, \quad i = 1, \ldots, n, \tag{64}$$

where D_i is the total derivative operator defined as

$$D_i = \frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots + u_{i i 1 i 2 \ldots i n} \frac{\partial}{\partial u_{i 1 i 2 \ldots i n}} + \cdots, \quad i = 1, 2, \ldots, n, \tag{65}$$

and here N_b^i, $b = 0, \ldots, k$ are Noether operators and the Euler-Lagrange operators are

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{j_1} \ldots D_{j_s} \frac{\partial}{\partial u_{i j_1 j_2 \ldots j_s}^\alpha}, \quad \alpha = 1, \ldots, m, \ i = 1, \ldots, n. \tag{66}$$

The Euler-Lagrange, approximate Lie-Bäcklund, and approximate Noether operators are connected by the operator identity

$$Y + D_i\left(\xi^i\right) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i N^i. \tag{67}$$

Definition 11 (see [10]). If there exists a function $L = L(x, u, u_{(1)}, \ldots, u_{(l)}) \in A, l < r$ and nonzero functions $f_\gamma^\beta \in A$ such that (1) which can be written as $\delta L / \delta u^\beta = \varepsilon f_\gamma^\beta E_1^\gamma$, where $f_\gamma^\beta = f_\gamma^\beta(x, u, u_{(1)}, \ldots, u_{(r-1)}), \beta, \gamma = 1, \ldots, m$ is an invertible matrix, then, provided that $E_1^\gamma \neq 0$, some L is called a partial Lagrangian; otherwise it is the standard Lagrangian. We term differential equations of the form

$$\frac{\delta L}{\delta u^\beta} = \varepsilon f_\gamma^\beta E_1^\gamma \tag{68}$$

approximate Euler-Lagrange-type equations.

Definition 12 (see [5]). An approximate Lie-Bäcklund symmetry operator Y is called an approximate Noether-type symmetry operator corresponding to a partial Lagrangian $L \in A$ if and only if there exists a vector $B = (B^1, B^2, \ldots, B^n), B^i \in A$ defined by

$$B^i = B_0^i + \varepsilon B_1^i + \cdots + \varepsilon^k B_k^i \tag{69}$$

such that

$$Y(L) + L D_i\left(\xi^i\right) = W^\alpha \frac{\delta L}{\delta u^\alpha} + D_i\left(B^i\right) + o\left(\varepsilon^{k+1}\right), \tag{70}$$

where $W = (W^1, W^2, \ldots, W^m), W^\beta \in A$ of Y is also the characteristic of the conservation law $D_i T^i = o(\varepsilon^{k+1})$, where

$$T^i = B^i - L\xi^i - W^\beta \frac{\partial L}{\partial u_i^\beta} + \cdots + o\left(\varepsilon^{k+1}\right) \tag{71}$$

of the approximate Euler-Lagrange-type (68).

Because (1) does not have a Lagrange function and if we put a transform $u = v_x$, then (1) becomes

$$\begin{aligned} &v_{xt} - 6 v_x v_{xx} + v_{xxxx} + \varepsilon\alpha\left(v_x + 2x v_{xx}\right) \\ &\quad - \varepsilon\beta\left(2 v_x + x v_{xx}\right) = 0. \end{aligned} \tag{72}$$

In order to write convenient, we can get

$$\begin{aligned} &u_{xt} - 6 u_x u_{xx} + u_{xxxx} + \varepsilon\alpha\left(u_x + 2x u_{xx}\right) \\ &\quad - \varepsilon\beta\left(2 u_x + x u_{xx}\right) = 0. \end{aligned} \tag{73}$$

Obviously,

$$u_{xt} - 6 u_x u_{xx} + u_{xxxx} = 0, \tag{74}$$

and the Lagrange function is

$$L = \frac{1}{2}\left(u_{xx}\right)^2 + \left(u_x\right)^3 - \frac{1}{2} u_x u_t. \tag{75}$$

And the approximate *Euler − Lagrange − type* equation is

$$\frac{\delta L}{\delta u} = -\varepsilon\alpha\left(u_x + 2xu_{xx}\right) + \varepsilon\beta\left(2u_x + xu_{xx}\right). \tag{76}$$

So the approximate Noether symmetry operator is $Y_0 + \epsilon Y_1$ for L

$$\begin{aligned}&\left(Y_0 + \varepsilon Y_1\right)L + D_i\left(\xi_0^i + \varepsilon\xi_1^i\right)L\\&\quad = \left[\left(\eta_0 - \xi_0^j u_j\right) + \varepsilon\left(\eta_1 - \xi_1^j u_j\right)\right]\\&\quad\cdot\left[-\epsilon\alpha\left(u_x + 2xu_{xx}\right) + \varepsilon\beta\left(2u_x + xu_{xx}\right)\right]\\&\quad + D_i\left(B_0^i + \varepsilon B_1^i\right),\end{aligned} \tag{77}$$

where

$$\begin{aligned}Y_0 + \varepsilon Y_1 &= \left(\xi_0^1 + \varepsilon\xi_1^1\right)\frac{\partial}{\partial t} + \left(\xi_0^2 + \varepsilon\xi_1^2\right)\frac{\partial}{\partial x} + \left(\eta_0\right.\\&\quad\left. + \varepsilon\eta_1\right)\frac{\partial}{\partial u} + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{xx}\frac{\partial}{\partial u_{xx}},\end{aligned} \tag{78}$$

$$\begin{aligned}\zeta_t &= \eta_{0t} + \left(\eta_{0u} - \xi_{0t}^1\right)u_t - \xi_{0t}^2 u_x - \xi_{0u}^1 u_t^2 - \xi_{0u}^2 u_x u_t\\&\quad + \xi_0^1 u_{tt} + \xi_0^2 u_{xt} + \varepsilon\left[\eta_{1t} + \left(\eta_{1u} - \xi_{1t}^1\right)u_t - \xi_{1t}^2 u_x\right.\\&\quad\left. - \xi_{1u}^1 u_t^2 - \xi_{1u}^2 u_x^2 + \xi_1^1 u_{tt} + \xi_1^2 u_{xt}\right],\end{aligned} \tag{79}$$

$$\begin{aligned}\zeta_x &= \eta_{0x} + \left(\eta_{0u} - \xi_{0x}^2\right)u_x - \xi_{0x}^1 u_t - \xi_{0u}^2 u_x^2 - \xi_{0u}^1 u_x u_t\\&\quad + \xi_0^1 u_{xt} + \xi_0^2 u_{xx} + \varepsilon\left[\eta_{1x} + \left(\eta_{1u} - \xi_{1x}^2\right)u_x - \xi_{1x}^1 u_t\right.\\&\quad\left. - \xi_{1u}^2 u_x^2 - \xi_{1u}^2 u_x u_t + \xi_1^1 u_{xzt} + \xi_1^2 u_{xx}\right],\end{aligned} \tag{80}$$

$$\begin{aligned}\zeta_{xx} &= \eta_{0xx} + \left(2\eta_{0xu} - \xi_{0xx}^2\right)u_x + \left(\eta_{0uu} - 2\xi_{0xu}^2\right)u_x^2\\&\quad - \xi_{0xx}^1 u_t - \xi_{0uu}^1 u_x^2 u_t - 2\xi_{0xu}^1 u_x u_t - \xi_{0u}^1 u_{xx} u_t\\&\quad - 2\xi_{0x}^1 u_{xt} - 2\xi_{0u}^1 u_x u_{xt} - \xi_{0uu}^2 u_x^3 - 3\xi_{0u}^2 u_x u_{xx}\\&\quad + \left(\eta_{0u} - 2\xi_{0x}^2\right)u_{xx} + \xi_0^1 u_{xxt} + \xi_0^2 u_{xxx} + \varepsilon\left[\eta_{1xx}\right.\\&\quad + \left(2\eta_{1xu} - \xi_{1xx}^2\right)u_x + \left(\eta_{1uu} - 2\xi_{1xu}^2\right)u_x^2 - \xi_{1xx}^1 u_t\\&\quad - \xi_{1uu}^1 u_x^2 u_t - 2\xi_{1xu}^1 u_x u_t - \xi_{1u}^1 u_{xx} u_t - 2\xi_{1x}^1 u_{xt}\\&\quad - 2\xi_{1u}^1 u_x u_{xt} - \xi_{1uu}^2 u_x^3 - 3\xi_{1u}^2 u_x u_{xx}\\&\quad\left. + \left(\eta_{1u} - 2\xi_{1x}^2\right)u_{xx} + \xi_1^1 u_{xxt} + \xi_1^2 u_{xxx}\right].\end{aligned} \tag{81}$$

So (58) becomes

$$\begin{aligned}&-\frac{1}{2}u_x\zeta_t - \frac{1}{2}u_t\zeta_x + 3u_x^2\zeta_x + u_{xx}\zeta_{xx}\\&\quad + L\left[\xi_{0t}^1 + \xi_{0u}^1 u_t + \varepsilon\left(\xi_{1t}^1 + \xi_{1u}^1 u_t\right)\right]\\&\quad + L\left[\xi_{0t}^2 + \xi_{0u}^2 u_t + \varepsilon\left(\xi_{1t}^2 + \xi_{1u}^2 u_t\right)\right]\\&\quad = \left[\eta_0 - \xi_0^1 u_t - \xi_0^2 u_x + \varepsilon\left(\eta_1 - \xi_1^1 u_t - \xi_1^2 u_x\right)\right]\\&\quad\cdot\left[-\varepsilon\alpha\left(u_x + 2xu_{xx}\right) + \varepsilon\beta\left(2u_x + xu_{xx}\right)\right]\\&\quad + \varepsilon\left(B_{1t}^1 + B_{1u}^1 u_t\right) + \varepsilon\left(B_{1t}^2 + B_{1u}^2 u_x\right)\\&\quad + \left(B_{0t}^1 + B_{0u}^1 u_t\right) + \left(B_{0x}^2 + B_{0u}^2 u_x\right).\end{aligned} \tag{82}$$

Put $L, \zeta_t, \zeta_x, \zeta_{xx}$ into (82), and let $\varepsilon^2 = 0$. We obtain

$$\begin{aligned}-\frac{1}{2}\eta_{0t} - B_{0u}^2 &= 0,\\\eta_{0x} &= 0,\\\eta_{0u} &= 0,\\\xi_0^1 &= 0,\\\xi_0^2 &= 0,\\B_{0u}^1 &= 0,\\B_{0x}^2 + B_{0t}^1 &= 0,\\-\frac{1}{2}\eta_{1x} - B_{1u}^1 &= 0,\\\eta_{1x} &= 0,\\\eta_{1u} &= 0,\\\xi_1^1 &= 0,\\\xi_1^2 &= 0,\\B_{0u}^1 &= 0,\\B_{1x}^2 + B_{1t}^1 &= 0,\\-\frac{1}{2}\eta_{1t} + \left(\alpha - 2\beta\right)\eta_0 - B_{1u}^2 &= 0,\\\eta_{1xx} + \left(2\alpha - \beta\right)\eta_0\cdot x &= 0.\end{aligned} \tag{83}$$

In the following we will consider three cases of α and β.

First Case. $\alpha \neq 2\beta$ and $\beta \neq 2\alpha$

$$\begin{aligned}\xi_0^1 &= 0,\\\xi_0^2 &= 0,\\\eta_0 &= 0,\\\xi_1^1 &= 0,\\\xi_1^2 &= 0,\\\eta_1 &= g\left(t\right),\\B_0^1 &= e\left(t, x\right),\\B_0^2 &= f\left(t, x\right),\end{aligned}$$

$$B_1^1 = i(t,x),$$
$$B_1^2 = -\frac{g_t(t)u}{2} + j(t,x), \tag{84}$$

where g and h are arbitrary function for t and $e(t,x)$, $f(t,x), i(t,x), j(t,x)$ are differential functions and $e_t(t,x) + f_x(t,x) = 0$ and $i_t(t,x) + j_x(t,x) = 0$.

Thus the equation has the following approximate Noether symmetric operators

$$Y = \varepsilon g(t)\frac{\partial}{\partial u}. \tag{85}$$

So the conservation vectors are

$$\begin{aligned} T^1 &= e(t,x) + \varepsilon i(t,x) + \frac{1}{2}\varepsilon u_x g(t), \\ T^2 &= f(t,x) + \varepsilon\left[-\frac{1}{2}g_t(t)u + j(t,x)\right] \\ &\quad - [\varepsilon g(t)]\left[3u_x^2 - \frac{1}{2}u_t\right] + D_x[\varepsilon g(t)]u_{xx}. \end{aligned} \tag{86}$$

Second Case. $\alpha \neq 2\beta$ and $\beta = 2\alpha$

$$\begin{aligned} \xi_0^1 &= 0, \\ \xi_0^2 &= 0, \\ \eta_0 &= h(t), \\ \xi_1^1 &= 0, \\ \xi_1^2 &= 0, \\ \eta_1 &= g(t), \\ B_0^1 &= e(t,x), \\ B_0^2 &= -\frac{h_t(t)u}{2} + f(t,x), \\ B_1^1 &= i(t,x), \\ B_1^2 &= -\frac{g_t(t)}{2}u - 3\alpha h(t) + j(t,x), \end{aligned} \tag{87}$$

where g and h are arbitrary function for t and $e(t,x)$, $f(t,x), i(t,x), j(t,x)$ are differential functions and $e_t(t,x) + f_x(t,x) = 0$, $i_t(t,x) + j_x(t,x) = 0$.

Thus the equation has the following approximate Noether symmetry operators

$$Y = h(t)\frac{\partial}{\partial u} + \varepsilon g(t)\frac{\partial}{\partial u}. \tag{88}$$

So the conservation vectors are

$$\begin{aligned} T^1 &= e(t,x) + \varepsilon i(t,x) + \frac{1}{2}u_x[h(t) + \varepsilon g(t)], \\ T^2 &= f(t,x) + \left[-\frac{1}{2}h_t(t)u\right] \\ &\quad + \varepsilon\left[-\frac{1}{2}g_t(t)u - 3\alpha h(t) + j(t,x)\right] \\ &\quad - [\varepsilon g(t) + h(t)]\left[3u_x^2 - \frac{1}{2}u_t\right] \\ &\quad + D_x[\varepsilon g(t) + h(t)]u_{xx}. \end{aligned} \tag{89}$$

Third Case. $\alpha = 2\beta$ and $\beta \neq 2\alpha$; the conservation laws are the same as the first case.

Data Availability

All data included in this study are available upon request by contact with the corresponding author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported in part by the Natural Science Foundation of Inner Mongolia of China under grant 2016MS0116.

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