Hindawi
Advances in Mathematical Physics
Volume 2018, Article ID 4925957, 11 pages
https://doi.org/10.1155/2018/4925957

Research Article

The Perturbed Riemann Problem with Delta Shock for a Hyperbolic System

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Received 30 June 2018; Accepted 26 August 2018; Published 5 September 2018

Academic Editor: Carlo Bianca

In this paper, we study the perturbed Riemann problem with delta shock for a hyperbolic system. The problem is different from the previous perturbed Riemann problems which have no delta shock. The solutions to the problem are obtained constructively. From the solutions, we see that a delta shock in the corresponding Riemann solution may turn into a shock and a contact discontinuity under a perturbation of the Riemann initial data. This shows the instability and the internal mechanism of a delta shock. Furthermore, we find that the Riemann solution of the hyperbolic system is instable under this perturbation, which is also quite different from the previous perturbed Riemann problems.

1. Introduction

In this paper, we are concerned with the following nonstrictly hyperbolic system of conservation laws in the form

$$\begin{aligned} u_t + \left(u^2\right)_x &= 0, \\ v_t + (uv)_x &= 0. \end{aligned} \tag{1}$$

Equations (1) can be derived from a two-dimensional hyperbolic system of conservation laws

$$\begin{aligned} u_t + \left(u^2\right)_x + (uv)_y &= 0, \\ v_t + (uv)_x + \left(v^2\right)_y &= 0. \end{aligned} \tag{2}$$

The above system is the mathematical simplification of Euler equations of gas dynamics. In 1991 and 1994, Yang and Zhang [1] and Tan and Zhang [2] obtained both numerical and analytical solutions to the Riemann problem of system (2). The form of Dirac delta functions supported on shocks was found necessary and was used as parts in the Riemann solutions. We call it a delta shock. A delta shock is the generalization of an ordinary shock. It is more compressive than an ordinary shock in the sense that more characteristics enter the discontinuity line. Mathematically, the delta shocks are new type singular solutions such that their components contain delta functions and their derivatives. Physically, they are interpreted as the process of formation of the galaxies in the universe, or the process of concentration of particles [3].

To investigate the validity of delta shock, in 1994, Tan et al. [4] considered the Riemann problem for the one-dimensional model (1). They found that there exist delta shock as the limit of vanishing viscosity for system (1). As for delta shock, there are numerous excellent papers. We refer readers to [3–14] and the references cited therein. There are still many open and complicated problems in the delta shock theories. Study of this area gives a new perspective in the theory of conservation law systems.

In this paper, we are interested in the internal mechanism and instability of a delta shock. For this purpose, we study system (1) with the following initial data:

$$(u,v)|_{t=0} = (u_0(x), v_0(x)) = \begin{cases} \left(u_0^-(x), v_0^-(x)\right), & x<0, \\ \left(u_0^+(x), v_0^+(x)\right), & x>0, \end{cases} \tag{3}$$

where $u_0^{\pm}(x)$ and $v_0^{\pm}(x)$ are all bounded C^1 functions with the following property:

$$\left(u_0^{\pm}(0\pm), v_0^{\pm}(0\pm)\right) = \left(\hat{u}^{\pm}, \hat{v}^{\pm}\right). \tag{4}$$

Here $\hat{u}^{\pm}$ and $\hat{v}^{\pm}$ are constants with $(\hat{u}^{-}, \hat{v}^{-}) \neq (\hat{u}^{+}, \hat{v}^{+})$. The initial value (3) is a perturbation of Riemann initial value (5) at the neighborhood of the origin in the $x - t$ plane. The perturbation on the Riemann initial data is reasonable. For example, error is unavoidable in computation and the error forms a perturbation of the initial data.

We divide our work into two parts according to the presence of delta shock or not. When the delta shock is not involved, the perturbed Riemann problem (1) and (3) is classical. More importantly, there is no delta shock in the corresponding Riemann solutions for the previous work on the perturbed Riemann problem. Therefore, we only pay attention to the perturbed Riemann problem (1) and (3) when the delta shock is involved. To overcome the difficulty caused by delta shock, we adopt the method of characteristic analysis and the local existence and uniqueness theorem proposed by Li Ta-tsien and Yu Wen-ci [15]. We construct the solution to the perturbed problem (1) and (3) locally in time.

Our result shows that a perturbation of initial data may bring essential change when a delta shock appears in the corresponding Riemann solution. A delta shock may turn into a shock and a contact discontinuity. This shows the instability of the delta shock, which allows us to better investigate the internal mechanism of a delta shock. Furthermore, the previous works [15, 16] about the perturbed Riemann problem pay more attention to the stability of the corresponding Riemann solution. In other words, the Riemann solution has a local structure stability with respect to the perturbation of Riemann initial data. A distinctive feature for this paper is that, for some initial data (3), the preceding local structure stability fails. We pay more attention to the differences between Riemann solution and perturbed Riemann solution.

The paper is organized as follows. In Section 2, we present some preliminary knowledge about the hyperbolic system (1). Then, the construction and proof of the solution to the perturbed Riemann problem (1) and (3) with delta shock are presented in Section 3.

2. Preliminaries

In this section, we recall the main properties of system (1) with Riemann initial data

$$(u, v)(x, 0) = \left(\hat{u}^{\pm}, \hat{v}^{\pm}\right), \quad \pm x > 0, \tag{5}$$

where $\hat{u}^{\pm}$ and $\hat{v}^{\pm}$ are constants with $(\hat{u}^{-}, \hat{v}^{-}) \neq (\hat{u}^{+}, \hat{v}^{+})$ (see [4, 17, 18] for a more detailed study of the model).

The eigenvalues of the hyperbolic system (1) are

$$\begin{aligned} \lambda_1(u, v) &= u, \\ \lambda_2(u, v) &= 2u, \end{aligned} \tag{6}$$

with the corresponding left eigenvectors,

$$\begin{aligned} l_1(u, v) &= (v, -u), \\ l_2(u, v) &= (1, 0), \end{aligned} \tag{7}$$

and the corresponding right eigenvectors,

$$\begin{aligned} r_1(u, v) &= (0, 1)^T, \\ r_2(u, v) &= (u, v)^T. \end{aligned} \tag{8}$$

By a direct calculation,

$$\begin{aligned} \nabla\lambda_1 \cdot r_1 &\equiv 0, \\ \nabla\lambda_2 \cdot r_2 &\equiv 2u. \end{aligned} \tag{9}$$

Therefore, λ_1 is always linearly degenerate; λ_2 is genuinely nonlinear if $u \neq 0$ and linearly degenerate if $u = 0$.

The Riemann invariants of system (1) along the characteristic fields are

$$\begin{aligned} \zeta(u, v) &= u, \\ \varsigma(u, v) &= \frac{u}{v}. \end{aligned} \tag{10}$$

Definition 1 (see [3, 19]). A pair of (u, v) is called a generalized delta shock solution to (1) with the initial data (3) on local time $[0, T)$, if there exists a smooth curve $l = \{(x_\delta(t), t) : 0 \leq t < T\}$ and a weight $\omega(x, t)$ such that u and v are represented in the following form:

$$\begin{aligned} u &= U(x, t), \\ v &= V(x, t) + \omega(x, t)\,\delta(l), \end{aligned} \tag{11}$$

in which $\delta(x)$ is the delta function, $\omega \in C^1(l)$, $U, V \in L^\infty(R \times [0, T); R)$ and satisfy

$$\begin{aligned} &\int_0^T \int_{-\infty}^{+\infty} \left(U\phi_t + U^2\phi_x\right) dx\,dt \\ &\quad + \int_{-\infty}^{+\infty} u_0(x)\,\phi(x, 0)\,dx = 0, \end{aligned} \tag{12}$$

$$\begin{aligned} &\int_0^T \int_{-\infty}^{+\infty} \left(V\phi_t + UV\phi_x\right) dx\,dt \\ &\quad + \int_0^T \omega\left(x_\delta(t), t\right) \frac{\partial\phi(x, t)}{\partial l} \sqrt{1 + \sigma_\delta^2}\,dt \\ &\quad + \int_{-\infty}^{+\infty} v_0(x)\,\phi(x, 0)\,dx = 0 \end{aligned} \tag{13}$$

for all the test functions $\phi \in C_0^\infty((-\infty, +\infty) \times [0, T))$. Here σ_δ is the tangential derivative of the curve l, and $\partial\phi(x, t)/\partial l$ stands for the tangential derivative of the function ϕ on the curve l.

Definition 2. For an $n \times n$ matrix $A = (a_{ij})$, define

$$\|A\| = \max_{i=1,\cdots,n} \sum_{i=1}^{n} \left|a_{ij}\right| \tag{14}$$

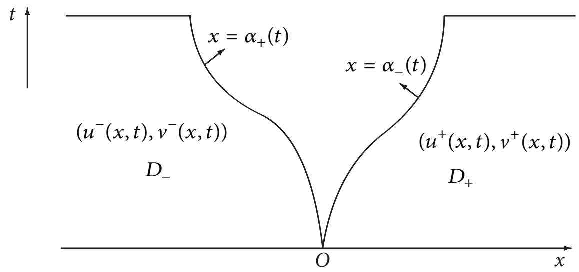


Figure 1: The local time.

and

$$\|A\|_{\min} = \inf\left\{\left\|\gamma A\gamma^{-1}\right\|; \gamma = \operatorname{diag}\{\gamma_i\}, \gamma_i \neq 0, i = 1, \cdots, n\right\}. \tag{15}$$

3. The Perturbed Riemann Problem with Delta Shock

In this section, we construct the perturbed Riemann solutions of hyperbolic system (1) with initial data (3) for local time and investigate the internal mechanism and instability of delta shock. About the perturbed Riemann problem, we have six cases according to the different constructions of the solutions to the corresponding Riemann problem (1) and (5) as follows:

(1) When $\widehat{u}^+ < \widehat{u}^- < 0$, the Riemann solution is $\overleftarrow{S} + J$

(2) When $\widehat{u}^+ \leq 0 \leq \widehat{u}^-$, the Riemann solution is delta shock δS

(3) When $\widehat{u}^- < \widehat{u}^+ \leq 0$, the Riemann solution is $\overleftarrow{R} + J$

(4) When $\widehat{u}^- < 0 < \widehat{u}^+$, the Riemann solution is $\overleftarrow{R}_1 + \overrightarrow{R}_2$

(5) When $\widehat{u}^+ > \widehat{u}^- > 0$, the Riemann solution is $J + \overrightarrow{R}$

(6) When $\widehat{u}^- > \widehat{u}^+ > 0$, the Riemann solution is $J + \overrightarrow{S}$

Here "+" means "followed by"; the capitals S, J, and R denote shock, contact discontinuity, and rarefaction wave, respectively.

Since the perturbed Riemann problem (1) and (3) with no delta shock is classical and well known, it will not be pursued here. In this section, we mainly study the differences between the perturbed Riemann solution and the corresponding Riemann solution. Thus we only consider the perturbed Riemann problem (1) and (3) with delta shock, a.e., $\widehat{u}^+ \leq 0 \leq \widehat{u}^-$.

It is known from classical theory that the classical solution $(u^-(x,t), v^-(x,t))$ and $(u^+(x,t), v^+(x,t))$ can be defined in a strip domains D_- and D_+ for local time, respectively (see Figure 1). Here $(u^-(x,t), v^-(x,t))$ and $(u^+(x,t), v^+(x,t))$ are local smooth solutions to the initial problem (1) with corresponding initial data $(u_0^-(x), v_0^-(x))$ and $(u_0^+(x), v_0^+(x))$ on both sides of $x = 0$, respectively. The right boundary of domain D_- is a I−characteristic : $x = \alpha_-(t)$; namely,

$$\frac{u^-(\alpha_-(t), t)}{v^-(\alpha_-(t), t)} = \frac{\widehat{u}^-}{\widehat{v}^-}, \qquad \frac{d\alpha_-(t)}{dt} = u^-(\alpha_-(t), t). \tag{16}$$

The left boundary of domain D_+ is I−characteristic : $x = \alpha_+(t)$; namely,

$$\frac{u^+(\alpha_+(t), t)}{v^+(\alpha_+(t), t)} = \frac{\widehat{u}^+}{\widehat{v}^+}, \qquad \frac{d\alpha_+(t)}{dt} = u^+(\alpha_+(t), t). \tag{17}$$

Now we turn our attention to the solution of (1) and (3) between the right boundary of domain D_- and the left boundary of domain D_+. We note that the corresponding Riemann solution is a delta shock δS with speed $\widehat{u}^- + \widehat{u}^+$ separating two states $(\widehat{u}^-, \widehat{v}^-)$ and $(\widehat{u}^+, \widehat{v}^+)$ (see Figure 2(a)). If the perturbed Riemann problem (1) and (3) has a solution by using a delta shock δS connecting two states (u^-, v^-) and (u^+, v^+) on local time $[0, T)$, we must choose u and v to be

$$u = u^+ + [u] H(-x + x_\delta(t)), \qquad v = v^+ + [v] H(-x + x_\delta(t)) + \omega(x_\delta(t), t)\,\delta(l), \tag{18}$$

Where, here and below, we use the usual notation $[u] = u^- - u^+$ with u^- and u^+ the values of the function u on the left-hand and right-hand sides of the discontinuity $x = x_\delta(t)(x_\delta(0) = 0)$, etc.; $H(x)$ is the Heaviside function, that is, 0 when $x < 0$ and 1 when $x > 0$; $\omega(x_\delta(t), t)$ and σ_δ are the weight and the tangential derivative of curve $l \triangleq \{(x_\delta(t), t) : 0 \leqslant t < T\}$, which can be defined by

$$0 = -\sigma_\delta [u] + \left[u^2\right], \qquad \frac{d\sqrt{1+\sigma_\delta^2}\,\omega(x_\delta(t), t)}{dt} = -\sigma_\delta [v] + [uv], \qquad \omega(0,0) = 0. \tag{19}$$

From (19), one can get that the propagating speed of the delta shock

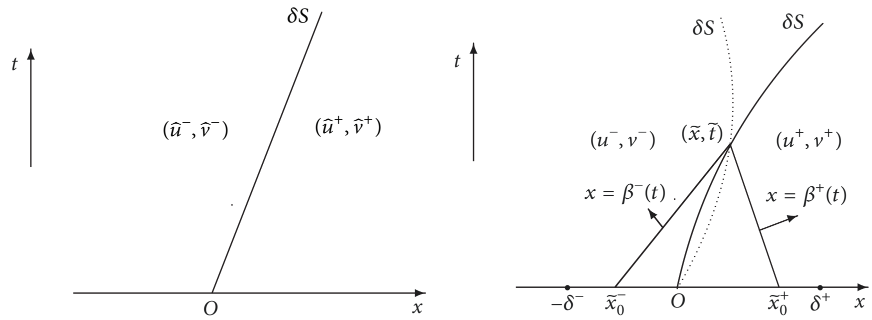
$$\sigma_\delta = \frac{dx_\delta(t)}{dt} = u^-(x_\delta(t), t) + u^+(x_\delta(t), t). \tag{20}$$

We will prove that the delta shock solution constructed in (18) is a solution of the initial value problem (1) and (3) in the sense of distributions on $[0, T)$.

Proposition 3. *The delta shock solution constructed in (18) satisfies (1) and (3) in the sense of distributions on a domain*

$$R(\epsilon) = \{(x,t) \mid -\infty < x < \infty, 0 \leq t < T\}, \tag{21}$$

where $T > 0$ is a finite time.



(a) The Riemann solution

(b) The perturbed Riemann solution

FIGURE 2: Case 1, Case 2 with $\dot{u}_0^-(0) < 0$, and Case 3 with $\dot{u}_0^+(0) < 0$.

Proof. Let

$$\begin{aligned} U(x,t) &= u^+ + [u] H(-x + x_\delta(t)), \\ V(x,t) &= v^+ + [v] H(-x + x_\delta(t)). \end{aligned} \tag{22}$$

Then the delta shock solution (18) can be reduced to

$$\begin{aligned} u &= U(x,t), \\ v &= V(x,t) + \omega(x_\delta(t), t)\, \delta(l). \end{aligned} \tag{23}$$

We need to check that (u, v) satisfies (1), which is (12) and (13).

For any test function $\phi \in C_0^\infty((-\infty, +\infty) \times [0, T))$, we plug (23) into the left-hand side of equation (12) and get

$$\begin{aligned} &\int_0^T \int_{-\infty}^{+\infty} \left(U\phi_t + U^2\phi_x\right) dx\,dt + \int_{-\infty}^{+\infty} u_0(x)\,\phi(x,0)\,dx \\ &= \int_0^T \int_{-\infty}^{x_\delta(t)} \left(u^-\phi_t + (u^-)^2\phi_x\right) dx\,dt \\ &+ \int_0^T \int_{x_\delta(t)}^{+\infty} \left(u^+\phi_t + (u^+)^2\phi_x\right) dx\,dt \\ &+ \int_{-\infty}^{+\infty} u_0(x)\,\phi(x,0)\,dx. \end{aligned} \tag{24}$$

Now using the fact that (u^-, v^-) and (u^+, v^+) are C^1 solutions to problem (1) with initial data $(u_0^-(x), v_0^-(x))$ and $(u_0^+(x), v_0^+(x))$ in the domains D_- and D_+, respectively, the divergence theorem gives

$$\begin{aligned} &\int_0^T \int_{-\infty}^{+\infty} \left(U\phi_t + U^2\phi_x\right) dx\,dt + \int_{-\infty}^{+\infty} u_0(x)\,\phi(x,0)\,dx \\ &= \int_0^T \int_{-\infty}^{x_\delta(t)} \left(\left(u^-\phi\right)_t + \left(\left(u^-\right)^2\phi\right)_x\right) dx\,dt \\ &+ \int_0^T \int_{x_\delta(t)}^{+\infty} \left(\left(u^+\phi\right)_t + \left(\left(u^+\right)^2\phi\right)_x\right) dx\,dt \\ &+ \int_{-\infty}^{+\infty} u_0(x)\,\phi(x,0)\,dx = -\int_{-\infty}^{0} u_0^-(x) \\ &\cdot \phi(x,0)\,dx + \int_0^T \phi(x_\delta(t), t) \\ &\cdot \left(-u^-(x_\delta(t), t)\,\sigma_\delta(x_\delta(t), t)\right. \\ &\left.+ \left(u^-(x_\delta(t), t)\right)^2\right) dt - \int_0^{+\infty} u_0^+(x) \\ &\cdot \phi(x,0)\,dx - \int_0^T \phi(x_\delta(t), t) \\ &\cdot \left(-u^+(x_\delta(t), t)\,\sigma_\delta(x_\delta(t), t)\right. \\ &\left.+ \left(u^+(x_\delta(t), t)\right)^2\right) dt + \int_{-\infty}^{+\infty} u_0(x) \\ &\cdot \phi(x,0)\,dx = -\int_{-\infty}^{+\infty} u_0(x)\,\phi(x,0)\,dx \\ &+ \int_0^T \phi(x_\delta(t), t)\left(\left[u^2\right] - [u]\,\sigma_\delta\right) dt \\ &+ \int_{-\infty}^{+\infty} u_0(x)\,\phi(x,0)\,dx = 0. \end{aligned} \tag{25}$$

Similarly, we plug (23) into the left-hand side of (13) and get

$$\begin{aligned} &\int_0^T \int_{-\infty}^{+\infty} \left(V\phi_t + UV\phi_x\right) dx\,dt + \int_0^T \omega(x_\delta(t), \\ &t)\,\frac{\partial\phi(x,t)}{\partial l}\sqrt{1 + \sigma_\delta^2}\,dt \\ &+ \int_{-\infty}^{+\infty} v_0(x)\,\phi(x,0)\,dx \\ &= \int_0^T \int_{-\infty}^{x_\delta(t)} \left(v^-\phi_t + u^-v^-\phi_x\right) dx\,dt \\ &+ \int_0^T \int_{x_\delta(t)}^{+\infty} \left(v^+\phi_t + u^+v^+\phi_x\right) dx\,dt + \int_0^T \omega(x_\delta(t), \end{aligned}$$

$$t)\frac{\partial\phi(x,t)}{\partial l}\sqrt{1+\sigma_\delta^2}dt$$

$$+\int_{-\infty}^{+\infty}v_0(x)\phi(x,0)\,dx$$

$$=\int_0^T\int_{-\infty}^{x_\delta(t)}\left((v^-\phi)_t+(u^-v^-\phi)_x\right)dxdt$$

$$+\int_0^T\int_{x_\delta(t)}^{+\infty}\left((v^+\phi)_t+(u^+v^+\phi)_x\right)dxdt$$

$$+\int_0^T\omega(x_\delta(t),$$

$$t)\frac{\partial\phi(x,t)}{\partial l}\sqrt{1+\sigma_\delta^2}dt$$

$$+\int_{-\infty}^{+\infty}v_0(x)\phi(x,0)\,dx=-\int_{-\infty}^{0}v_0^-(x)\phi(x,$$

$$0)\,dx+\int_0^T\phi(x_\delta(t),t)\left(-v^-(x_\delta(t),t)\right.$$

$$\left.\cdot\sigma_\delta(x_\delta(t),t)+u^-(x_\delta(t),t)\,v^-(x_\delta(t),t)\right)dt$$

$$-\int_0^{+\infty}v_0^+(x)\phi(x,0)\,dx-\int_0^T\phi(x_\delta(t),t)$$

$$\cdot\left(-v^+(x_\delta(t),t)\,\sigma_\delta(x_\delta(t),t)+u^+(x_\delta(t),t)\right.$$

$$\left.\cdot v^+(x_\delta(t),t)\right)dt+\int_0^T\omega(x_\delta(t),$$

$$t)\frac{\partial\phi(x,t)}{\partial l}\sqrt{1+\sigma_\delta^2}dt$$

$$+\int_{-\infty}^{+\infty}v_0(x)\phi(x,0)\,dx=-\int_{-\infty}^{+\infty}v_0(x)\phi(x,$$

$$0)\,dx+\int_0^T\phi(x_\delta(t),t)\left([uv]-[v]\,\sigma_\delta\right)dt$$

$$+\int_0^T\omega(x_\delta(t),$$

$$t)\frac{\partial\phi(x,t)}{\partial l}\sqrt{1+\sigma_\delta^2}dt$$

$$+\int_{-\infty}^{+\infty}v_0(x)\phi(x,0)\,dx=\int_0^T\phi(x_\delta(t),t)\left([uv]\right.$$

$$\left.-[v]\,\sigma_\delta\right)dt+\int_0^T\omega(x_\delta(t),t)$$

$$\cdot\sqrt{1+\sigma_\delta^2}d\phi(x_\delta(t),t)=\int_0^T\phi(x_\delta(t),t)\left([uv]-[v]\right.$$

$$\left.\cdot\sigma_\delta\right)dt+\omega(x_\delta(t),t)\sqrt{1+\sigma_\delta^2}\phi(x_\delta(t),t)\Big|_{t=0}^{t=T}$$

$$-\int_0^T\phi(x_\delta(t),$$

$$t)\,d\left(\omega(x_\delta(t),t)\sqrt{1+\sigma_\delta^2}\right)$$

$$=0, \tag{26}$$

which gets equality (13). Then we complete the proof of the proposition. □

Furthermore, to guarantee uniqueness of the solution, the delta shock solution constructed in (18) should satisfy the entropy condition (27) on the discontinuity $x=x_\delta(t)$.

Definition 4. The delta shock solution constructed in (18) is an admissible solution of the initial value problem (1) and (3) in the sense of distributions on $[0,T)$, if (u,v) satisfies Definition 1 and the entropy condition

$$\lambda_2(u^+,v^+)\le\lambda_1(u^+,v^+)\le\frac{dx_\delta(t)}{dt}\le\lambda_1(u^-,v^-)\le\lambda_2(u^-,v^-) \tag{27}$$

on the discontinuity $x=x_\delta(t)$.

We now turn to check the entropy condition (27) on $x=x_\delta(t)$. The discussion is divided into three cases: $\widehat{u}^+<0<\widehat{u}^-$, $\widehat{u}^+<0=\widehat{u}^-$, and $\widehat{u}^+=0<\widehat{u}^-$. We start with the following case.

Case 1 ($\widehat{u}^+<0<\widehat{u}^-$). Due to $\widehat{u}^+<0<\widehat{u}^-$, there exist constants $\delta^->0$ and $\delta^+>0$ so small that, for any $x_0^-\in(-\delta^-,0)$ and $x_0^+\in(0,\delta^+)$, the C^1 functions $u_0^-(x)$ and $u_0^+(x)$ satisfy the following condition:

$$u_0^+(x_0^+)<0<u_0^-(x_0^-). \tag{28}$$

Let $x=\beta^-(t)$ (resp., $x=\beta^+(t)$) be the downwards left (resp., right) II−characteristic from any point $(\widetilde{x},\widetilde{t})$ on the delta shock curve $x=x_\delta(t)$ (see Figure 2(b)). When the time $\widetilde{t}$ is small enough, the II−characteristic $x=\beta^-(t)$ (resp., $x=\beta^+(t)$) starting at $(\widetilde{x},\widetilde{t})$ will intersect at a point $(\widetilde{x}_0^-,0)$ (resp., $(\widetilde{x}_0^+,0)$) on the initial axis $t=0$ with $0>\widetilde{x}_0^->-\delta^-$ (resp., $0<\widetilde{x}_0^+<\delta^+$). Since the Riemann invariant $\zeta(u,v)=u$ must be a constant along II−characteristic $x=\beta^-(t)$ and $x=\beta^+(t)$, we have

$$\begin{aligned}\zeta\left(u^-(\beta^-(t),t),v^-(\beta^-(t),t)\right)&=u^-(\beta^-(t),t)\\&=u^-(\widetilde{x},\widetilde{t})=\zeta\left(u_0^-(\widetilde{x}_0^-),v_0^-(\widetilde{x}_0^-)\right)=u_0^-(\widetilde{x}_0^-),\end{aligned} \tag{29}$$

and

$$\begin{aligned}\zeta\left(u^+(\beta^+(t),t),v^+(\beta^+(t),t)\right)&=u^+(\beta^+(t),t)\\&=u^+(\widetilde{x},\widetilde{t})=\zeta\left(u_0^+(\widetilde{x}_0^+),v_0^+(\widetilde{x}_0^+)\right)=u_0^+(\widetilde{x}_0^+).\end{aligned} \tag{30}$$

Using Rankine-Hugoniot condition and the above two expressions, we obtain the propagation speed of the delta shock $x=x_\delta(t)$ at the point $(\widetilde{x},\widetilde{t})$

$$\begin{aligned}\sigma_\delta(\widetilde{x},\widetilde{t})=\left.\frac{dx_\delta(t)}{dt}\right|_{t=\widetilde{t}}&=u^-(\widetilde{x},\widetilde{t})+u^+(\widetilde{x},\widetilde{t})\\&=u_0^-(\widetilde{x}_0^-)+u_0^+(\widetilde{x}_0^+).\end{aligned} \tag{31}$$

Together with (28) and (31), we get the entropy condition

$$\begin{aligned}\lambda_2(u^+(\tilde{x},\tilde{t}),v^+(\tilde{x},\tilde{t})) &= 2u^+(\tilde{x},\tilde{t}) = 2u_0^+(\tilde{x}_0^+)\\ &< \lambda_1(u^+(\tilde{x},\tilde{t}),v^+(\tilde{x},\tilde{t})) = u_0^+(\tilde{x}_0^+) < \sigma_\delta(\tilde{x},\tilde{t})\\ &= u_0^-(\tilde{x}_0^-) + u_0^+(\tilde{x}_0^+) < \lambda_1(u^-(\tilde{x},\tilde{t}),v^-(\tilde{x},\tilde{t}))\\ &= u_0^-(\tilde{x}_0^-) = u^-(\tilde{x},\tilde{t}) < \lambda_2(u^-(\tilde{x},\tilde{t}),v^-(\tilde{x},\tilde{t}))\\ &= 2u_0^-(\tilde{x}_0^-) = 2u^-(\tilde{x},\tilde{t}),\end{aligned} \tag{32}$$

which is valid at any point $(\tilde{x},\tilde{t})$ on the curve $x = x_\delta(t)$ locally in time. Then we have the following.

Theorem 5. *In case of $\hat{u}^+ < 0 < \hat{u}^-$, the perturbed Riemann problem (1) and (3) has a delta shock solution constructed in (18) for local time. The admissible solution has a structure similar to that of the corresponding Riemann problem (1) and (5) (see Figure 2). Furthermore, the delta shock curve $x = x_\delta(t)$ possesses the following property:*

(a) *If $\dot{u}_0^-(0) < \dot{u}_0^+(0)$, then the curve is convex*

(b) *On the other hand, if $\dot{u}_0^-(0) > \dot{u}_0^+(0)$, then the curve is concave*

Proof. By Proposition 3 and inequality (32), we know that the delta shock solution constructed in (18) satisfies (1) and (3) in the sense of distributions and the entropy condition (27) for local time, respectively. Obviously, the delta shock curve $x = x_\delta(t)$ retains its form in a neighborhood of the origin $(0,0)$. Namely, the solution of the perturbed Riemann problem (1) and (3) has a structure similar to the corresponding Riemann solution of (1) and (5) for local time.

In order to show the behavior of the delta shock curve $x = x_\delta(t)$ near the origin, we need a certain a priori estimate on the value of $\ddot{x}_\delta(0)$. From (20), it can be easily checked that

$$\frac{dx_\delta(\tilde{t})}{d\tilde{t}} = u^-(x_\delta(\tilde{t}),\tilde{t}) + u^+(x_\delta(\tilde{t}),\tilde{t}). \tag{33}$$

Differentiating the above equality with respect to $\tilde{t}$ and letting $\tilde{t} = 0$, one obtains

$$\ddot{x}_\delta(0) = \left.\frac{du^-(x_\delta(\tilde{t}),\tilde{t})}{d\tilde{t}}\right|_{\tilde{t}=0} + \left.\frac{du^+(x_\delta(\tilde{t}),\tilde{t})}{d\tilde{t}}\right|_{\tilde{t}=0}. \tag{34}$$

Next, we compute the values of $(du^-(x_\delta(\tilde{t}),\tilde{t})/d\tilde{t})|_{\tilde{t}=0}$ and $(du^+(x_\delta(\tilde{t}),\tilde{t})/d\tilde{t})|_{\tilde{t}=0}$, respectively. From (1), it can be checked that

$$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = -\frac{\partial u^2}{\partial x} + \frac{\partial u}{\partial x}\frac{dx}{dt}\\ &= -2u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\frac{dx}{dt} = \left(-2u + \frac{dx}{dt}\right)\frac{\partial u}{\partial x}.\end{aligned} \tag{35}$$

Along the delta shock curve $x = x_\delta(t)$, by using the fact that $\dot{x}_\delta(0) = \hat{u}^- + \hat{u}^+$ and condition (35), we have

$$\begin{aligned}\left.\frac{du^-(x_\delta(\tilde{t}),\tilde{t})}{d\tilde{t}}\right|_{\tilde{t}=0} &= (-2\hat{u}^- + \dot{x}_\delta(0))\cdot\dot{u}_0^-(0)\\ &= (\hat{u}^+ - \hat{u}^-)\cdot\dot{u}_0^-(0)\end{aligned} \tag{36}$$

and

$$\begin{aligned}\left.\frac{du^+(x_\delta(\tilde{t}),\tilde{t})}{d\tilde{t}}\right|_{\tilde{t}=0} &= (-2\hat{u}^+ + \dot{x}_\delta(0))\cdot\dot{u}_0^+(0)\\ &= (\hat{u}^- - \hat{u}^+)\cdot\dot{u}_0^+(0).\end{aligned} \tag{37}$$

Substituting (36) and (37) into (34), we transform (34) into the form

$$\begin{aligned}\ddot{x}_\delta(0) &= (\hat{u}^+ - \hat{u}^-)\cdot\dot{u}_0^-(0) + (\hat{u}^- - \hat{u}^+)\cdot\dot{u}_0^+(0)\\ &= (\hat{u}^+ - \hat{u}^-)\cdot(\dot{u}_0^-(0) - \dot{u}_0^+(0)).\end{aligned} \tag{38}$$

With $\hat{u}^+ - \hat{u}^- < 0$ in mind, (38) shows the second derivative of the delta shock curve at origin $\ddot{x}(0) > 0$ when $\dot{u}_0^-(0) < \dot{u}_0^+(0)$; otherwise, $\ddot{x}(0) < 0$ when $\dot{u}_0^-(0) > \dot{u}_0^+(0)$. Thus the proof of Theorem 5 is completed. □

Case 2 ($\hat{u}^+ < 0 = \hat{u}^-$). When $\dot{u}_0^-(0) < 0$, due to $\hat{u}^+ < 0 = \hat{u}^-$, there exist constants $\delta^- > 0$ and $\delta^+ > 0$ so small that, for any $x_0^- \in (-\delta^-, 0)$ and $x_0^+ \in (0,\delta^+)$, the C^1 functions $u_0^-(x)$ and $u_0^+(x)$ also satisfy the inequality (28). Then a discussion similar to that for Case 1 shows that the delta shock δS solution defined by (18) also satisfies Definition 1 and the entropy condition (27). The delta shock δS can retain its form in a neighborhood of the origin. That is, the perturbed Riemann solution of (1) and (3) has a structure similar to the Riemann solution of (1) and (5) for this case (see Figure 2).

When $\dot{u}_0^-(0) > 0$, we can prove (29)~(31) correspondingly for this subcase. In the same way as Case 1, by virtue of $\hat{u}^- = 0$, there exist constants $\delta^- > 0$ and $\delta^+ > 0$ so small that, for any $x_0^- \in (-\delta^-, 0)$ and $x_0^+ \in (0,\delta^+)$, we have

$$u_0^+(x_0^+) < u_0^-(x_0^-) < 0. \tag{39}$$

Then, from (39), we get

$$\begin{aligned}\lambda_1(u^+(\tilde{x},\tilde{t}),v^+(\tilde{x},\tilde{t})) &= u_0^+(\tilde{x}_0^+) > \sigma_\delta(\tilde{x},\tilde{t})\\ &= u_0^-(\tilde{x}_0^-) + u_0^+(\tilde{x}_0^+)\end{aligned} \tag{40}$$

on the delta shock curve $x = x_\delta(t)$. Inequality (40) shows that the entropy condition (27) does not hold on the curve $x = x_\delta(t)$. Therefore, the perturbed Riemann solution to (1) and (3) should not be a delta shock solution for this subcase.

Now we prove that the perturbed Riemann solution should be a backward shock $\overleftarrow{S}$ followed by a contact discontinuity J for this subcase. The structure of the perturbed solution can be indicated in Figure 3(b), in which $\overleftarrow{S} : x = x_s(t)$ and $J : x = x_c(t)$ are free boundaries. Furthermore, on OA_1, we have

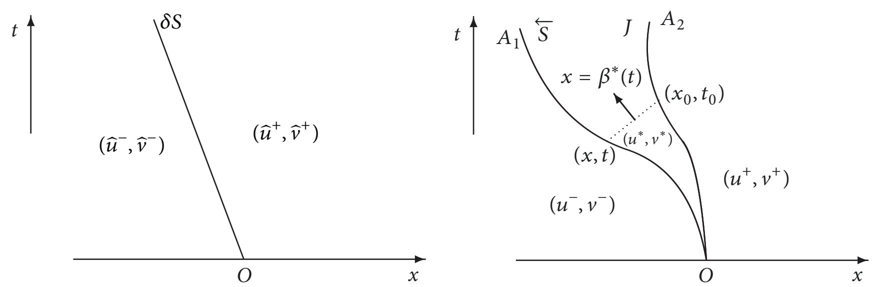
$$\frac{dx_s(t)}{dt} = u^- + u^*, \tag{41}$$

$$u^-v^* = u^*v^-. \tag{42}$$

On OA_2, we have

$$\frac{dx_c(t)}{dt} = u^*, \tag{43}$$

$$u^* = u^+. \tag{44}$$



(a) The Riemann solution (b) The perturbed Riemann solution

FIGURE 3: Case 2 with $\dot u_0^-(0) > 0$.

On the domain $\{(x,t) \mid x < x_s(t), 0 \le t < \epsilon\}$ ($\epsilon > 0$ so small), the perturbed Riemann solution to (1) and (3) is $(u^-(x,t), v^-(x,t))$. On the domain $\{(x,t) \mid x > x_c(t), 0 \le t < \epsilon\}$, the perturbed Riemann solution is $(u^+(x,t), v^+(x,t))$. On the domain $\{(x,t) \mid x_s(t) < x < x_c(t), 0 \le t < \epsilon\}$, the perturbed Riemann solution is denoted by $(u^*(x,t), v^*(x,t))$, which is an unknown regular solution to problem (1) and (3); moreover,

$$\lim_{(x,t)\longrightarrow(0,0)} u^*(x,t) = \hat u^* = \hat u^+,$$
$$\lim_{(x,t)\longrightarrow(0,0)} v^*(x,t) = \hat v^* = +\infty. \quad (45)$$

Noticing the corresponding results in [15, 16], the above perturbed Riemann problem is equivalent to the free boundary problem (1) with boundary conditions (41)~(44) on the fan-shaped domain $\{(x,t) \mid x_s(t) < x < x_c(t), 0 \le t < \epsilon\}$ ($\epsilon > 0$ so small).

Set

$$\begin{pmatrix} \mu^*(x,t) \\ \psi^*(x,t) \end{pmatrix} = \begin{pmatrix} l_2(\hat u^*, \hat v^*) \\ l_1(\hat u^*, \hat v^*) \end{pmatrix} \begin{pmatrix} u^*(x,t) \\ v^*(x,t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \hat v^* & -\hat u^* \end{pmatrix} \begin{pmatrix} u^*(x,t) \\ v^*(x,t) \end{pmatrix}. \quad (46)$$

Boundary condition (42) on $x = x_s(t)$ then reduces to

$$\psi^* = \frac{(u^-\hat v^* - v^-\hat u^*)\cdot \mu^*}{u^-}. \quad (47)$$

Boundary condition (44) on $x = x_c(t)$ can be written as

$$\mu^* = u^+. \quad (48)$$

Thus, the characterizing matrix H of this problem is of the form [15]

$$H = \begin{pmatrix} 0 & 0 \\ \dfrac{u^-\hat v^* - v^-\hat u^*}{u^-} & 0 \end{pmatrix}. \quad (49)$$

According to the local existence and uniqueness theorem (c.f. Chapter 6 in Li Ta-tsien and Yu Wen-ci [15]), if the minimal characterizing number

$$\|H\|_{\min} < 1, \quad (50)$$

then there exists a unique C^1 solution on the fan-shaped domain $\{(x,t) \mid x_s(t) < x < x_c(t), 0 \le t < \epsilon\}$, where $\epsilon > 0$ is suitably small.

By Remark 4.4 in the introduction of [15], it is not hard to prove that the minimal characterizing number of this problem for this case is

$$\|H\|_{\min} = 0 < 1. \quad (51)$$

Then the free boundary problem under consideration admits a unique piecewise smooth solution on the fan-shaped domain $\{(x,t) \mid x_s(t) < x < x_c(t), 0 \le t < \epsilon\}$. Hence we have the following.

Theorem 6. *In case of $\hat u^+ < 0 = \hat u^-$ and $\dot u_0^-(0) > 0$, the solution to the perturbed Riemann problem (1) and (3) is composed of a backward shock $x = x_s(t)$ followed by a contact discontinuity $x = x_c(t)$ for local time. The perturbed solution is dramatically different from the corresponding Riemann solution of (1) and (5), which is a delta shock (see Figure 3).*

Next, we proceed to prove that the perturbed Riemann problem (1) and (3) for this subcase admits a solution that contains a backward shock $\overleftarrow{S}$ and a contact discontinuity J near the origin in another way.

Let $x = \beta^*(t)$ be the upwards right $II-$characteristic from any point (x,t) on the shock curve $x = x_s(t)$ (see Figure 3(b)). The point (x_0, t_0) is the intersection point of the $II-$characteristic curve $x = \beta^*(t)$ and the contact discontinuity $x = x_c(t)$. Since the Riemann invariant $\zeta(u,v) = -u + v$ must be a constant along $II-$characteristic, we have

$$\zeta(u^*, v^*) = u^*(x,t) = u^*(\beta^*(t), t) = u^*(x_0, t_0) = u^*(x_s(t), t) = u^*(x_c(t_0), t_0). \quad (52)$$

From (52), it is easy to see that the propagating speed of $II-$characteristic

$$\frac{d\beta^*(t)}{dt} = 2u^*(\beta^*(t), t) = 2u^*(x_0, t_0) \quad (53)$$

is a constant. That is, the characteristic $x = \beta^*(t)$ is a straight line. Using the above equality, we arrive at

$$\frac{x_s(t) - x_c(t_0)}{t - t_0} = 2u^*(x_s(t), t) = 2u^*(x_c(t_0), t_0). \quad (54)$$

Noting (54), we get

$$x_s(t) - x_c(t_0) = 2(t - t_0) \cdot u^*(x_s(t), t). \tag{55}$$

Differentiate the above equation with respect to t and let $t = 0$; then one obtains

$$\dot{x}_s(0) - \dot{x}_c(0) \left.\frac{dt_0}{dt}\right|_{t=0} = 2\left(1 - \left.\frac{dt_0}{dt}\right|_{t=0}\right) \cdot \hat{u}^*. \tag{56}$$

Substituting

$$\dot{x}_s(0) = \dot{x}_c(0) = \hat{u}^+ = \hat{u}^* \tag{57}$$

into (56), with $\hat{u}^* = \hat{u}^+ < 0$, we get

$$\left.\frac{dt_0}{dt}\right|_{t=0} = 1. \tag{58}$$

Moreover, differentiating the last equality in (52) with respect to t and letting $t = 0$, we obtain

$$\left.\frac{du^*(x_s(t), t)}{dt}\right|_{t=0} = \left.\frac{du^*(x_c(t_0), t_0)}{dt_0}\right|_{t_0=0} \cdot \left.\frac{dt_0}{dt}\right|_{t=0}. \tag{59}$$

In view of (58) and (59), it is easy to see that

$$\left.\frac{du^*(x_s(t), t)}{dt}\right|_{t=0} = \left.\frac{du^*(x_c(t_0), t_0)}{dt_0}\right|_{t_0=0}. \tag{60}$$

In the following, firstly, we will accumulate the second derivative of the shock at the origin $\ddot{x}_s(0)$. Along $x = x_s(t)$, differentiating the above equality (41) with respect to t and letting $t = 0$, by (44) and (52), we have

$$\ddot{x}_s(0) = \left.\frac{du^-(x_s(t), t)}{dt}\right|_{t=0} + \left.\frac{du^*(x_s(t), t)}{dt}\right|_{t=0}. \tag{61}$$

On the one hand, from (35), we have

$$\frac{du^-(x_s(t), t)}{dt} = \left(-2u^- + \frac{dx_s(t)}{dt}\right) \cdot \frac{\partial u^-}{\partial x}. \tag{62}$$

Noting (57), (62), and $\hat{u}^- = 0$, one can get that

$$\left.\frac{du^-(x_s(t), t)}{dt}\right|_{t=0} = \hat{u}^+ \cdot \dot{u}_0^-(0). \tag{63}$$

On the other hand, by (60) and (44), we have

$$\begin{aligned} \left.\frac{du^*(x_s(t), t)}{dt}\right|_{t=0} &= \left.\frac{du^*(x_c(t_0), t_0)}{dt_0}\right|_{t_0=0} \\ &= \left.\frac{du^+(x_c(t_0), t_0)}{dt_0}\right|_{t_0=0}. \end{aligned} \tag{64}$$

Along the contact discontinuity wave curve $x = x_c(t)$, from (35), (57), and (64), it follows that

$$\begin{aligned} \left.\frac{du^*(x_s(t), t)}{dt}\right|_{t=0} &= \left.\frac{du^*(x_c(t_0), t_0)}{dt_0}\right|_{t_0=0} \\ &= -\hat{u}^+ \cdot \dot{u}_0^+(0). \end{aligned} \tag{65}$$

Then, substituting (63) and (65) into (61), we get

$$\ddot{x}_s(0) = \hat{u}^+ \cdot \left(\left(\dot{u}_0^-(0) - \dot{u}_0^+(0)\right)\right). \tag{66}$$

Secondly, we now have an estimate of the second derivative of the contact discontinuity wave curve at the origin, a.e. $\ddot{x}_c(0)$. Along $x = x_c(t)$, with (43), it holds that

$$\frac{dx_c(t_0)}{dt_0} = u^*(x_c(t_0), t_0). \tag{67}$$

Differentiating (67) with respect to t_0 and letting $t_0 = 0$ give

$$\ddot{x}_c(0) = \left.\frac{du^*(x_c(t_0), t_0)}{dt_0}\right|_{t_0=0} = -\hat{u}^+ \cdot \dot{u}_0^+(0). \tag{68}$$

Thirdly, combining (66) and (68), it follows that

$$\ddot{x}_s(0) - \ddot{x}_c(0) = \hat{u}^+ \cdot \dot{u}_0^-(0) < 0. \tag{69}$$

Finally, due to $x_s(0) = x_c(0) = 0$, (57), and (69), the perturbed Riemann solution to (1) and (3) in this subcase clearly consists of a backward shock $\overleftarrow{S}$ connecting (u^-, v^-) to (u^*, v^*), followed by a contact discontinuity J connecting (u^*, v^*) to (u^+, v^+) near the origin (see Figure 3(b)). Thus we have completed the construction and proof of the perturbed Riemann solution for this subcase.

Case 3 ($\hat{u}^+ = 0 < \hat{u}^-$). When $\dot{u}_0^+(0) < 0$, there exist constants $\delta^- > 0$ and $\delta^+ > 0$ so small that, for any $x_0^- \in (-\delta^-, 0)$ and $x_0^+ \in (0, \delta^+)$, the C^1 functions $u_0^-(x)$ and $u_0^+(x)$ also satisfy inequality (28). By a similar argument used in Case 1, we get that the delta shock solution defined by (18) satisfies the entropy condition (27). That is, in a neighborhood of the origin, the perturbed Riemann solution of (1) and (3) has a structure similar to the Riemann solution to (1) and (5) for this subcase (see Figure 2).

When $\dot{u}_0^+(0) > 0$, by virtue of $\hat{u}^+ \equiv 0$, there exist constants $\delta^- > 0$ and $\delta^+ > 0$ so small that, for any $x_0^- \in (-\delta^-, 0)$ and $x_0^+ \in (0, \delta^+)$, we have

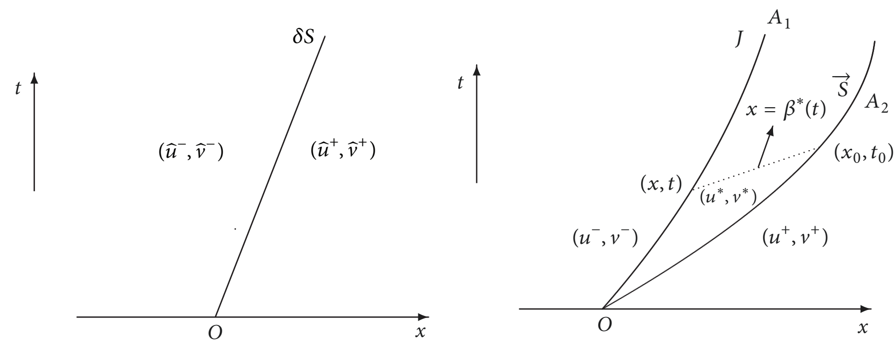
$$0 < u_0^+(x_0^+) < u_0^-(x_0^-). \tag{70}$$

Then in the same way as Case 1, we can prove (29)~(31) corresponding for this subcase. Finally, from (70), we get

$$\begin{aligned} u_0^-(\tilde{x}_0^-) + u_0^+(\tilde{x}_0^+) &= \sigma_\delta(\tilde{x}, \tilde{t}) \\ &> \lambda_1(u^-(\tilde{x}, \tilde{t}), v^-(\tilde{x}, \tilde{t})) \\ &= u_0^-(\tilde{x}_0^-) \end{aligned} \tag{71}$$

on the delta shock curve $x = x_\delta(t)$. Equality (53) shows that the entropy condition (27) does not hold for $x = x_\delta(t)$. Therefore, we should not expect the perturbed Riemann solution to (1) and (3) to be a delta shock solution for this subcase.

We will prove that the perturbed Riemann solution is a contact discontinuity J followed by a forward shock $\overrightarrow{S}$



(a) The Riemann solution (b) The perturbed Riemann solution

FIGURE 4: Case 3 with $\dot{u}_0^+(0) > 0$.

locally in time for this subcase. The structure of the perturbed solution to (1) and (5) can be indicated in Figure 4(b). As shown in Figure 4(b),

$$OA_1 : x = x_c(t) \quad (x_c(0) = 0) \tag{72}$$

and

$$OA_2 : x = x_s(t) \quad (x_s(0) = 0) \tag{73}$$

are free boundaries. Furthermore, OA_1 is a contact discontinuity, on which we have

$$\frac{dx_c(t)}{dt} = u^-, \tag{74}$$

$$u^- = u^*. \tag{75}$$

OA_2 is a forward shock, on which we have

$$\frac{dx_s(t)}{dt} = u^* + u^+, \tag{76}$$

$$u^* v^+ = v^* u^+. \tag{77}$$

On the domain $\{(x,t) \mid x < x_c(t), 0 \le t < \epsilon\}$ ($\epsilon > 0$ so small), the perturbed Riemann solution to (1) and (3) is $(u^-(x,t), v^-(x,t))$. On the domain $\{(x,t) \mid x > x_s(t), 0 \le t < \epsilon\}$, the perturbed Riemann solution is $(u^+(x,t), v^+(x,t))$. On the domain $\{(x,t) \mid x_c(t) < x < x_s(t), 0 \le t < \epsilon\}$, the perturbed Riemann solution is denoted by $(u^*(x,t), v^*(x,t))$, which is an unknown regular solution to problem (1) with (3) and satisfies condition (45).

Since $(u^-(x,t), v^-(x,t))$ and $(u^+(x,t), v^+(x,t))$ are known, in order to get the perturbed Riemann solution, we have to solve the free boundary problem (1) and (74)~(77) on the fan-shaped domain $\{(x,t) \mid x_c(t) < x < x_s(t), 0 \le t < \epsilon\}$ ($\epsilon > 0$ so small).

We introduce the change of variables

$$\begin{aligned}\begin{pmatrix}\mu^*(x,t)\\ \psi^*(x,t)\end{pmatrix} &= \begin{pmatrix} l_1(\hat{u}^*, \hat{v}^*)\\ l_2(\hat{u}^*, \hat{v}^*)\end{pmatrix}\begin{pmatrix}u^*(x,t)\\ v^*(x,t)\end{pmatrix}\\ &= \begin{pmatrix}\hat{v}^* & -\hat{u}^*\\ 1 & 0\end{pmatrix}\begin{pmatrix}u^*(x,t)\\ v^*(x,t)\end{pmatrix}.\end{aligned} \tag{78}$$

Boundary condition (75) on $x = x_c(t)$ then reduces to

$$\psi^* = u^-. \tag{79}$$

Boundary condition (77) on $x = x_s(t)$ can be written as

$$\mu^* = \frac{(u^+\hat{v}^* - v^+\hat{u}^*) \cdot \psi^*}{u^+}. \tag{80}$$

Hence, the characterizing matrix H of this problem is of the form [15]

$$H = \begin{pmatrix}\dfrac{u^+\hat{v}^* - v^+\hat{u}^*}{u^+} & 0\\ 0 & 0\end{pmatrix}. \tag{81}$$

According to the local existence and uniqueness theorem, what remains is to prove that $\|H\|_{\min} < 1$.

By Remark 4.4 in the introduction of [15], it is not hard to prove that for this subcase

$$\|H\|_{\min} = 0 < 1. \tag{82}$$

Then the free boundary problem under consideration admits a unique piecewise C^1 solution on the fan-shaped domain $\{(x,t) \mid x_c(t) < x < x_s(t), 0 \le t < \epsilon\}$ ($\epsilon > 0$ so small). We can conclude the following.

Theorem 7. *In case of $\hat{u}^+ = 0 < \hat{u}^-$ and $\dot{u}_0^+(0) > 0$, the solution of the perturbed Riemann problem (1) and (3) is composed of a contact discontinuity $x = x_c(t)$ followed by a forward shock $x = x_s(t)$ for local time. The perturbed solution is different from the corresponding Riemann solution of (1) and (5), which is a delta shock (see Figure 4).*

Next, we proceed to prove that the perturbed Riemann problem (1) and (3) for this subcase admits a solution which contains a contact discontinuity $x = x_c(t)$ followed by a forward shock $x = x_s(t)$ near the origin. Let $x = \beta^*(t)$ be the upwards right II−characteristic from any point (x,t) on the contact discontinuity curve $x = x_c(t)$ (see Figure 4). The point (x_0, t_0) is the intersection point of the II−characteristic curve $x = \beta^*(t)$ and the shock curve $x = x_s(t)$. Since the

Riemann invariant $\zeta(u,v) = u$ must be a constant along $II-$characteristic, we have

$$\begin{aligned}\zeta(u^*,v^*) &= u^*(x,t) = u^*(\beta^*(t),t) = u^*(x_0,t_0)\\ &= u^*(x_c(t),t) = u^*(x_s(t_0),t_0).\end{aligned} \tag{83}$$

From (83), it is easy to see that the propagating speed of the $II-$characteristic

$$\frac{d\beta^*(t)}{dt} = 2u^*(\beta^*(t),t) = 2u^*(x_0,t_0) \tag{84}$$

is a constant. That is, the characteristic $x = \beta^*(t)$ is a straight line. Using the above equality, we arrive at

$$\frac{x_c(t) - x_s(t_0)}{t - t_0} = 2u^*(x_c(t),t) = 2u^*(x_s(t_0),t_0). \tag{85}$$

Noting (85), we get

$$x_c(t) - x_s(t_0) = (t - t_0)\cdot 2u^*(x_c(t),t). \tag{86}$$

Differentiate the above equation with respect to t and let $t = 0$; then one obtains

$$\dot{x}_c(0) - \dot{x}_s(0)\left.\frac{dt_0}{dt}\right|_{t=0} = \left(1 - \left.\frac{dt_0}{dt}\right|_{t=0}\right)\cdot 2\widehat{u}^-. \tag{87}$$

Substituting

$$\dot{x}_s(0) = \dot{x}_c(0) = \widehat{u}^- \tag{88}$$

into (87), with $\widehat{u}^- > 0$, we get

$$\left.\frac{dt_0}{dt}\right|_{t=0} = 1. \tag{89}$$

Moreover, differentiating the last equality in (83) with respect to t and letting $t = 0$, we obtain

$$\left.\frac{du^*(x_c(t),t)}{dt}\right|_{t=0} = \left.\frac{du^*(x_s(t_0),t_0)}{dt_0}\right|_{t_0=0}\cdot\left.\frac{dt_0}{dt}\right|_{t=0}. \tag{90}$$

From (89) and (90), we derive

$$\left.\frac{du^*(x_c(t),t)}{dt}\right|_{t=0} = \left.\frac{du^*(x_s(t_0),t_0)}{dt_0}\right|_{t_0=0}. \tag{91}$$

In the following, firstly, we will accumulate the second derivative of the contact discontinuity curve at the origin $\ddot{x}_c(0)$. Along $x = x_c(t)$, differentiating (74) with respect to t and letting $t = 0$, it yields

$$\ddot{x}_c(0) = \left.\frac{du^-(x_c(t),t)}{dt}\right|_{t=0}. \tag{92}$$

From (35), we have

$$\frac{du^-(x_c(t),t)}{dt} = \left(-2u^-(x_c(t),t) + \frac{dx_c(t)}{dt}\right)\cdot\frac{\partial u^-}{\partial x}. \tag{93}$$

Noting (88) and (93), one can get that

$$\ddot{x}_c(0) = \left.\frac{du^-(x_c(t),t)}{dt}\right|_{t=0} = -\widehat{u}^-\cdot\dot{u}_0^-(0). \tag{94}$$

Secondly, we now estimate the second derivative of the shock curve at the origin $\ddot{x}_s(0)$. Along $x = x_s(t)$, with (76), it holds that

$$\frac{dx_s(t_0)}{dt_0} = u^*(x_s(t_0),t_0) + u^+(x_s(t_0),t_0). \tag{95}$$

Differentiating (95) with respect to t_0 and letting $t_0 = 0$ give

$$\begin{aligned}\ddot{x}_s(0) = {}& \left.\frac{du^*(x_s(t_0),t_0)}{dt_0}\right|_{t_0=0}\\ &+ \left.\frac{du^+(x_s(t_0),t_0)}{dt_0}\right|_{t_0=0}.\end{aligned} \tag{96}$$

On the one hand, by (75) and (91), we get

$$\begin{aligned}\left.\frac{du^*(x_s(t_0),t_0)}{dt_0}\right|_{t_0=0} &= \left.\frac{du^*(x_c(t),t)}{dt}\right|_{t=0}\\ &= \left.\frac{du^-(x_c(t),t)}{dt}\right|_{t=0}.\end{aligned} \tag{97}$$

Along the delta shock curve $x = x_s(t)$, it follows from (35), (88), and (97) that

$$\begin{aligned}\left.\frac{du^*(x_s(t_0),t_0)}{dt_0}\right|_{t_0=0} &= \left.\frac{du^*(x_c(t),t)}{dt}\right|_{t=0}\\ &= -\widehat{u}^-\cdot\dot{u}_0^-(0).\end{aligned} \tag{98}$$

On the other hand, based on (35) and (88), we obtain

$$\left.\frac{du^+(x_s(t_0),t_0)}{dt_0}\right|_{t_0=0} = \widehat{u}^-\cdot\dot{u}_0^+(0). \tag{99}$$

We substitute (98) and (99) into (96) and get

$$\ddot{x}_s(0) = \widehat{u}^-\cdot\left(\dot{u}_0^+(0) - \dot{u}_0^-(0)\right). \tag{100}$$

Thirdly, combining (94) and (100), it follows that

$$\ddot{x}_s(0) - \ddot{x}_c(0) = \widehat{u}^-\cdot\dot{u}_0^+(0) > 0. \tag{101}$$

Finally, by virtue of $x_c(0) = x_s(0) = 0$, (88), and (101), the perturbed Riemann solution to (1) and (3) in this subcase clearly consists of a contact discontinuity J connecting (u^-,v^-) to (u^*,v^*), followed by a forward shock $\overrightarrow{S}$ connecting (u^*,v^*) to (u^+,v^+) near the origin. Thus we have completed the construction and proof of the perturbed Riemann solution for this subcase (see Figure 4).

4. Conclusion

So far, we have locally constructed the solution to the perturbed Riemann problem (1) and (3) with delta shock. From the above discussion, we discover a delta shock in the corresponding Riemann solution may turn into a shock and a contact discontinuity after the perturbation of Riemann initial data. This shows the internal mechanism and instability of a delta shock. Then we can obtain the main theorem of this paper.

Theorem 8. *There is a unique local solution to the perturbed Riemann problem (1) and (3) with delta shock. The Riemann solution of (1) and (5) is instable under a perturbation of the Riemann initial data, which is quite different from the previous perturbed Riemann problems.*

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This paper was supported by NSF of China (11301264), China Scholarship Council, China Postdoctoral Science Foundation (2013M531343), the Fundamental Research Funds for the Central Universities (NZ2014107), Jiangsu Overseas Research & Training Program for University Prominent Young & Middle-aged Teachers and Presidents, and the NSF of Jiangsu Province of China (BK20130779).

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