Research Article

Preservers for the Tsallis Entropy of Convex Combinations of Density Operators

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Let H be a complex separable Hilbert space; we first characterize the unitary equivalence of two density operators by use of Tsallis entropy and then obtain the form of a surjective map on density operators preserving Tsallis entropy of convex combinations.

1. Introduction

In the mathematical framework of information theory, density operators (quantum states) are positive operators with trace 1 on complex separable Hilbert space H , and the set of all density operators is denoted by $S(H)$ on H . $S(H)$ is a compact convex set. A pure state is a rank one density operator, i.e., a rank one projection, and the set of all pure states is denoted by $P_1(H)$. We first introduce the definition of majorization given in [1]. Let $l^1(\mathbb{R}^+)$ be the set of all summable nonnegative real sequences and $l^1_1(\mathbb{R}^+)$ be the subset of $l^1(\mathbb{R}^+)$ consisting of all summable nonnegative real sequences whose sum is 1. Let $\vec{x}, \vec{y} \in l^1_1(\mathbb{R}^+)$ with $\vec{x} = (x_1, x_2, \dots, x_n, \dots)$, $\vec{y} = (y_1, y_2, \dots, y_n, \dots)$; then there exist (essentially unique) bijections $\pi, \omega : \{1, 2, \dots, n, \dots\} \rightarrow \{1, 2, \dots, n, \dots\}$ such that $x_{\pi(1)} \geq x_{\pi(2)} \geq \dots \geq x_{\pi(n)} \geq \dots$ and $y_{\omega(1)} \geq y_{\omega(2)} \geq \dots \geq y_{\omega(n)} \geq \dots$. We call $\vec{x} < \vec{y}$ if $\sum_{i=1}^k x_{\pi(i)} \leq \sum_{i=1}^k y_{\omega(i)}$ for each $k = 1, 2, \dots, n, \dots$, and $\sum_{i=1}^\infty x_{\pi(i)} = \sum_{i=1}^\infty y_{\omega(i)}$. We will denote $\vec{x}^\downarrow = (x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n)}, \dots)$ and $\vec{y}^\downarrow = (y_{\omega(1)}, y_{\omega(2)}, \dots, y_{\omega(n)}, \dots)$. Moreover, we have similar definitions and notations for $\vec{x}, \vec{y} \in \mathbb{R}^n$ with $\vec{x} = (x_1, x_2, \dots, x_n)$, $\vec{y} = (y_1, y_2, \dots, y_n)$.

Let $\rho, \sigma \in S(H)$ with $\rho = \sum_{i=1}^\infty x_i P_i$ and $\sigma = \sum_{i=1}^\infty y_i Q_i$, where $P_i, Q_i \in P_1(H)$; we denote $\rho < \sigma$ if $(x_1, x_2, \dots, x_n, \dots) =$

$\vec{x} < \vec{y} = (y_1, y_2, \dots, y_n, \dots)$. For $\rho \in S(H)$ and $r \in \mathbb{R} \setminus \{1\}$, Tsallis entropy $S^r(\rho)$ [2] is defined as follows:

$$S^r(\rho) = (1 - r)^{-1} (\text{tr}(\rho^r) - 1). \quad (1)$$

In this paper we will always assume that $r > 0$ and $r \neq 1$, i.e., $r \in \mathbb{R}^+ \setminus \{1\}$. If $r > 1$, then $\text{tr}(\rho^r) \leq 1$, for $\rho^r \leq \rho$. Otherwise, $S^r(\rho)$ can be infinite. Clearly $S^r(\rho) \geq 0$, and $S^r(\rho) = 0$ if and only if ρ is a pure state. It is well-known that $S^r(\rho)$ has concavity and subadditivity (see [3]). Using the spectral decomposition theorem, it is easy to see that $\lim_{r \rightarrow 1} S^r(\rho) = S(\rho)$, where $S(\rho) = -\text{tr} \rho \ln \rho$ is the von Neumann entropy. Therefore Tsallis entropy can be viewed as a generalization of the von Neumann entropy. Moreover Tsallis entropy is available for long-range interaction or fractal-type structure physical systems where the von Neumann entropy is not suitable and can be applied to thermostistical formalism [4] or to image thresholding segmentation [5]. The parameter r in $S^r(\rho)$ provides flexibility and universality in image processing.

It is well-known that quantum operation is a completely positive linear map [1], and some entanglement witnesses appear to be some special positive maps [6]. Therefore characterizing some maps on $S(H)$ is now important in quantum information. In [7], it is proved that, for a map $\phi : S(H) \rightarrow S(H)$, $\phi(\mu\rho + (1 - \mu)\sigma) = \mu\phi(\rho) + (1 - \mu)\phi(\sigma)$ ($\forall \rho, \sigma \in S(H)$, $\forall \mu \in [0, 1]$), if and only if there exists a unitary or antiunitary operator U acting on H such that

$\phi(\rho) = U\rho U^*$ for all $\rho \in S(H)$. The map of this form is widely studied in the context of relative entropy by Monlár et al. in a series of articles [8–11]. In the quantum context, Uhlmann proved that if $\rho, \sigma \in S(H)$ and $\dim H < \infty$, then $\rho < \sigma \Leftrightarrow \rho = \phi(\sigma)$, where ϕ is some mixed unitary quantum operation, i.e., $\phi(\rho) = \sum_{i=1}^k \lambda_i U_i \rho U_i^* (\forall \rho \in S(H))$, where $k < \infty$, $\lambda_i > 0$, $\sum_{i=1}^k \lambda_i = 1$, and all U_i are unitary operators acting on H . Uhlmann's theorem is widely used to study the role of majorization in quantum mechanics.

In the past decades, more attention has been paid to von Neumann entropy than Tsallis entropy. In [12, 13], He et al. characterized the unitary equivalence of maps on quantum states by use of the von Neumann entropy and obtained the form of maps on quantum states preserving the von Neumann entropy in finite dimensional Hilbert space. In [14], Li et al. proved that $S(\phi(\rho)) = S(\rho) (\forall \rho \in S(H))$ if and only if there exists a unitary operator U such that $\phi(\rho) = U\rho U^\dagger$, where $\phi(\rho)$ is a bistochastic quantum operation, i.e., $\phi(\rho) = \sum_i E_i \rho E_i^\dagger (\forall \rho \in S(H))$, and $\{E_i\}$ is a set of matrices known as operation elements. Moreover, in [15], Li et al. extended the result to infinite dimensional separable Hilbert space case. Recently, in [16], Bosyk et al. introduced a more general entropy, called the quantum (h, ϕ) -entropy $H_{(h, \phi)}(\rho)$, where (h, ϕ) is called a pair of entropic functionals, and obtained that, for two quantum states ρ, σ acting on finite dimensional Hilbert space H with $\rho < \sigma$, $H_{(h, \phi)}(\rho) = H_{(h, \phi)}(\sigma)$ holds for any pair (h, ϕ) of entropic functionals if and only if there exists a unitary and antiunitary operator U acting on H such that $\rho = U\sigma U^\dagger$ (see Definition 2 and Proposition 1 in [16]). In this paper, we would solve similar questions in the context of Tsallis entropy in complex separable Hilbert space.

The organization of this paper is as follows. In Section 2, we will characterize the unitary equivalence of two density operators by use of Tsallis entropy. In Section 3, the form of a surjective map on density operators preserving Tsallis entropy of convex combinations is obtained.

2. Tsallis Entropy Equivalence

The main result in this section is as follows.

Theorem 1. *Let H be a complex separable Hilbert space, $\rho, \sigma \in S(H)$ such that $\rho < \sigma$. Then the following are equivalent:*

- (1) $S^r(\rho) = S^r(\sigma)$ holds for any $r \in \mathbb{R}^+ \setminus \{1\}$.
- (2) $S^r(\rho) = S^r(\sigma)$ holds for some $r \in \mathbb{R}^+ \setminus \{1\}$ such that $S^r(\rho) + S^r(\sigma) < \infty$.
- (3) There exists a unitary operator U acting on H such that $\rho = U\sigma U^*$.

It should be noted that, for any $r > 1$, $S^r(\rho) + S^r(\sigma) < \infty$.

In addition, Theorem 1 can be revised in the case of finite dimensional Hilbert space without the conditions majorization $\rho < \sigma$. We present it as Theorem 2 and give a proof by use of similar method in [12].

Theorem 2. *Let H be a complex Hilbert space with $\dim H = n < +\infty$, $\rho, \sigma \in S(H)$, $0 < s \leq 1$, and I be the identity on H . Then the following are equivalent.*

(1)

$$S^r \left(t\rho + (1-t) \frac{I}{\text{tr}(I)} \right) = S^r \left(t\sigma + (1-t) \frac{I}{\text{tr}(I)} \right) \quad (2)$$

holds for any $r \in \mathbb{R}^+ \setminus \{1\}$ and any $t \in [0, 1]$.

(2) There exists $r \in \mathbb{R}^+ \setminus \{1\}$ such that

$$S^r \left(t\rho + (1-t) \frac{I}{\text{tr}(I)} \right) = S^r \left(t\sigma + (1-t) \frac{I}{\text{tr}(I)} \right) \quad (3)$$

holds for any $t \in (0, s)$.

(3) There exists a unitary operator U acting on H such that $\rho = U\sigma U^*$.

Proof. (1) $\Rightarrow$ (2). It is obvious.

(2) $\Rightarrow$ (3). Let $r \in \mathbb{R}^+ \setminus \{1\}$. Since $\rho, \sigma \in S(H)$, we may let $\rho = \sum_{i=1}^k x_i P_i$, $\sigma = \sum_{i=1}^l y_i Q_i$, where $1 \leq k, l \leq n$, $P_i, Q_i \in P_1(H)$, $0 < x_i, y_i \leq 1$, and $\sum_{i=1}^k x_i = \sum_{i=1}^l y_i = 1$. Hence we get

$$\begin{aligned} & \sum_{i=1}^k \left(tx_i + \frac{1-t}{n} \right)^r + (n-k) \left(\frac{1-t}{n} \right)^r \\ &= \text{tr} \left(\left(t\rho + (1-t) \frac{I}{n} \right)^r \right) = \text{tr} \left(\left(t\sigma + (1-t) \frac{I}{n} \right)^r \right) \quad (4) \\ &= \sum_{i=1}^l \left(ty_i + \frac{1-t}{n} \right)^r + (n-l) \left(\frac{1-t}{n} \right)^r. \end{aligned}$$

Taking the Taylor series of $(tx_i + (1-t)/n)^r$ and $(1-t)^r$ at $t = 0$, we have

$$\begin{aligned} & \left(tx_i + \frac{1-t}{n} \right)^r = \frac{1}{n^r} + (1 + (nx_i - 1)t)^r \\ &= \frac{1}{n^r} \left[1 + \sum_{p=1}^{\infty} \frac{r(r-1)\cdots(r-p+1)}{p!} (nx_i - 1)^p t^p \right] \quad (5) \\ &= \frac{1}{n^r} + \sum_{p=1}^{\infty} \frac{r(r-1)\cdots(r-p+1)}{n^{r-p} p!} \left(x_i - \frac{1}{n} \right)^p t^p, \end{aligned}$$

$$(1-t)^r = 1 + \sum_{p=1}^{\infty} \frac{r(r-1)\cdots(r-p+1)}{p!} (-1)^p t^p. \quad (6)$$

Thus we get

$$\begin{aligned} & \sum_{i=1}^k \left(tx_i + \frac{1-t}{n} \right)^r + (n-k) \left(\frac{1-t}{n} \right)^r = \sum_{i=1}^k \left(tx_i + \frac{1-t}{n} \right)^r \\ &+ \frac{n-k}{n^r} (1-t)^r = \frac{1}{n^{r-1}} \\ &+ \sum_{p=1}^{\infty} \frac{r(r-1)\cdots(r-p+1)}{n^{r-p} p!} \left(\sum_{i=1}^k \left(x_i - \frac{1}{n} \right)^p \right. \\ &+ \left. \frac{(-1)^p (n-k)}{n^p} \right) t^p = \frac{1}{n^{r-1}} \\ &+ \sum_{p=1}^{\infty} \frac{r(r-1)\cdots(r-p+1)}{n^{r-p} p!} \left(\sum_{i=1}^k \sum_{q=0}^{p-1} \frac{(-1)^q C_p^q}{n^q} x_i^{p-q} + \frac{(-1)^p}{n^{p-1}} \right) \\ &\cdot t^p, \end{aligned} \quad (7)$$

where $C_p^q = p!/q!(p-q)!$. Similarly, we can have the expansion of formula $\sum_{i=1}^l (ty_i + (1-t)/n)^r + (n-l)((1-t)/n)^r$.

$$\begin{aligned} & \sum_{i=1}^l \left(ty_i + \frac{1-t}{n} \right)^r + (n-l) \left(\frac{1-t}{n} \right)^r = \frac{1}{n^{r-1}} \\ & + \sum_{p=1}^{\infty} \frac{r(r-1)\cdots(r-p+1)}{n^{r-p} p!} \left(\sum_{i=1}^l \sum_{q=0}^{p-1} \frac{(-1)^q C_p^q}{n^q} y_i^{p-q} \right. \\ & \left. + \frac{(-1)^p}{n^{p-1}} \right) t^p. \end{aligned} \quad (8)$$

Thus we have

$$\sum_{i=1}^k x_i^d - \sum_{i=1}^l y_i^d = 0, \quad (d \in \mathbb{N}^+). \quad (9)$$

Without the loss of generality, let $x_1 \geq x_2 \geq \cdots \geq x_k > 0$, $y_1 \geq y_2 \geq \cdots \geq y_l > 0$. Dividing both sides of equality (9) by y_1^d and x_1^d , respectively, we have

$$\begin{aligned} & \left(\frac{x_1}{y_1} \right)^d + \left(\frac{x_2}{y_1} \right)^d + \cdots + \left(\frac{x_k}{y_1} \right)^d \\ & = 1 + \left(\frac{y_2}{y_1} \right)^d + \cdots + \left(\frac{y_l}{y_1} \right)^d, \end{aligned} \quad (10)$$

$$\begin{aligned} & 1 + \left(\frac{x_2}{x_1} \right)^d + \cdots + \left(\frac{x_k}{x_1} \right)^d \\ & = \left(\frac{y_1}{x_1} \right)^d + \left(\frac{y_2}{x_1} \right)^d + \cdots + \left(\frac{y_l}{x_1} \right)^d. \end{aligned} \quad (11)$$

If $x_1 > y_1$ and letting $d \rightarrow +\infty$, then the left side of equality (10) is infinite while its right side is less than l , which is a contradiction; therefore $x_1 \leq y_1$. Similarly, we have $x_1 \geq y_1$ by use of the equality (11), and so $x_1 = y_1$. Repeating the process, we get $k = l$ and $x_i = y_i$ ($i = 1, 2, \dots, k$). Hence ρ, σ have the same spectrum. This implies that ρ, σ are unitary equivalent.

(3) $\Rightarrow$ (1). Let $r \in \mathbb{R}^+ \setminus \{1\}$ and $t \in [0, 1]$. Denote $A_\rho = t\rho + (1-t)(I/n)$ and $A_\sigma = t\sigma + (1-t)(I/n)$. Since both A_ρ and A_σ are positive operators and $\rho = U\sigma U^*$, we have $A_\rho^r = (UA_\sigma U^*)^r = UA_\sigma^r U^*$. Thus $S^r(A_\rho) = (\text{tr}(A_\rho^r) - 1)/(1-r) = S^r(A_\sigma)$. $\square$

To prove the main Theorem 1, we need some preparation.

Firstly, the following lemma is a direct corollary of [1, Theorem 8.0.1] as the functions $f(x) = x^r$ ($r > 1$) and $f(x) = -x^r$ ($0 < r < 1$) are convex.

Lemma 3. Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ such that $x_i, y_i \geq 0$ and $\vec{x} < \vec{y}$. Then $\sum_{i=1}^n x_i^r \geq \sum_{i=1}^n y_i^r$ ($0 < r < 1$), and $\sum_{i=1}^n x_i^r \leq \sum_{i=1}^n y_i^r$ ($r > 1$).

Theorem 4. Let $\vec{x}, \vec{y} \in l^1(\mathbb{R}^+)$ such that $\vec{x} < \vec{y}$; then $\sum_{i=1}^\infty x_i^r \geq \sum_{i=1}^\infty y_i^r$ ($0 < r < 1$), and $\sum_{i=1}^\infty x_i^r \leq \sum_{i=1}^\infty y_i^r$ ($r > 1$).

Proof. Suppose that $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n, \dots)$, where $\alpha_i \geq 0$ and $\sum_{i=1}^\infty \alpha_i < \infty$. For arbitrary n , we have

$$(\alpha_1, \alpha_2, \dots, \alpha_n) < \left(\sum_{i=1}^n \alpha_i, 0, 0, \dots, 0 \right). \quad (12)$$

Then Lemma 3 implies that $\sum_{i=1}^n \alpha_i^r \geq (\sum_{i=1}^n \alpha_i)^r$ ($0 < r < 1$) and $\sum_{i=1}^n \alpha_i^r \leq (\sum_{i=1}^n \alpha_i)^r$ ($r > 1$). Let $n \rightarrow \infty$, by the continuity of the function $f(x) = x^r$, we get

$$\sum_{i=1}^\infty \alpha_i^r \geq \left(\sum_{i=1}^\infty \alpha_i \right)^r \quad (0 < r < 1), \quad (13)$$

$$\sum_{i=1}^\infty \alpha_i^r \leq \left(\sum_{i=1}^\infty \alpha_i \right)^r \quad (r > 1).$$

For any $m > 1$, we have that

$$\vec{x} < \vec{y} < \left(y_1, y_2, \dots, y_m, \sum_{i=m+1}^\infty y_i, 0, 0, \dots \right). \quad (14)$$

It is clear that we may assume $\sum_{i=1}^\infty x_i = \sum_{i=1}^\infty y_i \neq 0$. Since $\sum_{i=1}^\infty x_i < \infty$, there exists a positive integer $N \geq m$ such that

$$\sum_{i=N+1}^\infty x_i < \min \left\{ \left\{ y_1, y_2, \dots, y_m, \sum_{i=m+1}^\infty y_i \right\} - \{0\} \right\}. \quad (15)$$

Therefore

$$\begin{aligned} & \left(x_1, x_2, \dots, x_N, \sum_{i=N+1}^\infty x_i \right) \\ & < \left(y_1, y_2, \dots, y_m, \sum_{i=m+1}^\infty y_i, 0, \dots, 0 \right). \end{aligned} \quad (16)$$

Then Lemma 3 implies that

$$\sum_{i=1}^N x_i^r + \left(\sum_{i=N+1}^\infty x_i \right)^r \geq \sum_{i=1}^m y_i^r + \left(\sum_{i=m+1}^\infty y_i \right)^r \geq \sum_{i=1}^m y_i^r \quad (0 < r < 1), \quad (17)$$

$$\sum_{i=1}^N x_i^r + \left(\sum_{i=N+1}^\infty x_i \right)^r \leq \sum_{i=1}^m y_i^r + \left(\sum_{i=m+1}^\infty y_i \right)^r \quad (r > 1).$$

By inequalities (13), we have

$$\begin{aligned} & \sum_{i=1}^\infty x_i^r \geq \sum_{i=1}^m y_i^r \quad (0 < r < 1), \\ & \sum_{i=1}^\infty x_i^r \leq \sum_{i=1}^m y_i^r + \left(\sum_{i=m+1}^\infty y_i \right)^r \quad (r > 1). \end{aligned} \quad (18)$$

Let $m \rightarrow \infty$; we obtained the conclusion $\sum_{i=1}^\infty x_i^r \geq \sum_{i=1}^\infty y_i^r$ ($0 < r < 1$), $\sum_{i=1}^\infty x_i^r \leq \sum_{i=1}^\infty y_i^r$ ($r > 1$). $\square$

Proposition 5. Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ such that $x_i, y_i \geq 0$. If $\vec{x} < \vec{y}$ and $\sum_{i=1}^n x_i^r = \sum_{i=1}^n y_i^r$ for some $r \in \mathbb{R}^+ \setminus \{1\}$, then $\vec{x}^1 = \vec{y}^1$.

Proof. For $0 < \lambda < 1$, denote

$$\vec{z} = (\lambda x_1 + (1 - \lambda) y_1, \lambda x_2 + (1 - \lambda) y_2, \dots, \lambda x_n + (1 - \lambda) y_n). \quad (19)$$

It is obvious that $\vec{x} < \vec{z} < \vec{y}$. By Lemma 3 and $\sum_{i=1}^n x_i^r = \sum_{i=1}^n y_i^r$, we have $\sum_{i=1}^n (\lambda x_i + (1 - \lambda) y_i)^r = \sum_{i=1}^n x_i^r$. Taking the second derivative with respect to λ on both sides of this equality, we obtain

$$\sum_{i=1}^n r(r-1) (\lambda x_i + (1 - \lambda) y_i)^{r-2} (x_i - y_i)^2 = 0. \quad (20)$$

It is then easy to see that $x_i = y_i$ ($1 \leq i \leq n$); i.e., $\vec{x}^\downarrow = \vec{y}^\downarrow$. $\square$

Now we may generalize Proposition 5 to the infinite dimension case.

Theorem 6. Let $\vec{x}, \vec{y} \in l^1(\mathbb{R}^+)$ such that $\vec{x} < \vec{y}$ and $\sum_{i=1}^\infty x_i^r = \sum_{i=1}^\infty y_i^r < \infty$ for some $r \in \mathbb{R}^+ \setminus \{1\}$; then $\vec{x}^\downarrow = \vec{y}^\downarrow$.

Proof. Without the loss of generality, let $\vec{x} = \vec{x}^\downarrow = (x_1, x_2, \dots, x_n, \dots)$ and $\vec{y} = \vec{y}^\downarrow = (y_1, y_2, \dots, y_n, \dots)$. We first claim that there is $n_1 \geq 1$ such that $x_1 = y_{n_1}$. In fact, if it is not right, then $x_1 \neq y_n$ for any $n \geq 1$. Since $\vec{x} < \vec{y}$, we may assume that $\vec{x} \neq 0$, $0 \neq x_1 \leq y_1$. Then there exists $k \geq 1$ such that $x_1 \in (y_{k+1}, y_k)$, and $x_1 = ty_{k+1} + (1 - t)y_k$ for some $t \in (0, 1)$. For $s \in (0, t)$, let

$$\vec{z} = (y_1, y_2, \dots, y_{k-1}, sy_{k+1} + (1 - s)y_k, (1 - s)y_{k+1} + sy_k, y_{k+2}, \dots). \quad (21)$$

It is obvious that $\vec{x} < \vec{z} < \vec{y}$. By Theorem 4 and the condition $\sum_{i=1}^\infty x_i^r = \sum_{i=1}^\infty y_i^r < \infty$, we have

$$\begin{aligned} \sum_{i=1}^\infty x_i^r &= \sum_{i=1}^{k-1} y_i^r + (sy_{k+1} + (1 - s)y_k)^r \\ &\quad + ((1 - s)y_{k+1} + sy_k)^r + \sum_{i=k+2}^\infty y_i^r = \sum_{i=1}^\infty y_i^r \\ &< \infty. \end{aligned} \quad (22)$$

Therefore, for $s \in (0, t)$,

$$\begin{aligned} &(sy_{k+1} + (1 - s)y_k)^r + ((1 - s)y_{k+1} + sy_k)^r \\ &= y_k^r + y_{k+1}^r. \end{aligned} \quad (23)$$

Taking the second derivative with respect to s on both sides of this equality above, we obtain

$$\begin{aligned} &r(r-1) \\ &\cdot \left[(sy_{k+1} + (1 - s)y_k)^{r-2} + ((1 - s)y_{k+1} + sy_k)^{r-2} \right] \\ &\cdot (y_k - y_{k+1})^2 = 0. \end{aligned} \quad (24)$$

Therefore $y_k = y_{k+1}$, which is a contradiction.

Now let $x_1 = y_{n_1}$, and $\vec{x}' = (x_2, \dots, x_{n_1}, \dots)$, $\vec{y}' = (y_1, y_2, \dots, y_{n_1-1}, y_{n_1+1}, \dots)$; then $\vec{x}'$ and $\vec{y}'$ have the same properties as $\vec{x}$ and $\vec{y}$. By the claim above again, there is $n_2 \geq 1$ such that $x_2 = y_{n_2}$ and $n_2 \neq n_1$. Repeating this process, we have that, for any x_k , there is $n_k \geq 1$ such that $x_k = y_{n_k}$ and $n_k \neq n_l$ if $k \neq l$. Since $\vec{x}, \vec{y} \in l^1(\mathbb{R}^+)$ and $\sum_{i=1}^\infty x_i = \sum_{i=1}^\infty y_i < \infty$, we have $\vec{x}^\downarrow = \vec{y}^\downarrow$. $\square$

Proof of Theorem 1. (1) $\Rightarrow$ (2). It is obvious.

(2) $\Rightarrow$ (3). Since $\rho, \sigma \in S(H)$, we may let $\rho = \sum_{i=1}^\infty x_i P_i$, $\sigma = \sum_{i=1}^\infty y_i Q_i$, where $P_i, Q_i \in P_1(H)$ and $0 \leq x_i, y_i \leq 1$ such that $\sum_{i=1}^\infty P_i = \sum_{i=1}^\infty Q_i = id_H$ and $\sum_{i=1}^\infty x_i = \sum_{i=1}^\infty y_i = 1$. Let $\vec{x} = (x_1, x_2, \dots)$, $\vec{y} = (y_1, y_2, \dots)$. Noting that $\rho < \sigma$, then we have $\vec{x} < \vec{y}$. Since $S^r(\rho) = S^r(\sigma) < \infty$, then $\sum_{i=1}^\infty x_i^r = \sum_{i=1}^\infty y_i^r < \infty$. Therefore by Theorem 6, we have $\vec{x}^\downarrow = \vec{y}^\downarrow$. Thus there exists a unitary operator U acting on H such that $\rho = U\sigma U^*$.

(3) $\Rightarrow$ (1). Since both ρ and σ are positive operators and $\rho = U\sigma U^*$, then, for $r \in \mathbb{R}^+ \setminus \{1\}$, we have

$$\begin{aligned} S^r(\rho) &= (1 - r)^{-1} (\text{tr}(\rho)^r - 1) \\ &= (1 - r)^{-1} (\text{tr}(U\sigma U^*)^r - 1) \\ &= (1 - r)^{-1} (\text{tr}(U\sigma^r U^*) - 1) = S^r(\sigma). \end{aligned} \quad (25)$$

$\square$

3. The Characterization of Mappings That Preserve Tsallis Entropy

In this section, we shall determine the structure of surjective maps on density operators preserving Tsallis entropy of convex combinations. The following is our result.

Proposition 7. Let H be a complex separable Hilbert space and $\phi : S(H) \rightarrow S(H)$ be a surjective map such that, for any $\rho \in S(H)$, ρ and $\phi(\rho)$ are comparable, i.e., $\rho < \phi(\rho)$ or $\phi(\rho) < \rho$. Then the following statements are equivalent:

(1)

$$S^r(t\rho + (1 - t)\sigma) = S^r(t\phi(\rho) + (1 - t)\phi(\sigma)) \quad (26)$$

holds for any $\rho, \sigma \in S(H)$, any $r \in \mathbb{R}^+ \setminus \{1\}$, and any $t \in [0, 1]$.

(2) There exists $r > 1$ such that $S^r(\rho) = S^r(\phi(\rho))$ ($\forall \rho \in S(H)$) and that for any $\rho, \sigma \in S(H)$ there exists $t \in (0, 1)$ with

$$S^r(t\rho + (1 - t)\sigma) = S^r(t\phi(\rho) + (1 - t)\phi(\sigma)). \quad (27)$$

(3) There exists a unitary or antiunitary operator U on H such that $\phi(\rho) = U\rho U^*$ for all $\rho \in S(H)$.

Remark 8. In this proposition, scalar $t \in [0, 1]$ is defined by the arbitrariness of ρ and σ . We advanced this proposition into infinite dimension by use of comparable condition. It should be pointed out that a similar conclusion is discussed in [17]. Their result is as follows: for a given t , if $S^r(t\rho + (1 - t)\sigma) = S^r(t\phi(\rho) + (1 - t)\phi(\sigma))$, for any $\rho, \sigma \in S(H)$, then there exists a unitary or antiunitary operator U acting on H such that $\phi(\rho) = U\rho U^*$ for all $\rho \in S(H)$.

To prove Proposition 7, we need some preparation.

Lemma 9. Let H be a complex separable Hilbert space, A be a positive compact operator on H , and $x_1 \geq x_2 \geq \dots \geq x_n \geq \dots$ be all the eigenvalues of A ; then

$$x_n = \min_{K \subseteq H, \dim(K)=n-1} \sup_{h \in K^\perp, \|h\|=1} \langle Ah, h \rangle. \quad (28)$$

Proof. Let $K \subseteq H$ with $\dim(K) = n-1$, and let the unit vector e_i be the mutually orthogonal eigenvector of A with respect to x_i . Then $e_i = k_i \oplus h_i$, where $k_i \in K$ and $h_i \in K^\perp$. Since $\dim(K) = n-1$ there are $\alpha_1, \alpha_2, \dots, \alpha_n$ such that $\sum_{i=1}^n \alpha_i k_i = 0$ and $\sum_{i=1}^n |\alpha_i|^2 = 1$. Then $\sum_{i=1}^n \alpha_i e_i = \sum_{i=1}^n \alpha_i h_i \in (K^\perp \cap H_n) \setminus \{0\}$, where $H_n = \text{span}\{e_1, e_2, \dots, e_n\}$. Then

$$\left\langle A \left(\sum_{i=1}^n \alpha_i e_i \right), \sum_{i=1}^n \alpha_i e_i \right\rangle = \left\langle \sum_{i=1}^n x_i \alpha_i e_i, \sum_{i=1}^n \alpha_i e_i \right\rangle \geq x_n. \quad (29)$$

On the other hand it is easy to see that $x_n = \sup_{h \in H_{n-1}^\perp, \|h\|=1} \langle Ah, h \rangle$, and this completes the proof. $\square$

Corollary 10. Let H be a complex separable Hilbert space, A and B be positive compact operators on H such that $A \geq B$. If $x_1 \geq x_2 \geq \dots \geq x_n \geq \dots$ and $y_1 \geq y_2 \geq \dots \geq y_n \geq \dots$ are all the eigenvalues of A and B , respectively, then $x_i \geq y_i$ for $i = 1, 2, \dots$

Lemma 11. Let H be a complex separable Hilbert space, $\rho \in S(H)$, and U be a unitary operator acting on H . If there exist $r \in \mathbb{R}^+ \setminus \{1\}$ and $t \in (0, 1)$ such that $S^r(t\rho + (1-t)P) = S^r(tU\rho U^* + (1-t)P) < \infty$ holds for any $P \in P_1(H)$, then $\rho = U\rho U^*$.

Proof. Let $x_1 \geq x_2 \geq \dots \geq x_n \geq \dots$ be all the eigenvalues of ρ . More precisely, we assume $x_1 = \dots = x_{n_1} > x_{n_1+1} = \dots = x_{n_2} > \dots > x_{n_{k-1}+1} = \dots = x_{n_k} > \dots$. Let H_n be the eigenspace of ρ with respect to x_n and Q_n be the projection on H_n for any $n \geq 1$.

Assume that the unit vector e_1 is the eigenvector of ρ with respect to x_1 , i.e., $\rho e_1 = x_1 e_1$. Let $P = e_1 \otimes e_1$; then $\rho P = P\rho$, and all the eigenvalues of $t\rho + (1-t)P$ are $tx_1 + 1 - t \geq tx_2 \geq \dots \geq tx_n \geq \dots$ for $t \in (0, 1)$. Let $k = U^* e_1$; then $U^* P U = k \otimes k$. Now fix r and t as given in the theorem; by Corollary 10 there are $z_1 \geq 0, z_2 \geq 0, \dots, z_n \geq 0, \dots$ such that $\sum_{i=1}^\infty z_i = 1 - t$, and all the eigenvalues of $t\rho + (1-t)k \otimes k$ are $tx_1 + z_1 \geq tx_2 + z_2 \geq \dots \geq tx_n + z_n \geq \dots$. Let $\vec{x} = (tx_1 + 1 - t, tx_2, \dots, tx_n, \dots)$, $\vec{y} = (tx_1 + z_1, tx_2 + z_2, \dots, tx_n + z_n, \dots)$, then $\vec{y} < \vec{x}$. Since $S^r(t\rho + (1-t)P) = S^r(tU\rho U^* + (1-t)P) = S^r(t\rho + (1-t)k \otimes k) < \infty$, by Theorem 6 we have $\vec{y} = \vec{x}$. It follows that all the eigenvalues of $t\rho + (1-t)k \otimes k$ are $tx_1 + 1 - t \geq tx_2 \geq \dots \geq tx_n \geq \dots$. We claim that $k \in H_1$. In fact, otherwise, $k = k_1 \oplus k_2$ with $k_1 \in H_1, k_2 \perp H_1$, and

$k_2 \neq 0$. Since $tx_1 + 1 - t$ is the eigenvalue of $t\rho + (1-t)k \otimes k$, there is unit vector $h \in H$ such that

$$\begin{aligned} tx_1 + 1 - t &= \langle (t\rho + (1-t)k \otimes k)h, h \rangle \\ &= t \langle \rho h, h \rangle + (1-t) \langle (k \otimes k)h, h \rangle \\ &= tx_1 \|h_1\|^2 + t \langle \rho h_2, h_2 \rangle + (1-t) |\langle h, k \rangle|^2 \\ &\leq tx_1 \|h_1\|^2 + tx_{n_2} \|h_2\|^2 \\ &\quad + (1-t) |\langle h_1, k_1 \rangle + \langle h_2, k_2 \rangle|^2 \\ &\leq tx_1 - t(x_1 - x_{n_2}) \|h_2\|^2 \\ &\quad + (1-t) (\|h_1\| \|k_1\| + \|h_2\| \|k_2\|)^2 \\ &\leq tx_1 - t(x_1 - x_{n_2}) \|h_2\|^2 \\ &\quad + (1-t) (\|h_1\|^2 + \|h_2\|^2) (\|k_1\|^2 + \|k_2\|^2) \\ &= tx_1 - t(x_1 - x_{n_2}) \|h_2\|^2 + (1-t), \end{aligned} \quad (30)$$

where $h = h_1 \oplus h_2$ with $h_1 \in H_1$ and $h_2 \perp H_1$. Since $k_2 \neq 0$, the equality above holds only if $h_2 \neq 0$ and so

$$\begin{aligned} tx_1 + 1 - t &\leq tx_1 - t(x_1 - x_{n_2}) \|h_2\|^2 + (1-t) \\ &< tx_1 + 1 - t, \end{aligned} \quad (31)$$

which is a contradiction. This completes the proof of $k = U^* e_1 \in H_1$. By the arbitrariness of $e_1 \in H_1$ and the fact that $\dim(H_1) < \infty$ we have $U^* = U^{-1}$ is a bijection on H_1 and so $UQ_1 = Q_1 UQ_1, U^* Q_1 = Q_1 U^* Q_1$. Therefore $UQ_1 = Q_1 U$. Let $P = e_{n_2} \otimes e_{n_2}$, where e_{n_2} is a unit eigenvector of ρ with respect to x_{n_2} . Since $U(1 - Q_1) = (1 - Q_1)U$ and $P \leq 1 - Q_1, tx_1 = tx_2 = \dots = tx_{n_1}$ are the eigenvalues of both $t\rho + (1-t)U^* P U$ and $t\rho + (1-t)P$. Then we get $UQ_2 = Q_2 U$ by similar discussion as above with $(1 - Q_1)\rho(1 - Q_1)$ instead of ρ . Repeating this process, we have $UQ_n = Q_n U$ for any n , and so $U\rho = \rho U$; i.e., $U\rho U^* = \rho$. $\square$

Lemma 12. Let H be a complex Hilbert space with $\dim H = n < \infty$ and $\rho \in S(H)$. Then $\rho = I/n$ if and only if $S^r(\rho) = (n^{1-r} - 1)/(1 - r)$, where $r \in \mathbb{R}^+ \setminus \{1\}$ and I is the identity on H .

Proof. If $\rho = I/n$, then $S^r(\rho) = (\text{tr}((I/n)^r) - 1)/(1 - r) = (n^{1-r} - 1)/(1 - r)$ for any $r \in \mathbb{R}^+ \setminus \{1\}$. Let $\rho \in S(H)$; then $\rho = \sum_{i=1}^k x_i P_i$, where $P_i \in P_1(H), 1 \leq i \leq n, 0 \leq x_i \leq 1$, and $\sum_{i=1}^k x_i = 1$. Assume that $S^r(\rho) = (n^{1-r} - 1)/(1 - r)$ for some $r > 1$. By the convexity of the function $y = x^r$, we get

$$\text{tr}(\rho^r) = \sum_{i=1}^k x_i^r = k \cdot \frac{1}{k} \sum_{i=1}^k x_i^r \geq k \left(\sum_{i=1}^k \frac{1}{k} x_i \right)^r = k^{1-r}. \quad (32)$$

Equality holds if and only if $x_i = 1/k$ ($i = 1, 2, \dots, k$). So

$$\begin{aligned} \frac{n^{1-r} - 1}{1-r} &= S^r(\rho) = \frac{\text{tr}(\rho^r) - 1}{1-r} \leq \frac{k^{1-r} - 1}{1-r} \\ &\leq \frac{n^{1-r} - 1}{1-r} \quad (r > 1). \end{aligned} \quad (33)$$

Then $k = n$, $\text{tr}(\rho^r) = k^{1-r} = n^{1-r}$. Therefore $x_i = 1/n$ ($i = 1, 2, \dots, k$); i.e., $\rho = I/n$.

In the case $S^r(\rho) = (n^{1-r} - 1)/(1-r)$ for some $0 < r < 1$, the discussion is similar. $\square$

Lemma 13 (see [3] Lemma 1). *Let H be a complex separable Hilbert space, and, for $r \in \mathbb{R}^+ \setminus \{1\}$, let $S_r(H)$ be the subset of $S(H)$ consisting of all states ρ with $S^r(\rho) < \infty$. Then $S_r(H)$ is convex and the Tsallis entropy $S^r(\rho)$ is strictly concave on $S_r(H)$; i.e., for $t \in (0, 1)$, $S^r(t\rho + (1-t)\sigma) \geq tS^r(\rho) + (1-t)S^r(\sigma)$, where $\rho, \sigma \in S_r(H)$. Moreover equality holds if and only if $\rho = \sigma$.*

It should be noted that if $r > 1$ or $\dim(H) < \infty$, $S_r(H) = S(H)$.

Lemma 14 (see [3], Lemma 1). *With the notations as above, for $0 < t < 1$, $r \in \mathbb{R}^+ \setminus \{1\}$,*

$$\begin{aligned} S^r(t\rho + (1-t)\sigma) &\leq tS^r(\rho) + (1-t)S^r(\sigma) \\ &\quad + (r-1)^{-1}(1-t^r - (1-t)^r). \end{aligned} \quad (34)$$

If the involved quantities are finite, then the equality holds if and only if $\rho\sigma = 0$.

Proposition 15. *Let H be a complex Hilbert space with $\dim H = n < \infty$ and $\phi : S(H) \rightarrow S(H)$ be a surjective map. Then the following statements are equivalent:*

(1)

$$S^r(t\rho + (1-t)\sigma) = S^r(t\phi(\rho) + (1-t)\phi(\sigma)) \quad (35)$$

holds for any $\rho, \sigma \in S(H)$, any $r \in \mathbb{R}^+ \setminus \{1\}$, and any $t \in [0, 1]$.

(2) *There exists $r \in \mathbb{R}^+ \setminus \{1\}$ such that*

$$S^r(t\rho + (1-t)\sigma) = S^r(t\phi(\rho) + (1-t)\phi(\sigma)) \quad (36)$$

holds for any $\rho, \sigma \in S(H)$ and any $t \in [0, 1]$.

(3) *There exists a unitary or antiunitary operator V on H such that $\phi(\rho) = V\rho V^*$ for all $\rho \in S(H)$.*

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are obvious.

(2) $\Rightarrow$ (3): we divided it into two cases: $2 < \dim H = n$ and $\dim H = 2$.

Case 1 ($2 < \dim H = n$). Let $r \in \mathbb{R}^+ \setminus \{1\}$. Take $t = 1$ in equality (36); we have $S^r(\rho) = S^r(\phi(\rho))$ ($\forall \rho \in S(H)$).

Then ϕ is injective and so bijective. In fact, if $\rho, \sigma \in S(H)$ with $\phi(\rho) = \phi(\sigma)$, then, for $t \in (0, 1)$,

$$\begin{aligned} tS^r(\rho) + (1-t)S^r(\sigma) &= tS^r(\phi(\rho)) + (1-t)S^r(\phi(\sigma)) \\ &= S^r(t\phi(\rho) + (1-t)\phi(\sigma)) = S^r(t\rho + (1-t)\sigma). \end{aligned} \quad (37)$$

Thus we have $\rho = \sigma$ by Lemma 13.

Claim. ϕ preserves orthogonality in both directions; i.e., $\rho\sigma = 0 \Leftrightarrow \phi(\rho)\phi(\sigma) = 0$ for all $\rho, \sigma \in S(H)$.

In fact, if $\rho\sigma = 0$, by Lemma 14 we have

$$\begin{aligned} S^r(t\rho + (1-t)\sigma) &= t^r S^r(\rho) + (1-t)^r S^r(\sigma) \\ &\quad + (r-1)^{-1}(1-t^r - (1-t)^r). \end{aligned} \quad (38)$$

Since $S^r(\rho) = S^r(\phi(\rho))$, by equality (36) we get

$$\begin{aligned} S^r(t\rho + (1-t)\sigma) &= t^r S^r(\rho) + (1-t)^r S^r(\sigma) \\ &\quad + (r-1)^{-1}(1-t^r - (1-t)^r) \\ &= t^r S^r(\phi(\rho)) + (1-t)^r S^r(\phi(\sigma)) \\ &\quad + (r-1)^{-1}(1-t^r - (1-t)^r) \\ &= S^r(t\phi(\rho) + (1-t)\phi(\sigma)). \end{aligned} \quad (39)$$

By Lemma 13 again, we get $\phi(\rho)\phi(\sigma) = 0$. Similarly, we can show $\phi(\rho)\phi(\sigma) = 0 \Rightarrow \rho\sigma = 0$. This completes the proof of the claim.

Since

$$\begin{aligned} \rho \in P_1(H) &\iff \\ 0 = S^r(\rho) &\iff \\ 0 = S^r(\rho) = S^r(\phi(\rho)) &\iff \\ S^r(\phi(\rho)) = 0 &\iff \\ \phi(\rho) \in P_1(H), \end{aligned} \quad (40)$$

$\phi|_{P_1(H)}$ is a bijection from $P_1(H)$ onto $P_1(H)$.

By Corollary 1 in [18], there exists a unitary or antiunitary operator V on H such that $\phi(P) = VPV^*$ for any $P \in P_1(H)$.

Let $\psi(\rho) = V^*\phi(\rho)V$ for any $\rho \in S(H)$; it is obvious that $\psi(P) = V^*VPV^*V = P$ ($\forall P \in P_1(H)$). Since $S^r(I/n) = S^r(\phi(I/n))$, by Lemma 12 we have $\phi(I/n) = I/n$, where I is the identity on H . Then $\psi(I/n) = V^*\phi(I/n)V = V^*(I/n)V = I/n$.

To prove the Theorem, it is enough to prove $\psi(\rho) = \rho$ for each fixed $\rho \in S(H)$. By equality (36), we have

$$\begin{aligned} S^r\left(t\rho + (1-t)\frac{I}{n}\right) &= S^r\left(t\phi(\rho) + (1-t)\phi\left(\frac{I}{n}\right)\right) \\ &= S^r\left(tV\psi(\rho)V^* + (1-t)V\psi\left(\frac{I}{n}\right)V^*\right) \\ &= S^r\left(t\psi(\rho) + (1-t)\psi\left(\frac{I}{n}\right)\right) \\ &= S^r\left(t\psi(\rho) + (1-t)\frac{I}{n}\right). \end{aligned} \quad (41)$$

By Theorem 2, there exists a unitary operator U_ρ (depending on ρ and ψ) on H such that $\psi(\rho) = U_\rho\rho U_\rho^*$; i.e., $\rho =$

$U_\rho^* \psi(\rho) U_\rho$. By equality (36) again, we get, for any $P \in P_1(H)$,

$$\begin{aligned} S^r(t\rho + (1-t)P) &= S^r(t\phi(\rho) + (1-t)\phi(P)) \\ &= S^r(tV\psi(\rho)V^* + (1-t)V\psi(P)V^*) \\ &= S^r(t\psi(\rho) + (1-t)\psi(P)) \\ &= S^r(t\psi(\rho) + (1-t)P) \\ &= S^r(tU_\rho\rho U_\rho^* + (1-t)P). \end{aligned} \quad (42)$$

By Lemma 11, we have $\rho = U_\rho\rho U_\rho^*$. Thus $\rho = U_\rho\rho U_\rho^* = U_\rho U_\rho^* \psi(\rho) U_\rho U_\rho^* = \psi(\rho) = V^* \phi(\rho) V$; i.e., $\phi(\rho) = V\rho V^*$.

Case 2 ($\dim H = 2$). If $\dim H = 2$, let us identify $S(H)$ with the subset of 2×2 density matrices

$$S(H) = \left\{ \frac{1}{2} \begin{pmatrix} 1+z & x+iy \\ x-iy & 1-z \end{pmatrix} \mid (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1 \right\}. \quad (43)$$

Note that (x, y, z) satisfies $x^2 + y^2 + z^2 = 1$ if and only if the corresponding matrix is a rank one projection.

By equality (36), we still have $S^r(\rho) = S^r(\phi(\rho))$ for all $\rho \in S(H)$ and some $r \in \mathbb{R}^+ \setminus \{1\}$. Since ϕ still preserves pure states in both directions, then there exists a unitary operator V such that

$$\phi\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = V\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}V^*. \quad (44)$$

Let $\psi(\rho) = V^* \phi(\rho) V$ for all $\rho \in S(H)$; then

$$\psi\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (45)$$

For any $\rho \in S(H)$, let

$$\begin{aligned} \rho &= \begin{pmatrix} \frac{1+z}{2} & \frac{x+iy}{2} \\ \frac{x-iy}{2} & \frac{1-z}{2} \end{pmatrix}, \\ \psi(\rho) &= \begin{pmatrix} \frac{1+z'}{2} & \frac{x'+iy'}{2} \\ \frac{x'-iy'}{2} & \frac{1-z'}{2} \end{pmatrix}. \end{aligned} \quad (46)$$

Taking $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in equality (36), then for any $t \in [0, 1]$ and some $r \in \mathbb{R}^+ \setminus \{1\}$, we have

$$\begin{aligned} S^r\left(t\begin{pmatrix} \frac{1+z}{2} & \frac{x+iy}{2} \\ \frac{x-iy}{2} & \frac{1-z}{2} \end{pmatrix} + (1-t)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \\ = S^r\left(t\begin{pmatrix} \frac{1+z'}{2} & \frac{x'+iy'}{2} \\ \frac{x'-iy'}{2} & \frac{1-z'}{2} \end{pmatrix} + (1-t)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right). \end{aligned} \quad (47)$$

Since, for two qubit states $(x_1, x_2), (y_1, y_2)$, we always have $(x_1, x_2) < (y_1, y_2)$ or $(y_1, y_2) < (x_1, x_2)$, noting equality (47) and by Proposition 5, we know that the following two states have the same eigenvalues and eigenpolynomials.

$$\begin{aligned} &\left(t\begin{pmatrix} \frac{1+z}{2} & \frac{x+iy}{2} \\ \frac{x-iy}{2} & \frac{1-z}{2} \end{pmatrix} + (1-t)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right), \\ &\left(t\begin{pmatrix} \frac{1+z'}{2} & \frac{x'+iy'}{2} \\ \frac{x'-iy'}{2} & \frac{1-z'}{2} \end{pmatrix} + (1-t)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right). \end{aligned} \quad (48)$$

Then we get their eigenpolynomials

$$\begin{aligned} \lambda^2 - \lambda - \frac{x^2 + y^2 + (1-z)^2}{4}t^2 + \frac{1-z}{2}t &= \left|\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \left(t\begin{pmatrix} \frac{1+z}{2} & \frac{x+iy}{2} \\ \frac{x-iy}{2} & \frac{1-z}{2} \end{pmatrix} + (1-t)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)\right| \\ &= \left|\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \left(t\begin{pmatrix} \frac{1+z'}{2} & \frac{x'+iy'}{2} \\ \frac{x'-iy'}{2} & \frac{1-z'}{2} \end{pmatrix} + (1-t)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)\right| \end{aligned} \quad (49)$$

It follows that $z = z'$ and $|x+iy| = |x'+iy'|$, so there exists a function $\theta : S(H) \rightarrow [0, 2\pi)$ such that $e^{i\theta(\rho)}(x+iy) = x'+iy'$ for all $\rho \in S(H)$.

Now, for arbitrary $\rho, \sigma \in S(H)$, let

$$\begin{aligned}\rho &= \begin{pmatrix} \frac{1+z_\rho}{2} & \frac{x_\rho+iy_\rho}{2} \\ \frac{x_\rho-iy_\rho}{2} & \frac{1-z_\rho}{2} \end{pmatrix}, \\ \sigma &= \begin{pmatrix} \frac{1+z_\sigma}{2} & \frac{x_\sigma+iy_\sigma}{2} \\ \frac{x_\sigma-iy_\sigma}{2} & \frac{1-z_\sigma}{2} \end{pmatrix},\end{aligned}\quad (50)$$

and then

$$\begin{aligned}\psi(\rho) &= \begin{pmatrix} \frac{1+z_\rho}{2} & \frac{x_\rho+iy_\rho}{2} e^{i\theta_\rho} \\ \frac{x_\rho-iy_\rho}{2} e^{-i\theta_\rho} & \frac{1-z_\rho}{2} \end{pmatrix}, \\ \psi(\sigma) &= \begin{pmatrix} \frac{1+z_\sigma}{2} & \frac{x_\sigma+iy_\sigma}{2} e^{i\theta_\sigma} \\ \frac{x_\sigma-iy_\sigma}{2} e^{-i\theta_\sigma} & \frac{1-z_\sigma}{2} \end{pmatrix}.\end{aligned}\quad (51)$$

By equality (36) again, we have

$$\begin{aligned}&\lambda^2 - \lambda - \left((x_\rho - x_\sigma)^2 + (y_\rho - y_\sigma)^2 + (z_\rho - z_\sigma)^2 \right) t^2 \\&+ (2x_\sigma^2 + 2y_\sigma^2 + 2z_\sigma^2 - 2x_\rho x_\sigma - 2y_\rho y_\sigma - 2z_\rho z_\sigma) t \\&+ (1 - x_\sigma^2 - y_\sigma^2 - z_\sigma^2) = \lambda^2 - \lambda - (x_\rho^2 + y_\rho^2 + z_\rho^2 \\&+ x_\sigma^2 + y_\sigma^2 + z_\sigma^2 - 2x_\rho x_\sigma \cos(\theta_\rho - \theta_\sigma) \\&- 2y_\rho y_\sigma \cos(\theta_\rho - \theta_\sigma) - 2x_\rho y_\sigma \sin(\theta_\rho - \theta_\sigma) \\&+ 2x_\sigma y_\rho \sin(\theta_\rho - \theta_\sigma) - 2z_\rho z_\sigma) t^2 + (2x_\sigma^2 + 2y_\sigma^2 \\&+ 2z_\sigma^2 - 2x_\rho x_\sigma \cos(\theta_\rho - \theta_\sigma) \\&- 2y_\rho y_\sigma \cos(\theta_\rho - \theta_\sigma) - 2z_\rho z_\sigma \\&- 2x_\rho y_\sigma \sin(\theta_\rho - \theta_\sigma) + 2x_\sigma y_\rho \sin(\theta_\rho - \theta_\sigma)) t \\&+ (1 - x_\sigma^2 - y_\sigma^2 - z_\sigma^2).\end{aligned}\quad (52)$$

It follows that

$$\begin{aligned}&(x_\rho x_\sigma + y_\rho y_\sigma) (1 - \cos(\theta_\rho - \theta_\sigma)) \\&= (x_\rho y_\sigma - x_\sigma y_\rho) \sin(\theta_\rho - \theta_\sigma).\end{aligned}\quad (53)$$

In equality (53), if $x_\rho, x_\sigma \neq 0, y_\rho = y_\sigma = 0$, we have $\cos(\theta_\rho - \theta_\sigma) = 1$. Noting $\theta_\rho, \theta_\sigma \in [0, 2\pi)$, so $\theta_\rho = \theta_\sigma$. It means that θ_ρ is a constant θ_0 for

$$\rho = \begin{pmatrix} \frac{1+z_\rho}{2} & \frac{x_\rho}{2} \\ \frac{x_\rho}{2} & \frac{1-z_\rho}{2} \end{pmatrix}.\quad (54)$$

If $x_\rho, y_\rho, x_\sigma \neq 0, y_\sigma = 0$ in equality (53), we have either

$$\cos(\theta_\rho - \theta_0) = 1,\quad (55)$$

$$\sin(\theta_\rho - \theta_0) = 0$$

or

$$\cos(\theta_\rho - \theta_0) = \frac{x_\rho^2 - y_\rho^2}{x_\rho^2 + y_\rho^2},\quad (56)$$

$$\sin(\theta_\rho - \theta_0) = \frac{-2x_\rho y_\rho}{x_\rho^2 + y_\rho^2}.$$

For any

$$\rho = \begin{pmatrix} \frac{1+z_\rho}{2} & \frac{x_\rho+iy_\rho}{2} \\ \frac{x_\rho-iy_\rho}{2} & \frac{1-z_\rho}{2} \end{pmatrix} \in S(H),\quad (57)$$

equality (55) means $(x_\rho + iy_\rho) e^{i(\theta_\rho - \theta_0)} = x_\rho + iy_\rho$ while equality (56) results in $(x_\rho + iy_\rho) e^{i(\theta_\rho - \theta_0)} = x_\rho - iy_\rho$. Next we shall show that either equality (55) holds for all states ρ or equality (56) holds for all states ρ . Assume on the contrary that there exist ρ, σ such that the equality (55) holds for $x_\rho + iy_\rho$ and equality (56) holds for $x_\sigma + iy_\sigma$. Let $y_\rho, y_\sigma \neq 0$, it follows from equality (53) that

$$\begin{aligned}&2(x_\rho x_\sigma - y_\rho y_\sigma) \\&= (x_\rho + iy_\rho)(x_\sigma + iy_\sigma) + (x_\rho - iy_\rho)(x_\sigma - iy_\sigma) \\&= (x_\rho + iy_\rho) e^{i(\theta_\rho - \theta_0)} (x_\sigma - iy_\sigma) e^{-i(\theta_\sigma - \theta_0)} \\&+ (x_\rho - iy_\rho) e^{-i(\theta_\rho - \theta_0)} (x_\sigma + iy_\sigma) e^{i(\theta_\sigma - \theta_0)} \\&= 2(x_\rho x_\sigma + y_\rho y_\sigma) \cos(\theta_\rho - \theta_\sigma) \\&+ 2(x_\rho y_\sigma - x_\sigma y_\rho) \sin(\theta_\rho - \theta_\sigma) \\&= 2(x_\rho x_\sigma + y_\rho y_\sigma).\end{aligned}\quad (58)$$

This is a contradiction. As a whole, if equality (55) holds for states $\rho \in S(H)$, we have

$$\begin{aligned}\psi(\rho) &= \begin{pmatrix} \frac{1+z_\rho}{2} & \frac{x_\rho + iy_\rho}{2} e^{i\theta_\rho} \\ \frac{x_\rho - iy_\rho}{2} e^{-i\theta_\rho} & \frac{1-z_\rho}{2} \end{pmatrix} = \begin{pmatrix} \frac{1+z_\rho}{2} & \frac{x_\rho + iy_\rho}{2} e^{i(\theta_\rho - \theta_0)} e^{i\theta_0} \\ \frac{x_\rho - iy_\rho}{2} e^{-i(\theta_\rho - \theta_0)} e^{-i\theta_0} & \frac{1-z_\rho}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1+z_\rho}{2} & \frac{x_\rho + iy_\rho}{2} e^{i\theta_0} \\ \frac{x_\rho - iy_\rho}{2} e^{-i\theta_0} & \frac{1-z_\rho}{2} \end{pmatrix} = \begin{pmatrix} e^{i(\theta_0/2)} & 0 \\ 0 & e^{-i(\theta_0/2)} \end{pmatrix} \begin{pmatrix} \frac{1+z_\rho}{2} & \frac{x_\rho + iy_\rho}{2} \\ \frac{x_\rho - iy_\rho}{2} & \frac{1-z_\rho}{2} \end{pmatrix} \begin{pmatrix} e^{-i(\theta_0/2)} & 0 \\ 0 & e^{i(\theta_0/2)} \end{pmatrix}.\end{aligned}\quad (59)$$

So $\psi(\rho) = W\rho W^*$, where

$$W = \begin{pmatrix} e^{i(\theta_0/2)} & 0 \\ 0 & e^{-i(\theta_0/2)} \end{pmatrix}. \quad (60)$$

If equality (56) holds for all states $\rho \in S(H)$, we have

$$\begin{aligned}\psi(\rho) &= \begin{pmatrix} \frac{1+z_\rho}{2} & \frac{x_\rho + iy_\rho}{2} e^{i\theta_\rho} \\ \frac{x_\rho - iy_\rho}{2} e^{-i\theta_\rho} & \frac{1-z_\rho}{2} \end{pmatrix} = \begin{pmatrix} \frac{1+z_\rho}{2} & \frac{x_\rho + iy_\rho}{2} e^{i(\theta_\rho - \theta_0)} e^{i\theta_0} \\ \frac{x_\rho - iy_\rho}{2} e^{-i(\theta_\rho - \theta_0)} e^{-i\theta_0} & \frac{1-z_\rho}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1+z_\rho}{2} & \frac{x_\rho - iy_\rho}{2} e^{i\theta_0} \\ \frac{x_\rho + iy_\rho}{2} e^{-i\theta_0} & \frac{1-z_\rho}{2} \end{pmatrix} = \begin{pmatrix} e^{i(\theta_0/2)} & 0 \\ 0 & e^{-i(\theta_0/2)} \end{pmatrix} \begin{pmatrix} \frac{1+z_\rho}{2} & \frac{x_\rho - iy_\rho}{2} \\ \frac{x_\rho + iy_\rho}{2} & \frac{1-z_\rho}{2} \end{pmatrix} \begin{pmatrix} e^{-i(\theta_0/2)} & 0 \\ 0 & e^{i(\theta_0/2)} \end{pmatrix}.\end{aligned}\quad (61)$$

So $\psi(\rho) = W\rho^T W^*$, where ρ^T is the transpose of the matrix ρ and

$$W = \begin{pmatrix} e^{i(\theta_0/2)} & 0 \\ 0 & e^{-i(\theta_0/2)} \end{pmatrix}. \quad (62)$$

Let $U = VW$, the proof is completed. $\square$

Proof of Proposition 7. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are obvious.

(2) $\Rightarrow$ (3). Since $r > 1$, all the involved quantities are finite. Then by the same discussion as the proof of Proposition 15 in Case 1, we have $S^r(\rho) = S^r(\phi(\rho))$, $\phi(P_1(H)) = P_1(H)$, and that ϕ is a bijective map which preserves orthogonality in both directions. By Corollary 1 in [18], there exists a unitary or antiunitary operator U on H such that $\phi(P) = UPU^*$ for any $P \in P_1(H)$.

Let $\psi(\rho) = U^* \phi(\rho) U$ for any $\rho \in S(H)$; it is obvious that $\psi(P) = U^* U P U U^* = P$ ($\forall P \in P_1(H)$). Since $\rho < \phi(\rho)$ or $\phi(\rho) < \rho$, we have $\rho < \psi(\rho)$ or $\psi(\rho) < \rho$. Now fix a $\rho \in S(H)$. Since $S^r(\rho) = S^r(\phi(\rho)) = S^r(U\psi(\rho)U^*) = S^r(\psi(\rho))$, by Theorem 1, there exists a unitary operator W_ρ (depending on ρ and ψ) on H such that $\psi(\rho) = W_\rho \rho W_\rho^*$; i.e., $\rho = W_\rho^* \psi(\rho) W_\rho$. By equality (27), we have, for any $P \in P_1(H)$,

$$\begin{aligned}S^r(t\rho + (1-t)P) &= S^r(t\phi(\rho) + (1-t)\phi(P)) \\ &= S^r(tU\psi(\rho)U^* + (1-t)U\psi(P)U^*) \\ &= S^r(t\psi(\rho) + (1-t)\psi(P)) \\ &= S^r(t\psi(\rho) + (1-t)P) \\ &= S^r(tW_\rho \rho W_\rho^* + (1-t)P).\end{aligned}\quad (63)$$

By Lemma 11, we know $\rho = W_\rho \rho W_\rho^* = W_\rho W_\rho^* \psi(\rho) W_\rho W_\rho^* = \psi(\rho)$; then $\phi(\rho) = U\psi(\rho)U^* = U\rho U^*$. $\square$

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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