Decay Rate and Energy Gap for the Singularity Solution of the Inhomogeneous Landau-Lifshitz Equation on $S^2$

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1. Introduction

The nonlinear ferromagnetic chain model (FCM) has attracted the attention of physicists and mathematicians. For physicists, FCM represents the possibility of describing magnetic field density evolution with various external fields, while mathematicians study the various solutions or excitations in geometrically motive nonlinear models. Quantitative models depicting FCM were proposed by Landau and Lifshitz in 1935 [1], and the Landau-Lifshitz equation (LLE) was proposed to the dynamics of the nonequilibrium magnetism system:

$$S_t = \alpha S \wedge (\Delta S + H) - \beta S \wedge (S \wedge (\Delta S + H)),$$

where $H$ is the variational sum of magnetic anisotropy energy and magnetic field energy, $S = (S_1, S_2, S_3)$ denotes magnetization intensity, $\wedge$ is the cross product of Euclidean 3-space $R^3$, and $\alpha$ and $\beta$ are constants. If $H = 0$ and $\alpha = 0$, then LLE is the harmonic map heat flow (HMF). Otherwise, when $\beta = 0$, the Gilbert damping vanishes, and when $\beta = 0$ and $H = 0$, the LLE degenerates into the Schrödinger map heat flow (SMF), which is an important equation of differential geometry. SMF can be regarded as a nonlinear Schrödinger equation that contains a derivative term. Although the existence, uniqueness, and the blow-up problem of some nonlinear Schrödinger equations [2–4] are clear, the theorem of SMF becomes more complicated and some further work still needs to be done. Similarly, comparing HMF and some general harmonic system (or even biharmonic equation) [5–8], the mapping system is more complicated than the nonmapping system due to the curvature flow of the Riemannian manifolds.

For clarity, we set $f$ and $h$ as the complex structure and metric on the Kähler manifold, respectively. Different manifolds and their covariant derivatives are $M$ and $D$, respectively, and we define SMF as

$$\frac{\partial S}{\partial t} = \sum_{l=1}^{3} D^l \partial_l S.$$  (2)

Thus, (2) defines a mapping $S: R^3 \times R \rightarrow (M, h, J)$, which can degenerate into other subcases; for example, when $M = S^2$ and $J = SA$, (2) is the isotropic LLE without Gilbert damping, and if the manifold and complex structure are $H^2$ and $\lambda (\overrightarrow{a} \wedge \overrightarrow{b} = (\overrightarrow{a} \wedge \overrightarrow{b}) \text{diag}[1, 1, -1])$, respectively, then (2) is the hyperbolic isotropic LLE without Gilbert damping.

If $M = S^2$ and $J = SA$, then (2) can transform into a Schrödinger-type equation using the Hashimoto transform.
Under $n$ dimensional cylindrical coordinates, let $r = |\mathbf{x}|$, and the curvature and torsion are, respectively,

$$
\kappa = (S_r \cdot S_t)^{1/2},
\tau = \frac{S \cdot (S_t \Lambda S_r)}{\kappa^2}.
$$

Then, employing the Hashimoto transform,

$$
Q = \frac{\kappa}{2} \exp \left[ i \int \tau (t, \mathbf{r}) \, dr \right],
$$

and (2) transforms to

$$
i \frac{\partial}{\partial t} Q + \frac{\partial^2}{\partial r^2} Q + \frac{1}{r} (n-1) \frac{\partial}{\partial r} Q - \frac{1}{r^2} (n-1) Q
+ 2 |Q|^2 Q = 0.
$$

If $n = 1$, (5) is the standard cubic Schrödinger equation for which the Lax pair is obtained, and explicit solutions can be constructed using the Bäcklund transformation. Hence, (5) can be regarded as a simple case of (2). Similar to the inhomogeneous Schrödinger equation, (2) (or (1)) can be extended to the inhomogeneous expression. If we add the inhomogeneous term $q(t, \mathbf{x})$ to (2), SME can be generalized into the inhomogeneous SME (ILL) proposed by Balakrishnan [9]:

$$
\frac{\partial}{\partial t} S = q(t, \mathbf{x}) S \wedge \Delta S + \nabla q(t, \mathbf{x}) (S \wedge \nabla S),
$$

where the inhomogeneous term, $q(t, \mathbf{x})$, is a scalar function. Under radially symmetrical coordinates, (6) can be expressed as

$$
\frac{\partial}{\partial t} S - q(t, r) S \wedge \left( \frac{\partial^2}{\partial r^2} S + \frac{n-1}{r} \frac{\partial}{\partial r} S \right)
- \frac{\partial}{\partial r} q(t, r) (S \wedge \frac{\partial}{\partial r} S) = 0.
$$

Using the Hashimoto transform (4), (7) transforms into a nonlinear inhomogeneous Schrödinger equation:

$$
\begin{align*}
&i \frac{d}{dr} Q + q \left( \frac{d^2}{dr^2} Q + \frac{1}{r} (n-1) \frac{d}{dr} G - \frac{1}{r^2} (n-1) Q \\
&+ 2 |Q|^2 Q \right) + Q \left( \frac{d^2}{dr^2} q + \frac{1}{r} (n-1) \frac{d}{dr} q \right) \\
&+ 2 \left( \frac{d}{dr} q \right) |Q|^2 dr + 4 (n-1) \int \frac{1}{r} |Q|^2 \, dr \\
&+ 2 \left( \frac{d}{dr} \frac{d}{dr} q \right) \frac{\partial}{\partial r} Q = 0.
\end{align*}
$$

Daniel et al. [10] identified the integrable model (8) by analyzing the singularity structure of its solutions and also discussed Lax pairs, Bäcklund transformation, and soliton-like solutions of (7) (or (8)). They showed that inhomogeneous terms play an important role, and Painlevé analysis indicated that (7) (or (8)) was integrable in arbitrary $n$ dimensions if and only if $q(t, r) = \varepsilon_t r^{-2(n-1)} + \varepsilon_r r^{-(n-2)}$.

As all know, the bifurcation structures of the general integrable systems and autonomous differential systems [11–14] are clear due to the properties of Hamiltonian systems [15–18]. However, the structure for the unintegrable system is not so clear. If $q(t, r)$ takes other forms, what would be the solution of (7)? The regularity of the unintegrable case has been somewhat clarified [19, 20] in regard of the singularity behavior of the equation. Two different finite time blowup solutions of (7) were constructed in [20], one being an explicit form and the other being an implicit solution. Careful estimation of the two spatial dimension implicit solutions indicates that the energy density decay rate is $1/r$. However, if the spatial dimension is any integer $n$, does a global smooth (or blowup) solution of (7) exist? Does (7) also present similar regularity aspects under specific smooth initial data? The regularity of $n$ dimensional ILL remains open.

Many physical phenomena develop singular behavior, for example, boundary layers or blowup solutions. Liu [21] analyzed the concentration set of the stationary weak solutions to LLE for the ferromagnetic spin chain. Based on the Ginzburg-Landau approximation, Wang [22] established the existence of a global weak solution for LLE, $n < 4$, with respect to smooth initial boundary data, which is smooth from a closed set with locally finite $n$ dimensional Hausdorff measure. In 2008, Huh [23] constructed infinite energy explicit blowup solutions for the modified LLE. Ding [24] constructed an infinite energy blowup solution for LLE on a hyperbolic target.

The finite energy blowup solution was firstly studied by perturbation methods [25, 26], and it was found that the collapse of a symmetric case with large initial energy formed a singularity where the blowup rate could be estimated by scale invariance. To obtain the required resolution for evolving similar solutions, adaptive mesh refinement (AMR) is required [26–28], which dynamically and locally adds numerical resolution. Although Van Den Berg et al. [26] present a formal analysis and AMR simulation for LLE with the Gilbert term, rigorous proof of blowup for 2-dimensional (2D) LLE remains open. However, Ding et al. [29] proved that $n = 3$ or $n = 4$ dimensional LLE with the Gilbert term will lead to a finite time blowup under specific initial boundary conditions. Some regularity and blowup results for LLE were derived considering the Maxwell field [30, 31].

Although the blowup problem is clear for specific settings (initial state or initial boundary conditions), the blowup problem for ILL is unclear. The one-dimensional ILL will not form a singularity as the inhomogeneous term is periodic [32]. However, if the inhomogeneous term is in some other specific format [33], blowup occurs for the inhomogeneous HMF.

We studied the blowup and energy gap for $n$ dimensional aperiodic ILL on $S^n$ target and investigated what happens as $t$ tends to infinity when the initial data is smooth and
sufficiently large, in particular whether the solution develops distinct behaviors (finite or global time singularity) under the large data. Global smooth (or blowup) theory for ILL was not established, but we discuss some special solutions that form a singularity in finite time and classify these solutions and analyses based on their energy density.

In Section 2, we obtain a blowup solution for the ILL and derive the exact decay rate of the energy density for that solution. In Section 3, we obtain another local blowup solution that contains an energy gap under the inhomogeneous term and classify this solution into four types. In Section 4, we obtain the decay rate and prove the decreasing (or increasing) property of the local blowup solution for one of these types under some specific coefficient settings. Finally, in Section 5, we summarize the paper and present our conclusions.

2. Blowup Solution on a Sphere

2.1. Blowup Solution Derivation. Selecting the appropriate solution to construct the blowup solution is difficult. Group invariance (under $O(3,1)$ or $SL(2,C)$) can be applied to search for exact solutions of LLE [34–36], but ILL will not admit any group invariance. Following [19, 20, 37], we use cylindrical coordinates,

\begin{align}
S_1 (t, r) &= \cos (m (t, r)) \sin (f (r)) , \\
S_2 (t, r) &= \sin (m (t, r)) \sin (f (r)) , \\
S_3 (r) &= \cos (f (r)) ,
\end{align}

where $r = | \vec{x} |$, $m(t, r)$, and $f(r)$ are functions to be determined.

For simplicity, consider the case where $q(t, \vec{x})$ does not contain $t$ and denote $q(r), m(t, r)$, and $f(r)$ as $q$, $m$, and $f$, respectively. Then, from (9), (7) will be greatly simplified and maybe is transformed into

\begin{align}
2 \cos (f) \left( \frac{d}{dr} f \right) \left( \frac{m}{\partial / \partial \tau} \right) m &+ q \frac{\partial^2 m}{\partial r^2} m \\
&+ \left( \frac{d}{dr} g \right) \frac{\partial m}{\partial r} m + (n - 1) q \frac{\partial (f)}{\partial r} m \frac{1}{r} = 0
\end{align}

and

\begin{align}
q \sin (f) \cos (f) \left( \frac{\partial f}{\partial r} \right) ^2 &+ \left( \frac{\partial f}{\partial m} \right) \sin (f) r \\
&- q \left( \frac{d^2}{dr^2} f \right) r - \left( \frac{d}{dr} q \right) \left( \frac{d}{dr} f \right) r \\
&- (n - 1) q \frac{d}{dr} f = 0,
\end{align}

which are nonlinear partial differential equations. To solve these equations, let us assume that $m$ is a variable separation function:

\begin{equation}
m = g (r) h (t) + C,
\end{equation}

where $g(r)$ and $h(t)$ are dependent on $r$ and $t$, respectively; and $C$ is a constant. Since $q(t, \vec{x})$ does not contain $t$, $g(r)$ and $h(t)$ are denoted as $g$ and $h$, respectively. Substituting (12) into (10) and (11),

\begin{align}
2 \left( \frac{d}{dr} g \right) q \cos (f) \left( \frac{d}{dr} f \right) r &+ \left( \frac{d}{dr} g \right) \sin (f) \left( \frac{d}{dr} q \right) r + q \left( \frac{d^2}{dr^2} g \right) \sin (f) r \\
&+ (n - 1) \left( \frac{d}{dr} g \right) q \sin (f) = 0,
\end{align}

\begin{align}
q \left( \frac{d^2}{dr^2} f \right) r + \left( \frac{d}{dr} q \right) \left( \frac{d}{dr} f \right) r + (n - 1) q \frac{d}{dr} f = 0,
\end{align}

\begin{align}
\cos (f) \left( \frac{d}{dr} g \right) ^2 q - g = 0,
\end{align}

and

\begin{equation}
h = \frac{1}{1 - T},
\end{equation}

where $T > 0$. Solving (13)–(15),

\begin{align}
f &= 2 \arctan (e^F), \\
q &= \frac{C_1 (e^{-F} + e^F)}{2 ((d/dr) F) r^{n-1}},
\end{align}

and

\begin{equation}
g = \frac{1}{2} C_3 \left( -1 + e^{-2F} \right) e^F,
\end{equation}

where $C_1$ and $C_2$ are constants, $F$ is a function of $r$, and $F$ satisfies

\begin{equation}
\begin{aligned}
\frac{1}{n} r^n &- \frac{1}{2} C_3 C_1 F - \frac{1}{8} C_3 C_1 e^{2F} + \frac{1}{8} C_3 C_1 e^{-2F} + C_4 = 0,
\end{aligned}
\end{equation}

where $C_4$ is a constant.

The $F$ derivation of (20) is

\begin{equation}
\frac{1}{4} C_3^2 C_1 \left( e^{2F} + 2 + e^{-2F} \right).
\end{equation}

If $C_1$ and $C_3$ are nonzero constants, regardless of the value of $F$, the left sections of (20) and (21) are continuous functions, and (21) is nonzero. Therefore, according to the implicit function theorem, there exists a function $F$ that satisfies (20), and we have the following conclusion.
Theorem 1. Assuming that \( q \) takes the form of (18) and \( F \) satisfies (20), there is a solution of (7):

\[
S_1 (t, r) = \frac{2e^F}{1 + e^{2F}} \cos \left( \frac{C_3 (-e^F + e^{-F})}{2 (t - T)} + C \right),
\]
\[
S_2 (t, r) = \frac{2e^F}{1 + e^{2F}} \sin \left( \frac{C_3 (-e^F + e^{-F})}{2 (t - T)} + C \right),
\]
\[
S_3 (r) = \frac{-1 + e^{2F}}{1 + e^{2F}},
\]

where \( C, C_1, C_3, \) and \( C_4 \) are arbitrary constants; and constant \( T > 0 \).

On \( S^2 \), from (9),
\[
\left| S \right|^2 = S_1^2 + S_2^2 + S_3^2 = \left( \frac{\partial}{\partial r} f \right)^2 + \sin^2 (f) \left( \frac{\partial}{\partial r} m \right)^2,
\]
and \((\partial/\partial r) m)^2\) of (23) is
\[
\left( \frac{\partial}{\partial r} m \right)^2 = \left( \cot^2 (f) + 1 \right)^2 \left( \frac{\partial}{\partial r} f \right)^2 C_2^2 \frac{2}{(t + T)^2}
\]
\[
= \frac{C_2^2 (\partial/\partial r) F^2 (e^{2F} + 2 + e^{-2F})}{4 (t + T)^2},
\]
which indicates that (22) is a solution of (7) which blows up at \( T \).

The energy density of (22) will also blow up, as can be seen from the energy density:
\[
W_E = \frac{C_1 (\partial/\partial r) f}{r^{n+1}} + \frac{C_3 \sin^2 (f) (\partial/\partial r) m^2}{(\partial/\partial r) F r^n}.
\]

2.2. Decay Rate of the Blowup Solution. From the solutions and the energy density of the upper section, it is difficult to see the decay behavior of energy density. The decay characteristics of the solution determine whether solution energy can be finite value across the whole space. Therefore, we investigate the decay rate of the solution using the implicit solution from Section 2.1.

If we take the derivative of (20) with respect to \( r \),
\[
\frac{d}{dr} F = \frac{4r^n}{C_3 C_1 (e^{2F} + 2 + e^{-2F})},
\]
and, from (17), (24), and (25),
\[
W_E = \frac{8e^F}{C_3 (e^{2F} + 2 + e^{-2F}) (1 + e^{2F})}
\]
\[
= \frac{2C_3 (e^{2F} + e^{-F} + 2e^F)}{(e^{2F} + 2 + e^{-2F}) (1 + e^{2F}) (-t + T)^2}.
\]
From (26), \((\partial/\partial r) F \big|_{1} = 0\), so if \( r > 0 \) and \( C_2 C_3 > 0 \),
\[
\frac{d}{dr} F > 0,
\]
where, in this case, \( F \) is a strictly increasing function in the interval \([0, +\infty)\).

Substituting \( F = (n/2) \ln (Kr) \) \((K > 0)\) into the left side of (20),
\[
H (r) = \frac{r^n}{n} - \frac{1}{8} C_1 C_3 (Kr)^n - \frac{1}{4} C_1 C_3 n \ln (Kr)
\]
\[
+ \frac{C_1 C_3}{8 (Kr)^n} + C_4.
\]
If \( r \to +\infty \) and \((1/8) C_3 C_1 K^n - 1/n \neq 0\), then, from (20),
\[
\lim_{r \to +\infty} H (r) = -\frac{n}{4} C_3 C_1 \cdot \infty = -\infty.
\]

Equations (30) and (31) mean that \( K \) determines the limit to be a positive or a negative value of \( H (r) \). Hence,
\[
\frac{n}{2} \ln (K_1 r) < F < \frac{n}{2} \ln (K_2 r) \quad (0 < K_1 < K_2).
\]
Combining \( F = (n/2) \ln (Kr) \) and (27),
\[
W_E
\]
\[
= \frac{8 (Kr)^{3/2} n}{C_4 \left( (Kr)^{2n} + 2 (Kr)^n + 1 \right) (1 + (Kr)^n)}
\]
\[
+ \frac{2C_3 \left( (Kr)^{5n/2} + 2 (Kr)^{3n/2} + (Kr)^n \right)}{(1 + (Kr)^n) (1 + (Kr)^n) (-t + T)^2}.
\]
From (33), as \( r \to +\infty \), the decay rate of the energy density is
\[
W_E \sim \frac{\bar{C}(t)}{r^{n/2}},
\]
where \( \bar{C}(t) \) is function of \( t \); and as \( t \to T \), \( \bar{C}(t) = \infty \).

Thus, we may obtain the decay rate of the \( C_1 C_3 > 0 \) case. If \( C_1 C_3 < 0 \) and \( r \to +\infty \), then the decay rate of the energy density can be similarly proven. Therefore,
\[
F \to -\infty,
\]
where \( F \) satisfies
\[
-\frac{n}{2} \ln (K_1 r) > F > -\frac{n}{2} \ln (K_2 r),
\]
and from (27), if \( r \to +\infty \), we may estimate the solution
\[
W_E \sim \frac{\bar{C}(t) e^F}{r^{n/2}},
\]
where \( \bar{C}(t) \) is a function of \( t \), where if and only if \( t = T \), then \( \bar{C}(t) = \infty \), which has the limit
\[
\lim_{r \to +\infty} W_E = 0.
\]
Thus, $W_E$ is a continuous function with the limit at $+\infty$ of 0. Hence, whether the energy is a finite value depends on the convergence of $W_E$ at 0. From (20), if $r \to 0$,

$$\lim_{r \to 0} F = \tilde{C},$$

where $\tilde{C}$ is constant. Thus,

$$\lim_{r \to 0} W_E = \frac{8 e^C}{C_3 (e^{2C} + 2 + e^{-2C})} \left(1 + e^{2\tilde{C}}\right) + \frac{2C_3 (e^{\tilde{C}} + e^{-\tilde{C}} + 2e^C)}{(e^{2\tilde{C}} + 2 + e^{-2\tilde{C}}) \left(1 + e^{2\tilde{C}}\right)} (t + T)^2,$$

and if $\tilde{C} = 0$, then

$$\lim_{r \to 0} W_E = \frac{T^2 - 2(T + 2 + C_3^2)}{T^2C_3 - 2T^2C_3 + t^2C_3}. \quad (41)$$

We can obtain the initial energy density from (41) by setting $t = 0$:

$$\lim_{r \to 0} W_{E_0} = \frac{T^2 + C_3^2}{T^2C_3}. \quad (42)$$

From (40), the initial energy density is a finite value and if and only if $t = T$, $W_E$ will be in infinite value. Combining this and the decay behavior as $r \to +\infty$, the energy of the finite spatial area can be estimated as

$$\int_0^A q (r) \left| S_i (t, r) \right|^2 r^{n-1} dr < \tilde{C} (t), \quad (43)$$

where $A$ is a constant and $\tilde{C}(t) = \infty$ if and only if $t = T$. Furthermore, the total energy of the spatial region is

$$\int_0^\infty q (r) \left| S_i (t, r) \right|^2 r^{n-1} dr = \infty. \quad (44)$$

From the above analysis, we may conclude that, assuming that $q$ takes the form of (18) and $F$ satisfies (20), there is the following solution of (7).

**Theorem 2.** If $C_i C_3 \neq 0$, $q$ takes the form of (18) and $F$ satisfies (20), the energy density of (22) is

$$W_E \sim \frac{\tilde{C} (t)}{r^{n/2}}, \quad (45)$$

as

$$r \to +\infty, \quad (46)$$

and the energy in $r \in [0, A]$ satisfies

$$E [0, A] = \int_0^A q (r) \left| S_i (t, r) \right|^2 r^{n-1} dr \leq \tilde{C} (t), \quad (47)$$

where $A$ is constant; and $\tilde{C}(t)$ depends on $t$, such that as $t \to T$, $\tilde{C}(t) = \infty$.

Similar to the proof of [20], the decay rate of the norm of gradient can now be estimated. Combining (18) and (26),

$$q = \frac{1}{8} C_1^2 r^{-2n+2} e^{-F} \left(1 + e^{2F}\right) C_3 \left(e^{2F} + 2 + e^{-2F}\right), \quad (48)$$

and to estimate the decay rate as $r \to +\infty$, we substitute $F = (n/2) \ln(Kr) \; (K > 0)$:

$$q = \frac{1}{8} C_1^2 C_3 \left((Kr)^{n/2} + (Kr)^{-n/2} + 3 (Kr)^{n/2}\right) + 3 (Kr)^{-n/2}. \quad (49)$$

Hence,

$$q \sim r^{-n/2+2}, \quad r \to +\infty. \quad (50)$$

The gradient norm and energy density are related:

$$W_E = q \left| S_i \right|^2, \quad (51)$$

so that, combining Theorem 1 and $q \sim r^{-n/2+2}$, as $r \to +\infty$,

$$\left| S_i \right|^2 \sim \frac{\tilde{C} (t)}{r^2}, \quad (52)$$

where $\tilde{C}(t)$ is function of $t$, such that if and only if $t \to T$, $\tilde{C}(t) = \infty$.

### 3. Multiple Branches of the Blowup Solutions

We investigate the classification of the blowup solution. Since (7) may contain many solutions different from (22), we extend (22) to a general form to find other solutions. In particular, we use (9) as an undetermined solution and assume that $C_i (i = 1, 2, 3, 4), C$, and $T > 0$ are constant. We set

$$m = gh + C, \quad (53)$$

where

$$g = C_2 + \frac{1}{2} C_3 \left(-1 + e^{-2F}\right) e^F, \quad (54)$$

$$h = \frac{1}{t - T}, \quad (55)$$

and

$$q = \frac{C_1 \left(e^{-F} + e^F\right)}{2((d/dr) F) r^{n-1}}. \quad (56)$$

Therefore,

$$\frac{d}{dr} F = \frac{4 r^{n-1} \left(e^{2F} C_3 - 2 e^F C_2 - C_3\right)}{C_1 C_3^2 \left(-e^{-2F} + e^{2F} + e^F - 1\right)}. \quad (57)$$
If we set
\[ C_4 = \frac{1}{8} C_1 C_3 \left( e^{2F} + e^{-2F} \right) + \frac{1}{2} C_1 C_2 \left( e^F + e^{-F} \right) + \frac{C_1 B C_2^2}{C_3} + \frac{1}{2} C_1 C_3 B \]
\[ + \left( \frac{2 C_1 C_2 \sqrt{C_2^2 + C_2^2}}{C_3} \right) \arctanh \left( \frac{e^F C_3 - C_2}{\sqrt{C_2^2 + C_2^2}} \right) \]
then we can obtain a solution of (56) which satisfies the initial condition \( F(0) = B \) if and only if
\[ C_3^n - \frac{1}{8} C_3^2 C_1 e^{2F} - \frac{1}{2} C_1 C_2 C_3 e^F - \frac{1}{2} C_3^2 C_1 F + \frac{1}{4} C_3^2 C_1 e^{-2F} - \frac{1}{2} C_3 C_1 e^{-F} - C_1 B C_2^2 + C_3 C_4 \]
\[ + \left( \frac{2 C_1 C_2 \sqrt{C_2^2 + C_2^2}}{C_3} \right) \arctanh \left( \frac{e^F C_3 - C_2}{\sqrt{C_2^2 + C_2^2}} \right) = 0. \]

The derivative of the left of (58) is
\[ -1 \frac{C_3^2}{4} \left( e^{4F} + e^{2F} - e^{-2F} - 1 \right) C_1. \]

We set \( C_1, C_2, \) and \( C_3 \neq 0. \) Thus, (59) is 0 if and only if \( F = 0, \) and
\[ F \neq \ln \left( \frac{C_2 \pm \sqrt{C_2^2 + C_2^2}}{C_3} \right). \]

Regardless of \( F \) value, (59) and the left section of (58) are continuous functions, and (59) is not zero. According to the existence theorem of implicit functions, for any
\[ F \in R \setminus \left\{ 0, \ln \left( \frac{C_2 \pm \sqrt{C_2^2 + C_2^2}}{C_3} \right) \right\}, \]
there is an \( F \) that satisfies (58).

From the above process, we obtain the following conclusion:

**Theorem 3.** Assuming that \( q \) takes the form of (18) and \( F \) satisfies (58), then there is a solution of (7):

\[ S_1(t, r) = \frac{2e^F}{1 + e^{2F}} \cos \left( \frac{C_3 \left(-e^F + e^{-F}\right) + C_2 t - T + C}{2 \left(t - T\right)} \right), \]
\[ S_2(t, r) = \frac{2e^F}{1 + e^{2F}} \sin \left( \frac{C_3 \left(-e^F + e^{-F}\right) + C_2 t - T + C}{2 \left(t - T\right)} \right), \]
\[ S_3(r) = \frac{-1 + e^{2F}}{1 + e^{2F}}, \]

where \( C, C_1, C_2, C_3, \) and \( C_4 \) are any constants; and \( T > 0 \) is constant.

The energy density of the solution from Theorem 3 is
\[ W_r = \frac{2e^{-F} \left( e^{2F} C_1 - 2e^{2F} C_2 - C_3 \right) \left(C_3^2 (1 + e^{2F}) + 4e^{2F} \left(t - T\right)^2\right)}{C_3 \left(1 + e^{2F}\right) \left(-e^{2F} + e^{2F} + e^{4F} - 1\right) \left(-t + T\right)^2}. \]

Although (62) is similar to (22), their respective energies ((63) and (27)) are different. In (62), \( F \) satisfies (58) which contains \( \arctanh, \) and this defines the range of \( F:\n\]
\[ -1 < \frac{e^F C_3 - C_2}{\sqrt{C_2^2 + C_2^2}} < 1. \]

If
\[ \frac{e^F C_3 - C_2}{\sqrt{C_2^2 + C_2^2}} = \pm 1, \]
then
\[ F = \ln \left( \frac{\pm \sqrt{C_2^2 + C_2^2} + C_2}{C_3} \right). \]

For (56), if \( F \) takes the form of (66), the right of (56) is 0, and \( F_1 \) is 0.

We must solve
\[ \frac{C_1 e^{2F} - 2C_2 e^F - C_3}{\left(-e^{-2F} + e^{2F} + e^{4F} - 1\right) C_1} > 0, \]
which can be expressed as
\[ \frac{C_3 \left(e^F - \left(C_2 + \sqrt{C_2^2 + C_3^2} \right) / C_3\right) \left(e^F - \left(C_2 - \sqrt{C_2^2 + C_3^2} \right) / C_3\right)}{(e^{4F} - 1)(e^{-2F} + 1)} > 0. \]
If $C_3 > 0$, then
\[ \frac{\sqrt{C_2^2 + C_3^2 + C_2}}{C_3} > 0, \] (69)
and if $C_3 > 0$ and $C_2 < 0$, then
\[ \ln \left( \frac{\sqrt{C_2^2 + C_3^2 + C_2}}{C_3} \right) < 0. \] (76)

and
\[ -\frac{\sqrt{C_2^2 + C_3^2 + C_2}}{C_3} < 0. \] (70)

Solving
\[ \left( e^{4F} - 1 \right) \left( e^{-2F} + 1 \right) > 0, \] (71)
we obtain
\[ F > 0. \] (72)

Similarly, solving
\[ \left( e^{4F} - 1 \right) \left( e^{-2F} + 1 \right) < 0, \] (73)
we obtain
\[ F < 0. \] (74)

If $C_3 > 0$ and $C_2 > 0$, then the inequality is
\[ \ln \left( \frac{\sqrt{C_2^2 + C_3^2 + C_2}}{C_3} \right) > 0, \] (75)
and if $C_3 > 0$ and $C_2 < 0$, then

Thus, if $C_3 > 0$, $C_2 > 0$, $C_1 < 0$, and
\[ 0 < F < \ln \left( \frac{\sqrt{C_2^2 + C_3^2 + C_2}}{C_3} \right), \] (77)
then (68) holds, and since $r > 0$,
\[ F_r > 0. \] (78)

If $r > 0$, $C_3 > 0$, $C_2 < 0$, $C_1 > 0$, and
\[ -\infty < F < \ln \left( \frac{\sqrt{C_2^2 + C_3^2 + C_2}}{C_3} \right), \] (79)
then
\[ F_r > 0. \] (80)

Similarly, if $C_3 < 0$, we can solve $F$ from $F_r > 0$, and we can solve $F_r < 0$ for the $C_3 > 0$ (or $C_3 < 0$) case. From (58) and (56), if $C_3 > 0$, the change of $F_r$ is shown in Table 1.

Similar to Table 1, if $C_3 < 0$, the change of $F_r$ is shown in Table 2.

From Tables 1 and 2, $F$ satisfying (58) can be subdivided into the following categories:

<table>
<thead>
<tr>
<th>$C_1 &gt; 0$</th>
<th>$C_3 &gt; 0$, $C_2 &gt; 0$</th>
<th>$C_3 &gt; 0$, $C_2 &lt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1 &gt; 0$</td>
<td>If $0 &lt; F &lt; \ln \left( \frac{\sqrt{C_2^2 + C_3^2 + C_2}}{C_3} \right)$, $F_r &lt; 0$</td>
<td>If $-\infty &lt; F &lt; \ln \left( \frac{\sqrt{C_2^2 + C_3^2 + C_2}}{C_3} \right)$, $F_r &gt; 0$</td>
</tr>
<tr>
<td>$C_1 &lt; 0$</td>
<td>If $0 &lt; F &lt; \ln \left( \frac{-\sqrt{C_2^2 + C_3^2 + C_2}}{C_3} \right)$, $F_r &gt; 0$</td>
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</tr>
</tbody>
</table>
(1) $C_3 > 0$, $C_2 > 0$ and $C_1 > 0$ (or $C_3 < 0$, $C_2 < 0$ and $C_1 > 0$).

$F$ is a continuous curve. If $F = 0$, solving (58),

\[ r_0 = \left[ \frac{n}{C_3} \left( 2 \operatorname{arctanh} \left( \frac{-C_3 + C_2}{\sqrt{C_2^2 + C_3^2}} \right) \right) \cdot \sqrt{C_2^2 + C_3^2 C_1 C_2 + C_1 C_2 C_3 - C_3 C_4} \right]^{1/n}, \]  

(81)

and (56) can be expressed as

\[ r^{n-1} \frac{dr}{dF} = \frac{C_1 C_3}{4} \frac{\left( -e^{-2F} + e^{2F} + e^{4F} - 1 \right)}{e^{2F} C_3 - 2 e^F C_2 - C_3}. \]  

(82)

If the right section of (82) is 0, then $F = 0$. Hence, $r_F = 0$ as $r = r_0$. Thus, the tangent of $F$ is perpendicular to the $r-$axis at $(r_0, 0)$.

If $r = 0$, then $F_r = 0$ from (56). Hence, the tangent of $F$ is parallel to the $r-$axis at $(0, F(0))$. Specifically, in the fourth quadrant of the $rF$ coordinate system, $F$ is monotonically increasing and $r \in [0, r_0]$. In the first quadrant, $F$ is monotonically decreasing and $r \in [0, r_0]$.

For example, if $n = 2$, $C_1 = C_2 = C_3 = 1$, and $C_4 = 0$, then (58) is

\[ \frac{1}{2} r^2 - \frac{1}{8} e^{2F} - \frac{1}{2} e^F - \frac{3}{2} F - \frac{1}{2} e^{-F} + \frac{1}{8} e^{-2F} + 2 \sqrt{2} \operatorname{arctanh} \left( \frac{1}{2} e^F \sqrt{2} - \frac{1}{2} \sqrt{2} \right) = 0, \]  

(83)

and the evolution of $F$, determined by (83), is shown in Figure 1.

(2) $C_3 > 0$, $C_2 > 0$ and $C_1 < 0$ (or $C_3 < 0$, $C_2 < 0$ and $C_1 < 0$).

The evolution of $F$ in the first and the fourth quadrant are opposite that of case (1). In the fourth quadrant, $F$ is monotonically decreasing, and in the first quadrant, $F$ is monotonically increasing. The tangent of $F$ is perpendicular to the $F-$axis at $(0, F(0))$. If we set

\[ F_{\max} = \max \left( \ln \left( \frac{\sqrt{C_2^2 + C_3^2 + C_2}}{C_3} \right), \right) \]  

\[ \ln \left( \frac{-\sqrt{C_2^2 + C_3^2 + C_2}}{C_3} \right) \right), \]  

(84)

the maximum value of $F$ in first quadrant will not exceed $F_{\max}$.

Specifically, if $n = 2$, $C_1 = -1$, $C_2 = C_3 = 1$, and $C_4 = 0$, then (58) is

\[ \frac{1}{2} r^2 + \frac{1}{8} e^{2F} + \frac{1}{2} e^F + \frac{3}{2} F + \frac{1}{2} e^{-F} - \frac{1}{8} e^{-2F} + 2 \sqrt{2} \operatorname{arctanh} \left( \frac{1}{2} e^F \sqrt{2} + \frac{1}{2} \sqrt{2} \right) = 0, \]  

(85)

and the evolution of $F$ from (85) is shown in Figure 2.

(3) $C_3 > 0$, $C_2 < 0$ and $C_1 > 0$ (or $C_3 < 0$, $C_2 > 0$ and $C_1 < 0$).

$F$ is completely in the fourth quadrant, and is monotonically increasing. The maximum value of $F$ will not
Figure 3: Evolution of $F$ from (86) with $r \in [0,5]$ and $F \in [-4,4]$.

Figure 4: Evolution of $F$ from (87) with $r \in [0,10]$ and $F \in [-4,4]$.

$F$ exceed $F_{\max}$, and the curve in this quadrant has lower bound $F(0)$.

For example, when $n = 2$, $C_1 = 1$, $C_2 = -1$, $C_3 = 1$, and $C_4 = 1$, (58) is

$$
\frac{1}{2} r^2 - \frac{1}{8} e^{2F} + \frac{1}{2} e^F - \frac{3}{2} F + \frac{1}{2} e^{-F} + \frac{1}{8} e^{-2F} + 1
$$

and Figure 3 shows the evolution of $F$ from (86) with $r \in [0,5]$ and $F \in [-4,4]$.

(4) $C_3 > 0$, $C_2 < 0$ and $C_1 < 0$ (or $C_3 < 0$, $C_2 > 0$ and $C_1 > 0$).

$F$ is monotonically decreasing in the fourth quadrant, with upper bound $F(0)$.

Cases (1) and (2) are multi-value, i.e., for each $r$, there are two values of $F$. We call these multi-branch functions. However, there is only a single monotonic function for cases (3) and (4).

Let us select some specific parameters and investigate the characteristic solutions. If $n = 2$, $C_1 = C_2 = -1$, $C_3 = 1$, and $C_4 = 1$, then (58) is

$$
\frac{1}{2} r^2 - \frac{1}{8} e^{2F} + \frac{1}{2} e^F - \frac{3}{2} F + \frac{1}{2} e^{-F} + \frac{1}{8} e^{-2F} + 1
$$

and Figure 3 shows the evolution of $F$ from (86) with $r \in [0,5]$ and $F \in [-4,4]$.

$F(z)$ and $r(z)$ cannot be expressed as a Hamiltonian system; that is, there is no function $H(F,r)$ such that

$$
\frac{\partial}{\partial F} H(F,r) = \frac{d}{dz} r, \\
\frac{\partial}{\partial r} H(F,r) = \frac{d}{dz} F.
$$

Therefore, (90) are incompatible systems. Solving the first equation of (90),

$$
H(F,r) = -C_1 C_3 2^{2F} \left(-\frac{1}{2} e^{-2F} + \frac{1}{2} e^F - \frac{1}{4} e^{4F} \right) - C_1 C_3 F + F_1 (r),
$$

where $F_1$ is a function that is only dependent on $r$; and solving the second equation of (90),

$$
H(F,r) = -\frac{4r^n}{n} \left(2 e^F C_3 - e^{2F} C_2 + C_3 \right) + F_2 (F),
$$

where $F_2$ is a function that is only dependent on $F$.

Since (91) and (92) are contradictory expressions, (90) is an incompatible system.

Equations (88)-(89) contain more solutions than (58). The evolution of these equations can also be classified into four types as follows:

1. Multibranch elliptic case I: $C_3 > 0$, $C_2 > 0$, and $C_1 > 0$ (or $C_3 < 0$, $C_2 < 0$, and $C_1 < 0$).

2. Multibranch hyperbolic case I: $C_3 > 0$, $C_2 > 0$, and $C_1 < 0$ (or $C_3 < 0$, $C_2 < 0$, and $C_1 < 0$).
3. Multibranch hyperbolic case II: \( C_3 > 0, C_2 < 0 \), and \( C_1 < 0 \) (or \( C_3 < 0, C_2 > 0 \), and \( C_1 > 0 \)).

4. Multibranch elliptic case II: \( C_3 > 0, C_2 < 0 \), and \( C_1 < 0 \) (or \( C_3 < 0, C_2 > 0 \), and \( C_1 > 0 \)).

To see the evolution of these four solutions directly, we set \( n = 2 \) and \( C_1, C_2, \) and \( C_3 \) as follows:

1. \( C_1 = C_2 = C_3 = 1 \).
2. \( C_1 = -1 \) and \( C_2 = C_3 = 1 \).
3. \( C_1 = 1, C_2 = -1, \) and \( C_3 = 1 \).
4. \( C_1 = C_2 = -1 \) and \( C_3 = 1 \).

Figures 5–8 show the orientation fields for these four cases in the \( rF \) coordinate, respectively.

The direction of vector field in Figure 5 is opposite to that of Figure 8, and the direction of Figure 6 is opposite to that of Figure 7. Figures 5 and 8 (or Figures 6 and 7) are very similar. However, in Section 4, we show that the energy densities suggest that all of these four multibranch cases are distinct.

4. Blowup Solution and Energy Gap

The solutions presented in Sections 2 and 3 will blow up in finite time, but the energy of the solution presented in Section 2 is infinite on all spatial regions. However, can (7) (or (6)) contain some finite energy solutions on the sphere, since solution (62) presented in Section 3 contains some finite energy subcases? In this section, we investigate the blowup solution (62) and particularly the finite energy subcases.

Let us analyze the properties of the energy density (63) of solution (62). Since (63) contains more parameters than (27), the evolution of \( F_r \) is more complex for (56) than (26), which leads to more complex evolutionary behavior of (63).

If \( C_3 > 0, C_2 < 0 \), and \( C_1 < 0 \) (or \( C_3 < 0, C_2 > 0 \), and \( C_1 > 0 \)), then \( F_r \) is negative in \([0, +\infty)\), as shown in Figures 4 and 8. Substituting \( F = -(n/2) \ln(Lr) \) \((L > 0)\) into the left of (58), we obtain a function \( J(r) \), with the limit

\[
\lim_{r \to r_{bc}} J(r) = \frac{1}{8n} C_3 \text{sign} \left( C_1 C_3 + 8 \right) \cdot \infty,
\]

where \( K^n C_1 C_3 + 8 \neq 0 \).

From (93), the positive and negative of \( \lim_{r \to r_{bc}} J(r) \) can be determined by \( L \); that is,

\[
-\frac{n}{2} \ln(L_1 r) > F > -\frac{n}{2} \ln(L_2 r)
\]

\((0 < L_1 < L_2), \; r \to +\infty\).
Substituting $F = -\frac{n}{2} \ln(Lr)$ into (63), the decay rate of the energy density as $r \to +\infty$ is

$$W_E \sim \frac{\overline{C}(t)}{r^{n/2}},$$  \hspace{1cm} (95)$$

where $\overline{C}(t)$ is a function of $t$, such that as $t \to T$, $\overline{C}(t) = \infty$.

Similar to the proof of Theorem 2, we have the following corollary.

**Corollary 4.** If $F$ satisfies (58), $C_3 > 0$, $C_2 < 0$, and $C_1 < 0$ (or $C_3 < 0$, $C_2 > 0$, and $C_1 > 0$), then $r \to +\infty$; the energy density of (62) is

$$W_E \sim \frac{\overline{C}(t)}{r^{n/2}},$$  \hspace{1cm} (96)$$

and the energy in $r \in [0, A]$ satisfies

$$E[0, A] = \int_0^A q(r) S_r(t, r) I^{r-n-1} dt \leq \overline{C}(t),$$  \hspace{1cm} (97)$$

where $A$ is constant; and $\overline{C}(t)$ is only dependent on $t$, such that as $t \to T$, $\overline{C}(t) = \infty$.

In Corollary 4, parameters $C_3$, $C_2$, and $C_1$ belong to one type from Table 1 (or Table 2). The other three parameters types were divided into two types: $C_3 C_2 > 0$ and $C_3 C_2 < 0$.

If $t - T = C_3$, then (63) may be expressed as

$$\overline{W}_E = \frac{2 \left( e^{2F} C_3 - 2e^F C_2 - C_3 \right) \left( C_3^2 (1 + e^{2F})^2 + 4e^{2F} C_5^2 \right)}{C_5^2 e^{2F} \left( 1 + e^{2F} \right) (-e^{-2F} + e^{-F} + e^F - 1)},$$  \hspace{1cm} (98)$$

and the monotonicity of (63) and that of (98) are the same under $t - T = C_3$. Substituting $F = \ln(x)$ into (98),

$$\overline{W}_E = \frac{2 (x^2 C_3 - 2x C_2 - C_3) \left( C_3^2 x^4 + 2 C_5^2 x^2 + 4x^2 C_5^2 + C_1^2 \right)}{C_5^2 (x^4 + 1) (x^4 + x^2 C_5^2 + 1)}.$$

Let us separate one equation from this, such as

$$x^2 C_3 - 2x C_2 - C_3 = 0,$$  \hspace{1cm} (100)$$

and solve to obtain

$$x_{1,2} = \frac{\pm \sqrt{C_2^2 + C_3^2 + C_2}}{C_3}. $$  \hspace{1cm} (101)$$

Thus, if $x = C_2/C_3$, the left section of (100) achieves its maximum or minimum value.

The derivative of the left of (99) is

$$\left( \overline{W}_E \right)_x = -2C_5^{-2} \left( x^6 + x^4 - x^2 - 1 \right)^{-2} \left( x^{10} C_3^3 - 4x^9 C_3 C_5^2 - x^8 C_5^3 + 12x^8 C_3 C_5 C_2^2 - 8x^7 C_2 C_3^2 - 32x^7 C_2 C_5 C_3^2 - 2x^6 C_5^3 - 36x^6 C_3 C_5^2 - 8x^5 C_2 C_3 C_5^2 + 32x^5 C_5^2 + 2x^4 C_3^3 + 36x^4 C_3 C_5 C_2^2 - 32x^3 C_3 C_5^2 + x^2 C_3^3 - 12x^2 C_3 C_5^2 - 4x C_2 C_3 C_5^2 - C_3^3 \right).$$

If $C_1 = C_2 = C_3 = C_5 = 1$, then the exact relationship of $\overline{W}_E$ and $x$ (or $\overline{W}_E$ and $F$) is shown in Figure 9 (Figure 10).

Figure II shows the evolution of energy density from (63) for multi ranch elliptic case I with $C_1 = C_2 = C_3 = 1$.
and $C_4 = 0$, and Figure 12 shows multibranch hyperbolic case I with $C_1 = -1$, $C_2 = C_3 = 1$, and $C_4 = 0$. In these two cases, $n = 2$ and $T = 1$. The behavior is shown for $t = 0, 1/4, 1/2, 3/4$, and the function $W_E = 1/r$.

The multibranch elliptic case I (Figure 11) is a local solution that terminates at some point of $r$. The multibranch hyperbolic case I (Figure 12) is also a local solution. However, there are some energy gaps on $W_E$. Figure 13 shows the evolution of multibranch hyperbolic case II with $C_1 = C_3 = -1$ and $C_2 = C_4 = 1$ is also local with some energy gaps.

The situation for the multibranch elliptic case II is different from the other three cases. As predicted in Corollary 4, this solution diverges, as shown in Figure 14 with $C_1 = C_2 = -1$ and $C_3 = C_4 = 1$. $W_E$ is a global solution and is close to zero as $r \to \infty$. Near the $W_E$ peaks, there are also some energy gaps.

We present some connections of ILL with real physics in the ending of this section. Clearly, we return the continuous ILL model back to the discrete Heisenberg model in the real
physics. We plan to use the inhomogeneous Heisenberg chain model to discuss some physically relevant potential then.

ILL model is the continuum limitation of the discrete inhomogeneous Heisenberg chain, where the corresponding Heisenberg Hamiltonian (representing nearest neighbor interaction between $S_i$ and $S_{i+1}$) is as follows:

$$H = -\Sigma_i q_i S_i S_{i+1},$$

(103)

where $S_i$ and $S_{i+1}$ are on-site spins and $q_i$ denotes the on-site ferromagnetic parameter which is the inhomogeneous physical interactions at the position $i$.

Equation (7) is a continuum model that can predicate any value of the site spin vector in a mathematical simulation way. Equation (7) plays a similar role as a Schrödinger equation for the Heisenberg model in quantum physics. As we can see in Section 1, the continuum limit of this system is also a nonlinear inhomogeneous Schrödinger equation (8) under the radial coordinates. For the discrete inhomogeneous Heisenberg chain, $H$ or $S_i$ has been used in the description of the magnetic properties and spin states of conjugated hydrocarbons and a large variety of metallic and organometallic compounds. What does the blowup solution (terminated in a finite time) represent in this discrete lattice system? In the continuum model ((7) or (8)), the blowup solution means that the gradient will be an infinite value, while the time is a finite time $T$. Accordingly, this finite time means that there are distortion and destruction in every position of $i$ sites of a lattice. In this time $T$, there is a phase transition in the ferromagnetic solid.

Theorem 2 indicates that the phase transition (distortion and destruction) is possible under some special $q_i$. More clearly, if the exact quantitative scale of $q$ is $r^{-n/2+2}$ as $r \to +\infty$, this breaking down will happen in the ferromagnetic material. From a discrete point of view, $i$ represents a correspondence of $r$ by spatial position. The distribution $|i| \to r$ means that the value of $q_i$ is the value of $q(r)$. Hence, we expect that the phase transition in real ferromagnetic structure ($n = 3$) will begin in some blowup time $T$ under the setting of $q_i \sim \sqrt{|i|}$ (as $|i| \to +\infty$).

Multiple branches solution (see solution (62) in Section 3) allows a finite time blowup too. Accordingly, the discrete inhomogeneous Heisenberg chain of this system will generate a phase transition under some special setting of $q_i$. In particular, if $n = 3$ and $q_i \sim \sqrt{|i|}$ (as $|i| \to +\infty$), the phase transition is a possible case in the multibranch elliptic case II according to Corollary 4. However, although (22) and (62) share a same breakdown in their discrete inhomogeneous Heisenberg chain system, the situation of multiple branches solution (62) is more complex than solution (22). For example, an energy gap can be found in the multibranch elliptic case II, while there is not any gap in solution (22). Why the energy gap will form in solution (62)? The most important reason is that there is a function arctanh in (58). If $F$ touches the edges of the definition domain of arctanh, $F$ jumps discontinuously from one value to another. In fact, if $F$ touches the edges of the definition domain of arctanh, $F$ will go to another track that belongs to the same multibranch. We can see that the difference between (22) and (62) is just a phase shift of $C_j/(t-T)$ in the solution. From a point of the discrete inhomogeneous Heisenberg chain, the energy gap is inevitable due to the phase shift. This means that if we force the phase of the site spin vector to do a parallel movement, the jumping of the spin vector can be found in some location $|i|$ in any time before the blowup time. Furthermore, the scale of the jumping enlarges as the time goes on.

In the homogeneous Heisenberg chain ($q_i = 1$), all its lattice sites are equivalent. However, the self-consistent many-body ground states need not preserve all symmetries of the Hamiltonian in the inhomogeneous case. After the balance is broken, some singularity and jumping behaviors are possible according to the examples presented in the preceding sections. As we can see in these solutions, the Hamiltonian itself comprises inequivalent lattice sites that lead to the blowup and the energy gap. The inequivalent lattice sites are the case, for example, the Heisenberg chains with impurities or, more generally, the systems with different types of magnetic atoms. In the real problems, it is very common to find inhomogeneities rather than the homogeneous case. Actually, external nonuniform magnetic fields, anisotropic crystal fields, finite-size effects, boundary conditions, local impurities, lattice defects, and so forth can be induced for generalized inhomogeneities instances.

5. Conclusions

We investigated the evolution of the singular solution of $n$ dimensional ILL on $S^2$. Two different ansätze were applied in constructing the exact solutions in radially symmetric coordinates. Accordingly, two singular solutions were obtained ((22) and (62)). If one coefficient of (62) is set to 0, (62) degenerates into (22). All solutions show the same blowup
rate \((T - t)^{-\frac{3}{2}}\). However, (22) is a global solution on the entire spatial domain, whereas (62) contains some local and global subcases.

Energy density plays an important role in the energy analysis of these two solutions. The energy density decay rate (22) scales as \(r^{\frac{n}{2}}\), which implies that the initial energy of this solution will be finite for \(n > 3\). However, (62) is somewhat more complicated, with four subcases. If the coefficients of the solution (\(C_1, C_2, C_3\) in (62)) are suitably selected, the solution can be divided into four types: multibranch elliptic cases I and II and multibranch hyperbolic cases I and II. In contrast to (22), these subcases may lead to a new energy gap class of exact solutions.

One of these four cases is the global solution in the spatial domain. In this case, since the decay rate also scales as \(r^{\frac{n}{2}}\), the initial energy is finite. The other three subcases can be local solutions, which means that the initial energy will be finite for any spatial dimension. All these multibranch cases lead to energy gaps in the energy density. The evolution of energy density for the cases was shown in Section 3, illustrating the decay property and monotonic increasing and decreasing behaviors. These evolutions were consistent with the theorem and corollary provided in Section 3. Thus, the multibranching of \(F\) for the solution plays a decisive role.

Integrability of the physical ILL has been investigated in [10]. The ILL discussed in this paper does not belong to the integrability system due to the inhomogeneous term. While many integrable spin systems are known, the integrability of ILL in the continuum limit has not been investigated. It will be of considerable interest to investigate the underlying singularity structures and energy gaps of such nonintegrable nonlinear evolution equations in detail. It would be also instructive to explore whether similar singular solutions can be obtained for LLE with Gilbert damping.

In conclusion, ILL and LLE display very interesting dynamic structures and further work is required to identify them.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author declares that he has no conflicts of interest.

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