**Research Article**

**Exact Traveling Wave Solutions to the (2 + 1)-Dimensional Jaulent-Miodek Equation**

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We derive exact traveling wave solutions to the (2 + 1)-dimensional Jaulent-Miodek equation by means of the complex method, and then we illustrate our main result by some computer simulations. It has presented that the applied method is very efficient and is practically well suited for the nonlinear differential equations that arise in mathematical physics.

1. Introduction and Main Results

Nonlinear differential equations widely describe many important dynamical systems in various fields of science, especially in nonlinear optics, plasma physics, solid state physics, and fluid mechanics. It has aroused widespread attention in the study of nonlinear differential equations [1–28]. Exact solutions of nonlinear differential equations play an important role in the study of mathematical physics phenomena. Hence, seeking explicit solutions of physics equations is an interesting and significant subject.

In 2001, Geng et al. [29] developed some (2 + 1)-dimensional models from the Jaulent-Miodek hierarchy [30]. Over the past few years, many research results for the (2 + 1)-dimensional Jaulent-Miodek equations have been generated [31–34], such as the algebraic-geometrical solutions, the bifurcation and exact solutions, the N-soliton solution, and Multiple kink solutions for the (2 + 1)-dimensional Jaulent-Miodek equations.

In 2012, Zhang et al. [35] studied the following (2 + 1)-dimensional Jaulent-Miodek equation:

\[ a_1 u_{xx} + a_2 u_{xx}^2 - a_3 u_{xxx} - a_4 u_x u_{xy} + a_5 u_{yy} = 0, \]  

where \( a_i \) are constants, \( i = 1, 2, \ldots, 5 \).

We say that a meromorphic function \( \zeta \) belongs to the class \( W \) if \( \zeta \) is an elliptic function, or a rational function of \( z \), or a rational function of \( e^{iz} \), \( \mu \in \mathbb{C} \). Only these functions can satisfy an algebraic addition theorem which was proved by Weierstrass, so the letter \( W \) was utilized [36]. In 2006, Eremenko [36] proved that all meromorphic solutions of the Kuramoto-Sivashinsky algebraic differential equation belong to the class \( W \). Recently, Kudryashov et al. [37, 38] used Laurent series to seek meromorphic exact solutions of some nonlinear differential equations. Following their work, the complex method was proposed by Yuan et al. [39, 40].

Substituting traveling wave transform

\[ u(x, y, t) = v(z), \quad z = x + ly + \lambda t, \]  

into (1), and then integrating it we get

\[ v''' - (a_1 \lambda + a_3 l^2) v' + \frac{lb}{2} (v')^2 - \frac{a_2}{3} (v')^3 - \delta = 0, \]  

where \( b = a_1 + a_4, l \) and \( \lambda \) are constants, and \( \delta \) is the integration constant. Setting \( w = v' \), (3) becomes

\[ w'' - (a_1 \lambda + a_5 l^2) w + \frac{lb}{2} w^2 - \frac{a_2}{3} w^3 - \delta = 0. \]  

We say that a meromorphic function \( \zeta \) belongs to the class \( W \) if \( \zeta \) is an elliptic function, or a rational function of \( z \), or a rational function of \( e^{iz} \), \( \mu \in \mathbb{C} \). Only these functions can satisfy an algebraic addition theorem which was proved by Weierstrass, so the letter \( W \) was utilized [36]. In 2006, Eremenko [36] proved that all meromorphic solutions of the Kuramoto-Sivashinsky algebraic differential equation belong to the class \( W \). Recently, Kudryashov et al. [37, 38] used Laurent series to seek meromorphic exact solutions of some nonlinear differential equations. Following their work, the complex method was proposed by Yuan et al. [39, 40].
They employed the Nevanlinna value distribution theory to investigate the existence of meromorphic solutions to some differential equations and then obtain the representations of meromorphic solutions to these differential equations [41, 42]. It shows that the complex method has a strong theoretical basis which can prove that meromorphic solutions of certain differential equations belong to the class $W$ and obtain exact solutions by the indeterminate forms of the solutions. Besides, this method can be applied to get all traveling wave exact solutions or general solutions of related differential equations [43, 44]. In this article, we would like to use the complex method to obtain exact traveling wave solutions to the $(2+1)$-dimensional Jaulent-Miodek equation.

**Theorem 1.** If $a_3 \neq 0$, then the meromorphic solutions $w$ of (4) belong to the class $W$. In addition, (4) has the following classes of solutions.

(i) The rational function solutions

$$w_{r,1}(z) = \pm \sqrt[3]{\frac{6}{a_2}} \left( \frac{1}{z - z_0} + \frac{lb}{2a_2} \right),$$

$$w_{r,2}(z) = \pm \sqrt[3]{\frac{6}{a_2}} \left( \frac{1}{z - z_0} - \frac{1}{z - z_0 - z_1} - \frac{1}{z_1} \right) + \frac{lb}{2a_2},$$

(ii) The simply periodic solutions

$$w_{s,1}(z) = \pm \sqrt[3]{\frac{3}{2a_2}} \left( -\varphi + E \right) \left( \frac{4\varphi E^2 + 4\varphi^2 E + 2\varphi^2 F - \varphi g_2 - E g_2}{(12E^2 - g_2)^2} \right) + \frac{lb}{2a_2},$$

$$w_{s,2}(z) = \pm \sqrt[3]{\frac{3}{2a_2}} \left( -\varphi + E \right) \left( \frac{4\varphi E^2 + 4\varphi^2 E + 2\varphi^2 F - \varphi g_2 - E g_2}{(12E^2 - g_2)^2} \right) + \frac{lb}{2a_2},$$

where $z_0 \in \mathbb{C}, z_1 \neq 0, \lambda = -l^2(4a_2a_3 - b^2)/4a_1a_2, \delta = -l^2b^2/24a_2^2$ in the former case, or $\lambda = -(4a_2a_3 - b^2)^2z_1^2 + 24a_1a_2^2, \delta = -(l^2b^2z_1^2 - 72lb_2a_2z_1 - 96\sqrt{6}a_2^3)/24a_2^3z_1^3$ in the latter case.

(iii) The elliptic function solutions

$$w_{s,3}(z) = \pm \sqrt[3]{\frac{3}{2a_2}} \left( \frac{3}{2a_2} \right) \left( \frac{6}{a_2z_1} \right) (z - z_0) + c_3,$$

$$w_{s,4}(z) = \pm \sqrt[3]{\frac{3}{2a_2}} \left( \frac{3}{2a_2} \right) \left( \frac{6}{a_2z_1} \right) (z - z_0) + c_4,$$

$$w_{s,5}(z) = \pm \sqrt[3]{\frac{3}{2a_2}} \left( \frac{3}{2a_2} \right) \left( \frac{6}{a_2z_1} \right) (z - z_0) + c_5,$$

$$w_{s,6}(z) = \pm \sqrt[3]{\frac{3}{2a_2}} \left( \frac{3}{2a_2} \right) \left( \frac{6}{a_2z_1} \right) (z - z_0) + c_6,$$

where $z_0 \in \mathbb{C}, z_1 \neq 0, c_1$ and $c_2$ are integral constants.

(ii) The simply periodic solutions

$$v_{s,1}(z) = \pm \sqrt[3]{\frac{3}{2a_2}} \left( \frac{6}{a_2} \right) \left( \frac{1}{z - z_0} \right) + \frac{lb}{2a_2},$$

$$v_{s,2}(z) = \pm \sqrt[3]{\frac{3}{2a_2}} \left( \frac{6}{a_2} \right) \left( \frac{1}{z - z_0} \right) + \frac{lb}{2a_2},$$

where $z_0 \in \mathbb{C}, z_1 \neq 0, c_1$ and $c_2$ are integral constants.

(iii) The elliptic function solutions

$$v_{s,3}(z) = \pm \sqrt[3]{\frac{3}{2a_2}} \left( \frac{6}{a_2} \right) \left( \frac{1}{z - z_0} \right) + \frac{lb}{2a_2},$$

$$v_{s,4}(z) = \pm \sqrt[3]{\frac{3}{2a_2}} \left( \frac{6}{a_2} \right) \left( \frac{1}{z - z_0} \right) + \frac{lb}{2a_2},$$

where $z_0 \in \mathbb{C}, z_1 \neq 0, c_1$ and $c_2$ are integral constants.

(iii) The elliptic function solutions

$$v_{s,5}(z) = \pm \sqrt[3]{\frac{3}{2a_2}} \left( \frac{6}{a_2} \right) \left( \frac{1}{z - z_0} \right) + \frac{lb}{2a_2},$$

$$v_{s,6}(z) = \pm \sqrt[3]{\frac{3}{2a_2}} \left( \frac{6}{a_2} \right) \left( \frac{1}{z - z_0} \right) + \frac{lb}{2a_2},$$

where $z_0 \in \mathbb{C}, z_1 \neq 0, c_1$ and $c_2$ are integral constants.

The rest of this paper is organized as follows. Section 2 introduces some preliminary theory and the complex
method. In Section 3, we will give the proof of Theorems 1 and 2. Some computer simulations will be given to illustrate our main results in Section 4. Conclusions are presented at the end of the paper.

2. Preliminary Theory and the Complex Method

At first, we give some notations and definitions, and then we introduce some lemmas and the complex method.

Let \( m \in \mathbb{N} = \{1, 2, 3, \ldots\}, r_j \in \mathbb{Z}, j = 0, 1, \ldots, m, r = (r_0, r_1, \ldots, r_m), \) and \( K_r[w](z) \)

\[
K_r[w](z) := [w(z)]^r [w'(z)]^r [w''(z)]^r \cdots [w^{(m)}(z)]^r,
\]
then \( r(d) = \sum_{j=0}^m r_j \) is the degree of \( K_r[w]. \) Let the differential polynomial be defined by

\[
F(w, w', \ldots, w^{(m)}) = \sum_{r \in J} a_r K_r[w],
\]
where \( J \) is a finite index set, and \( a_r \) are constants, then \( \text{deg} F(w, w', \ldots, w^{(m)}) = \max_{r \in J} |d(r)| \) is the degree of \( F(w, w', \ldots, w^{(m)}). \)

Consider the following differential equation:

\[
F(w, w', \ldots, w^{(m)}) = cw^n + d,
\]
where \( n \in \mathbb{N}, c \neq 0, d \) are constants.

Set \( p, q \in \mathbb{N} \), and meromorphic solutions \( w \) of (13) have at least one pole. If (13) has exactly \( p \) distinct meromorphic solutions, and their multiplicity of the pole at \( z = 0 \) is \( q \), then (13) is said to satisfy the \( (p, q) \) condition. It could be not easy to show that the \( (p, q) \) condition of (13) holds, so we need the weak \( (p, q) \) condition as follows.

Inserting the Laurent series

\[
w(z) = \sum_{r=-\infty}^{\infty} \beta_r z^r,
\]
into (13), we can determine exactly \( p \) different Laurent singular parts:

\[
\sum_{r=-q}^{\infty} \beta_r z^r; \tag{15}
\]
then (13) is said to satisfy the weak \( (p, q) \) condition.

Given two complex numbers \( v_1, v_2, \text{Im}(v_1/v_2) > 0 \), and let \( L \) be the discrete subset \( L[2v_1, 2v_2] = \{v | v = 2^i v_1 + 2^j v_2, c_1, c_2 \in \mathbb{Z}\} \), and \( L \) is isomorphic to \( \mathbb{Z} \times \mathbb{Z}. \) Let the discriminant \( \Delta = \Delta(b_1, b_2) = b_1^2 - 2b_2^2 \) and

\[
h_n = h_n(L) = \sum_{v \in L \setminus \{0\}} \frac{1}{v^2}; \tag{16}
\]
A meromorphic function \( \varphi(z) = \varphi(z, g_2, g_3) \) with double periods \( 2v_1, 2v_2 \), which satisfies the equation

\[
\left( \varphi'(z) \right)^2 = 4\varphi(z^3) - g_2 \varphi(z) - g_3,
\]
in which \( g_2 = 60h_4, g_3 = 140h_6, \) and \( \Delta(g_2, g_3) \neq 0, \) is called the Weierstrass elliptic function.

In 2009, Eremenko et al. [45] studied the \( m \)-order Briot-Bouquet equation (BBEq)

\[
F \left( w, w^{(m)} \right) = \sum_{j=0}^n F_j(w) \left( w^{(m)} \right)^j = 0, \tag{18}
\]
where \( F_j(w) \) are constant coefficient polynomials, \( m \in \mathbb{N}. \) For the \( m \)-order BBEq, we have the following lemma.

**Lemma 3** (see [37, 40, 46]). Let \( m, n, p, s \in \mathbb{N}, \text{deg} F(w, w^{(m)}) < n, \) and a \( m \)-order BBEq

\[
F \left( w, w^{(m)} \right) = cw^n + d \tag{19}
\]
satisfies the weak \( (p, q) \) condition; then the meromorphic solutions \( w \) belong to the class \( W. \) Suppose for some values of parameters such solution \( w \) exists; then other meromorphic solutions form a one-parametric family \( (z - z_0), z_0 \in C. \) Furthermore, each elliptic solution with pole at \( z = 0 \) can be written as

\[
w(z) = \sum_{i=1}^{s-1} \sum_{j=1}^{q} (-1)^j \beta_{-ij} d^{-1} \frac{d^{-2}}{(j-1)!} \left( \frac{\varphi'(z) + C_j}{\varphi(z) - D_j} \right)^2
\]

\[- \varphi(z) + \sum_{i=1}^{s-1} \beta_{-1i} \varphi'(z) + C_i \varphi(z) - D_i \]

\[- \sum_{j=1}^{q} (-1)^j \beta_{-ij} d^{-1} \frac{d^{-2}}{(j-1)!} \varphi(z) + \beta_0, \tag{20}
\]
where \( \beta_{-ij} \) are determined by (14), \( \sum_{i=1}^{s} \beta_{-1i} = 0, \) and \( C_i = 4D_i^2 - q_i D_i - g_3. \)

Each rational function solution \( w = R(z) \) is expressed as

\[
R(z) = \sum_{i=1}^{s} \sum_{j=1}^{q} \frac{\beta_{ij}}{(z - z_j)} + \beta_0, \tag{21}
\]
which has \( s(\leq p) \) distinct poles of multiplicity \( q. \)

Each simply periodic solution \( w = R(\eta) \) is a rational function of \( \eta = e^{i\mu} (\mu \in C) \) and is expressed as

\[
R(\eta) = \sum_{i=1}^{s} \sum_{j=1}^{q} \frac{\beta_{ij}}{(\eta - \eta_j)} + \beta_0, \tag{22}
\]
which has \( s(\leq p) \) distinct poles of multiplicity \( q. \)

**Lemma 4** (see [46, 47]). Weierstrass elliptic functions \( \varphi(z) \) have an addition formula as below:

\[
\varphi(z - z_0) = -\varphi(z_0) - \varphi(z)
\]

\[+ \frac{1}{4} \left( \varphi'(z_0) + \varphi'(z) \right)^2. \tag{23}\]
When \( g_2 = g_3 = 0 \), Weierstrass elliptic functions can be degenerated to rational functions according to
\[
\varphi(z, 0, 0) = \frac{1}{z^2}.
\]
(24)

When \( \Delta(g_2, g_3) = 0 \), it can also be degenerated to simple periodic functions according to
\[
\varphi(z, 3d^2, -d^3) = 2d - \frac{3d}{2} \coth^2 \sqrt{\frac{3d}{2} z}.
\]
(25)

By the above definitions and lemmas, we now present the complex method as below for the convenience of readers.

**Step 1.** Substitute the transformation \( T : u(x, t) \rightarrow w(z) \) defined by \( (x, t) \rightarrow z \) into a given partial differential equation (PDE) to yield a nonlinear ordinary differential equation (ODE).

**Step 2.** Substitute (14) into the ODE to determine whether the weak \( \langle p, q \rangle \) condition holds.

**Step 3.** Find out meromorphic solutions \( w(z) \) of the ODE with a pole at \( z = 0 \), in which we have \( m-1 \) integral constants.

**Step 4.** Obtain meromorphic solutions \( w(z - z_0) \) by Lemmas 3 and 4.

**Step 5.** Substituting the inverse transformation \( T^{-1} \) into the meromorphic solutions, we get the exact solutions for the original PDE.

### 3. Proof of Main Results

**Proof of Theorem 1.** Substituting (14) into (4) we have \( p = 2 \), \( q = 1 \), \( \beta_{-1} = \pm \sqrt{6}a_2 \), \( \beta_0 = lb/2a_2 \), \( \beta_1 = -\sqrt{6}(4a_3a_5 - b^2) \), \( \beta_2 = -(6b^2/2a_2 + 6lb\lambda a_5 + 12\lambda a_5^2)/(8a_2^3) \), and \( \beta_3 \) is an arbitrary constant.

Therefore, (4) is a second-order BBEq and satisfies weak (2, 1) condition. Hence, by Lemma 3, we obtain that meromorphic solutions of (4) belong to \( W \). We will show meromorphic solutions of (4) in the following.

By (21), we infer that the indeterminate rational solutions of (4) are
\[
R_1(z) = \frac{\beta_{-1}}{z} + \frac{\beta_1}{z - z_1} + \beta_{10},
\]
(26)
with pole at \( z = 0 \).

Substituting \( R_1(z) \) into (4), we have
\[
R_{1,1}(z) = \pm \frac{6}{a_2} \frac{1}{z} + \frac{lb}{2a_2},
\]
(27)
where \( \lambda = -(4a_2a_5 - b^2)/4a_2a_2 \) and \( \delta = -lb^3/24a_2^3 \).

\[
R_{1,2}(z) = \pm \frac{6}{a_2} \left( \frac{1}{z} - \frac{1}{z - z_1} - \frac{1}{z_1} \right) + \frac{lb}{2a_2},
\]
(28)
where \( \lambda = -((4a_2a_5 - b^2)/4a_2a_5) + 24a_2^3/4a_2a_2 \) and \( \delta = -(lb^3/24a_2^3) - 72lb_2a_2a_2 - 96\sqrt{6}a_2^3/24a_2^3 \).

So the rational solutions of (4) are
\[
w_{r,1}(z) = \pm \frac{6}{a_2} \frac{1}{z - z_0} + \frac{lb}{2a_2},
\]
\[
w_{r,2}(z) = \pm \frac{6}{a_2} \left( \frac{1}{z - z_0} - \frac{1}{z - z_0 - z_1} - \frac{1}{z_1} \right) + \frac{lb}{2a_2},
\]
where \( z_0 \in C, z_1 \neq 0 \). \( \lambda = -lb^3/24a_2^3 \) in the former case, or \( \lambda = -lb^3/24a_2^3 \) in the latter case.

To obtain simply periodic solutions, let \( \eta = e^{i\zeta} \), and substitute \( w = R(\eta) \) into Eq. (4), then
\[
\mu^2 (\eta R' + \eta^2 R'') - (a_1 \lambda + a_3 \delta) \eta + \frac{lb}{2} \eta^2 - \frac{a_2}{3} \eta^3 - \delta = 0.
\]
(30)

Substituting
\[
R_2(z) = \frac{\beta_{21}}{\eta - 1} + \frac{\beta_{22}}{\eta - \eta_1} + \frac{\beta_{20}}{\eta - \eta_1},
\]
(31)
into (30), we obtain that
\[
R_{2,1}(z) = \pm \frac{6}{a_2} \mu \left( \frac{1}{\eta - 1} - \frac{1}{2} \right) + \frac{lb}{2a_2},
\]
\[
R_{2,2}(z) = \pm \frac{6}{a_2} \mu \left( \frac{1}{\eta - 1} - \frac{1}{2} \right) - \frac{\eta_1}{\eta - \eta_1} + \frac{\eta_1 + 1}{2(\eta_1 - 1)}
\]
\[
+ \frac{lb}{2a_2},
\]
where \( \lambda = -((4a_2a_5 - b^2)/4a_2a_5) + 24a_2^3/4a_2a_5 \) in the former case, or \( \lambda = 3\mu^2(\eta_1 + 1)/2(\eta_1 - 1)^2 - lb^3/4a_2a_2 + 3(\eta_1 + 1)^2(\sqrt{2a_2^3/3\mu} + lb/2)\eta_1 + \sqrt{2a_2^3/3\mu} - lb/2)\mu^2/2a_2(\eta_1 - 1)^2 - lb^3/24a_2^3 \) in the latter case.

Inserting \( \eta = e^{i\zeta} \) into (32), we can get simply periodic solutions to (4) with pole at \( z = 0 \)
\[
w_{s,0,1}(z) = \pm \frac{3}{2a_2} \mu \coth \frac{\mu}{2} z + \frac{lb}{2a_2},
\]\n\[
w_{s,0,2}(z)
\]
\[
= \pm \frac{3}{2a_2} \mu \left( \coth \frac{\mu}{2} z - \coth \frac{\mu}{2} (z - z_1) - \coth \frac{\mu}{2} z_1 \right)
\]
\[
+ \frac{lb}{2a_2},
\]
(33)
where \( \lambda = -(4a_2 a_0 - b^2)^2 + 6a_2 \mu^2)/4a_1 a_2, \) \( \delta = -(l b^3 + 12 \sqrt{6} \mu^3 a_3^2 / 24a_2^2) \) in the former case, or \( \lambda = (3 \mu^2/2) \coth^2(\mu/2) z_1 - l b^2 / 4a_2, \) \( \delta = \sqrt{3 / 2} a_2 \mu^2 \coth^2(\mu/2) z_1 (\mu \coth(\mu/2) z_1 + \sqrt{3 / 2} a_2 (l b/2)) - l b^2 / 24a_2^2 \) in the latter case.

So simply periodic solutions of (4) are

\[
\begin{align*}
 w_{1,1}(z) &= \pm \sqrt{\frac{3}{2a_2}} \mu \coth \frac{\mu}{2} (z - z_0) + \frac{lb}{2a_2}, \\
 w_{1,2}(z) &= \pm \sqrt{\frac{3}{2a_2}} \mu \left( \coth \frac{\mu}{2} (z - z_0) - \coth \frac{\mu}{2} z_1 \right) + \frac{lb}{2a_2},
\end{align*}
\]

where \( z_0 \in C, z_1 \neq 0. \lambda = -(4a_2 a_0 - b^2)^2 + 6a_2 \mu^2)/4a_1 a_2, \) \( \delta = -(l b^3 + 12 \sqrt{6} \mu^3 a_3^2 / 24a_2^2) \) in the former case, or \( \lambda = (3 \mu^2/2) \coth^2(\mu/2) z_1 - l b^2 / 4a_2, \) \( \delta = \sqrt{3 / 2} a_2 \mu^2 \coth^2(\mu/2) z_1 (\mu \coth(\mu/2) z_1 + \sqrt{3 / 2} a_2 (l b/2)) - l b^2 / 24a_2^2 \) in the latter case.

From (20), we have the indeterminate relations to elliptic solutions of (4) with pole at \( z = 0 \)

\[
w_{d1}(z) = \frac{\beta_{-1} \varphi'(z) + C_1}{2 \varphi(z) - D_1} + \beta_{30},
\]

where \( C_1 = 4D_1^3 - g_2 D_1 - g_3 \). Making use of Lemma 4 to \( w_{d1}(z) \), and considering the results obtained above, we infer that \( \beta_{30} = lb/2a_2, g_3 = 0, C_1 = D_1 = 0. \) So we obtain

\[
w_{d1}(z) = \pm \sqrt{\frac{3}{2a_2}} \frac{\varphi'(z)}{\varphi(z)} + \frac{lb}{2a_2},
\]

where \( g_3 = 0. \)

Thus, the elliptic function solutions of (4) are

\[
w_d(z) = \pm \sqrt{\frac{3}{2a_2}} \frac{\varphi'(z - z_0, g_3, 0)}{\varphi(z - z_0, g_3, 0)} + \frac{lb}{2a_2},
\]

where \( z_0 \in C, g_3 = 0, g_2 \) is arbitrary. Applying the addition formula, we can rewrite it as

\[
w_d(z) = \pm \sqrt{\frac{3}{2a_2}} \varphi'(z - z_0, g_3, 0) + \frac{lb}{2a_2},
\]

where \( g_3 = 0, F^2 = 4E^3 - g_2 E, E \) and \( g_2 \) are arbitrary.

**Proof of Theorem 2.** By Theorem 1, we can obtain the rational function solutions of (3) which are

\[
\begin{align*}
 v_{r,1}(z) &= \int w_{r,1}(z) \, dz = \int \left( \pm \sqrt{\frac{6}{a_2}} \frac{1}{z - z_0} \right. \\
 &\left. + \frac{lb}{2a_2} \right) \, dz = \pm \frac{6}{a_2} \ln(z - z_0) + \frac{lb}{2a_2} (z - z_0) + c_1, \\
 v_{r,2}(z) &= \int w_{r,2}(z) \, dz
\end{align*}
\]

where \( z_0 \in C, z_1 \neq 0, c_1 \) and \( c_2 \) are integral constants.

The simply periodic solutions of (3) are

\[
\begin{align*}
 v_{s,1}(z) &= \int v_{r,1}(z) \, dz = \int \left( \pm \sqrt{\frac{6}{a_2}} \frac{1}{z - z_0} \right. \\
 &\left. + \frac{lb}{2a_2} \right) \, dz = \pm \frac{6}{a_2} \ln(z - z_0) + \frac{lb}{2a_2} (z - z_0) + c_2, \\
 v_{s,2}(z) &= \int v_{r,2}(z) \, dz = \int \left( \pm \sqrt{\frac{6}{a_2}} \frac{1}{z - z_0} - \frac{1}{z - z_0 - z_1} - \frac{1}{z_1} \right) \\
 &\left. + \frac{lb}{2a_2} \right) \, dz = \pm \frac{6}{a_2} \ln(z - z_0) - \frac{lb}{2a_2} (z - z_0) + c_2,
\end{align*}
\]

where \( z_0 \in C, z_1 \neq 0, c_1 \) and \( c_2 \) are integral constants.
The elliptic function solutions of (3) are
\[
\nu_{d,2}(z) = \int w_d(z) \, dz = \int \left( \pm \frac{3}{2a_2} \varphi'(z-z_0) + \frac{lb}{2a_2} \right) \, dz \\
= \pm \frac{3}{2a_2} \ln \varphi(z-z_0) + \frac{lb}{2a_2} (z-z_0) + c_5,
\]
where \(c_5\) is the integral constant, \(c_5^2 = 4H^3 - g_2H + g_3 = 0\). □

4. Computer Simulations

In this section, we illustrate our main results by some computer simulations. We carry out further analysis to the properties of the new solutions as in the following figures.

(1) By employing the complex method, we are able to obtain the rational solutions \(w_{r,1}(z)\) and \(w_{r,2}(z)\) of (4). Figure 1 shows shape of solutions \(w_{r,2}(z)\) for \(a_2 = 6, b = -24, l = 1, \lambda = -1, z_0 = 0, z_1 = 1\) within the interval \(-5 \leq x, y \leq 5\). Note that they have one generation pole which are showed by Figure 1.

(2) By applying the complex method, we achieve the simply periodic solutions \(w_{s,1}(z)\) and \(w_{s,2}(z)\) of (4). The solutions \(w_{s,1}(z)\) and \(w_{s,2}(z)\) come from the hyperbolic function. Figure 2 shows the shape of solutions \(w_{s,1}(z)\) for \(a_2 = 6, b = -24, l = 1, \lambda = -1, z_0 = 0, \mu = 1\) within the interval \(-2\pi \leq x, y \leq 2\pi.\)

(3) By using the complex method, we are able to get the rational solutions \(v_{r,1}(z)\) and \(v_{r,2}(z)\) of (3). Figure 3 shows the shape of solutions \(v_{r,2}(z)\) for \(a_2 = 6, b = 6, l = 1, \lambda = -1, z_0 = 0, z_1 = 6, c_2 = 0\) within the interval \(-5 \leq x, y \leq 5\). Note that they have one generation pole which are showed by Figure 3.

(4) By employing the complex method, we obtain the simply periodic solutions \(v_{s,1}(z)\) and \(v_{s,2}(z)\) of (3). The solutions \(v_{s,1}(z)\) and \(v_{s,2}(z)\) are in terms of the hyperbolic function solution. Figure 4 shows the shape of solutions \(v_{s,2}(z)\) for \(a_2 = 6, b = 6, l = 1, \lambda = -1, z_0 = 0, \mu = 1, c_5 = 0\) within the interval \(-2\pi \leq x, y \leq 2\pi.\) It may be observed from Figure 4 that when \(t\) increases, there would
be a delay for the appearance of the peak within the natural topology of traveling wave solution.

5. Conclusions

In summary, we have utilized the complex method to construct exact solutions of the nonlinear evolution equation. We first show that meromorphic solutions of the (2 + 1)-dimensional Jaulent-Miodek equation belong to the class \( W \), and then we obtain the exact traveling wave solutions for this equation. To our knowledge, the solutions in this paper have not been reported in former literature. The simply periodic solutions \( w_{s,2}(z) \), \( v_{s,2}(z) \) and the rational solutions \( w_{r,2}(z) \), \( v_{r,2}(z) \) are not only new but also not degenerated successively by the elliptic function solutions. We expand the results in [32, 33].

Based on the previous works [32, 33], the complex method allows us to confirm that meromorphic solutions of the differential equation belong to the class \( W \) easily. By the indeterminate forms of the solutions, we can find meromorphic solutions \( w(z) \) for the differential equation with a pole at \( z = 0 \); then we are able to obtain all meromorphic solutions \( w(z-z_0), z_0 \in \mathbb{C} \) for the differential equation with an arbitrary pole. The results demonstrate that the applied method is direct and efficient method, which allows us to do tedious and complicated algebraic calculation. We can apply the idea of this study to other nonlinear evolution equations.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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