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Research Article

Operator of Time and Generalized Schrödinger Equation

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The equation describing the change of the state of the quantum system with respect to energy is introduced within the framework of the self-adjoint operator of time in nonrelativistic quantum mechanics. In this proposal, the operator of time appears to be the generator of the change of the energy, while the operator of energy that is conjugate to the operator of time generates the time evolution. Two examples, one with discrete time and the other with continuous one, are given and the generalization of Schrödinger equation is proposed.

1. Introduction

Certainly, the most important equation in quantum mechanics is the Schrödinger equation [1]. It is seen as the equation that, for a particular choice of the Hamiltonian, determines how the state of the quantum mechanical system changes with respect to time. Usually, the Hamiltonian is a function of coordinate and momentum, $H(\hat{q}, \hat{p})$, but there are situations in which one is interested in time dependent Hamiltonians as well [2–8]. In these cases, one can find how the energy of the system changes in time, which is the consequence of the influence of the environment on the system under consideration. However, the Schrödinger equation is not appropriate for finding an answer to the question of how the state of the quantum system changes with respect to the change of energy acquired or lost by the system. In order to tackle this problem, one needs an equation in which, so to say, one differentiates with respect to the energy, not with respect to the time as in the Schrödinger equation. This is what we are going to discuss in the present article, and this will be done by using the formalism of the operator of time.

There are a whole variety of topics and approaches related to the operator of time (e.g., [9–11] and references therein), but let us shortly review the approach we proposed in [12–15]. Our approach is similar to the one in [16] and the references therein and [17].

2. Operators of Time and Energy and Schrödinger Equations

In order to fulfill the demand coming from the general relativity, that space and time should be treated on equal footing (just like for every spatial degree of freedom a separate Hilbert space is introduced), one should introduce Hilbert space where the operator of time $\hat{t}$ acts. More concretely, in the case of one degree of freedom, besides $\hat{q}$ and the conjugate momentum $\hat{p}$, acting nontrivially in $\mathscr{H}_q$, there should be $\mathscr{H}_t$ where $\hat{t}$, together with $\hat{s}$ that is conjugate to $\hat{t}$, acts nontrivially. So, in $\mathscr{H}_q \otimes \mathscr{H}_t$ for the self-adjoint operators $\hat{q}\otimes\hat{I}$, $\hat{p}\otimes\hat{I}$, $\hat{I}\otimes\hat{t}$, and $\hat{I}\otimes\hat{s}$, the following commutation relations hold:

$$\begin{aligned} \frac{1}{i\hbar}\left[\hat{q}\otimes\hat{I}, \hat{p}\otimes\hat{I}\right] &= \hat{I}\otimes\hat{I}, \\ \frac{1}{i\hbar}\left[\hat{I}\otimes\hat{t}, \hat{I}\otimes\hat{s}\right] &= -\hat{I}\otimes\hat{I}. \end{aligned} \tag{1}$$

The other commutators vanish. The operator of time $\hat{t}$ has a continuous spectrum $\{-\infty, +\infty\}$, just like the operators of coordinate and momentum $\hat{q}$ and $\hat{p}$. And so is the case for the operator $\hat{s}$, which is conjugate to time and is the operator of energy. After noticing the complete similarity between the coordinate and the momentum on the one hand and between time and energy on the other hand, one can introduce eigenvectors of $\hat{t}$:

$$\hat{t}\,|t\rangle = t\,|t\rangle\,, \quad \text{for every } t \in \mathbf{R}. \tag{2}$$

The question related to the norm and measurement of $|t\rangle$ is analyzed in detail in [16]. It was shown there that the measurements can be treated *à la* von Neumann, and how the standard predictions of quantum mechanics can be obtained.

In $|t\rangle$ representation, the operator of energy becomes $i\hbar(\partial/\partial t)$, while its eigenvectors $|E\rangle$ become $e^{(1/i\hbar)Et}$, for every $E \in \mathbf{R}$.

After Pauli, it has been well known that there is no self-adjoint operator of time that is conjugate to the Hamiltonian $H(\hat{q},\hat{p})$ which has the spectrum bounded from below. Within our proposal, the self-adjoint operator of time is conjugate to the operator of energy which has an unbounded spectrum. The Hamiltonian and the operator of energy are acting in different Hilbert spaces, but there is a subspace of the total Hilbert space where

$$\hat{s}\,|\psi\rangle = H(\hat{q},\hat{p})\,|\psi\rangle\,. \tag{3}$$

The states that satisfy this equation are physical, since they have nonnegative energies. The last equation is nothing else but the Schrödinger equation. By taking $|q\rangle \otimes |t\rangle$ representation of the previous equation, one gets the familiar form of Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\psi(q,t) = \widehat{H}\psi(q,t)\,, \tag{4}$$

with the shorthand notation $\widehat{H} = H(q, -i\hbar(\partial/\partial q))$. In other words, the operator of energy has negative eigenvalues as well as nonnegative ones, but the Schrödinger equation appears as a constraint that selects physically meaningful states; that is, the states with nonnegative energies, due to the nonnegative spectrum of $H(\hat{q},\hat{p})$, are selected by the Schrödinger equation. For the time independent Hamiltonian, the typical solution of the Schrödinger equation (4) is $\psi_E(q)e^{(1/i\hbar)Et}$, which is $|q\rangle \otimes |t\rangle$ representation of $|\psi_E\rangle \otimes |E\rangle$, where $H(\hat{q},\hat{p})|\psi_E\rangle = E|\psi_E\rangle$ and $\hat{s}|E\rangle = E|E\rangle$. For such states, the Heisenberg uncertainty relation for $\hat{s}$ and $\hat{t}$ obviously holds. The energy eigenvectors $|E\rangle$ have the same formal characteristics as, say, the momentum eigenvectors; that is, they are normalized to $\delta(0)$ and, for different values of energy, they are mutually orthogonal.

Therefore, the Schrödinger equation demands equal action of the operator of energy $\hat{s}$ and Hamiltonian $H(\hat{q},\hat{p})$ on the states of quantum system, and, on the other hand, due to the fact that the operator of energy in time representation is $i\hbar(\partial/\partial t)$, it is said for the Schrödinger equation that it describes how the state of a quantum system changes with time. In the Schrödinger equation, the Hamiltonian appears as the dynamical counterpart of energy. The Hamiltonian is often seen as the one that determines the time evolution of states of the quantum system since its eigenvalues appear in the phase factor $\langle t \mid E\rangle = e^{(1/i\hbar)Et}$.

In analogy with this, one can introduce

$$\hat{t}\,|\psi\rangle = G(\hat{q},\hat{p})\,|\psi\rangle\,. \tag{5}$$

By taking $|p\rangle \otimes |E\rangle$ representation of the previous equation, one gets

$$i\hbar\frac{\partial}{\partial E}\psi(p,E) = \widehat{G}\psi(p,E)\,, \tag{6}$$

with the shorthand notation $\widehat{G} = G(i\hbar(\partial/\partial p), p)$. (Instead of $|p\rangle$ representation, one can, of course, use $|q\rangle$ representation. The momentum representation is taken just because the energy and momentum form a quadrivector. A similar remark holds for the above $|q\rangle \otimes |t\rangle$ representation.)

In (5), one demands that the operator of time and its dynamical counterpart $G(\hat{q},\hat{p})$ should act equally on the states of the quantum mechanical system. As the original Schrödinger equation, this equation represents constraint as well. The original Schrödinger equation can be seen as the energy constraint, while this one is a time constraint that also selects a subspace in $\mathscr{H}_q \otimes \mathscr{H}_t$. After representing (5) in $|p\rangle \otimes |E\rangle$ basis, in analogy with the interpretation of the original Schrödinger equation, it could be said that this equation describes how the state of the quantum mechanical system changes with respect to the change of energy. Just like the time is treated as the evolution parameter for (4), the energy in (6) can be seen as the independent variable.

The typical solution of, let us call it, the second Schrödinger equation (6) is $\psi_t(p)e^{(-1/i\hbar)Et}$. It is $|p\rangle \otimes |E\rangle$ representation of $|\psi_t\rangle \otimes |t\rangle$, where $G(\hat{q},\hat{p})|\psi_t\rangle = t|\psi_t\rangle$ and $\hat{t}|t\rangle = t|t\rangle$. For such states, the Heisenberg uncertainty relation for $\hat{s}$ and $\hat{t}$ obviously holds.

Let us stress that the generators of the Lie algebra (beside coordinate and momentum) are $\hat{s}$ and $\hat{t}$ and not $H(\hat{q},\hat{p})$ and $G(\hat{q},\hat{p})$. For the former, $(1/i\hbar)[\hat{t},\hat{s}] = -\hat{I}$ holds, while for the latter ones, since they are just dynamical counterparts of the energy and time, there is no reason to demand noncommutativity *a priori*; that is, their commutator depends on the particular choice of these functions of coordinate and momentum.

As for the different systems the different Hamiltonians are appropriate, for different situations, one should use appropriate $G(\hat{q},\hat{p})$. As an illustration, let us briefly discuss two examples of $G(\hat{q},\hat{p})$. Due to the Big Bang as the beginning of time, it seems reasonable to assume $G(\hat{q},\hat{p})$ with a bounded-from-below spectrum. Without going into debate about the existence of time crystals, the first example offers a toy model of discrete time, while for the second one the quantum system is characterized with the continuous time.

If $G(\hat{q},\hat{p})$ is

$$G(\hat{q},\hat{p}) = \frac{\hbar}{m^2c^4}\left(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega\hat{q}^2\right), \tag{7}$$

then the solutions of the second Schrödinger equation (5) are $|\psi_n\rangle \otimes |t_n\rangle$, $n \in N$, where $|\psi_n\rangle$ are well known solutions of the eigenvalue problem for the Hamiltonian of the harmonic oscillator and

$$t_n = \frac{\hbar^2\omega}{m^2c^4}\left(n + \frac{1}{2}\right). \tag{8}$$

If the standard ladder operators $\hat{a}^\dagger$ and $\hat{a}$, which act on $|\psi_{t_n}\rangle$, are directly multiplied by the time translation operator,

$\hat{a}^\dagger \otimes e^{(1/i\hbar)\hat{s}\Delta t}$ and $\hat{a} \otimes e^{(1/i\hbar)\hat{s}\Delta t}$, where $\Delta t = \hbar^2\omega/m^2c^4$, then it stands that

$$\hat{a}^\dagger \otimes e^{-(1/i\hbar)\hat{s}\Delta t} \left|\psi_{t_n}\right\rangle \otimes \left|t_n\right\rangle = \sqrt{n+1} \left|\psi_{t_{n+1}}\right\rangle \otimes \left|t_{n+1}\right\rangle,$$
$$\hat{a} \otimes e^{(1/i\hbar)\hat{s}\Delta t} \left|\psi_{t_n}\right\rangle \otimes \left|t_n\right\rangle = \sqrt{n} \left|\psi_{t_{n-1}}\right\rangle \otimes \left|t_{n-1}\right\rangle. \quad (9)$$

So, these operators formally describe the transition between the states that represent nearby moments in time. The distinguished moments t_n are the ones in which the system can be located in time.

The unitary operator attached to the change of the energy is $e^{(1/i\hbar)\hat{t}\Delta E}$. Note that the unitary operator that represents the time evolution is $e^{(1/i\hbar)\hat{s}\Delta t}$, not $e^{(1/i\hbar)H(\hat{q},\hat{p})\Delta t}$, because $\hat{s}$ is the generator of time translation in $\mathscr{H}_t$, not $H(\hat{q},\hat{p})$, since it does not even act in $\mathscr{H}_t$. Of course, these two unitary operators are effectively the same when they act on the solutions of the original Schrödinger equation.

Another example is the case with the continuous time, when $G(\hat{q},\hat{p}) = (\hbar/m^3c^4)\hat{p}^2$. The solutions of the second Schrödinger equation are then $|p\rangle \otimes |t\rangle$, where $t = (\hbar/m^3c^4)p^2$. This $G(\hat{q},\hat{p})$ commutes with the Hamiltonian of a free particle, while the previous one obviously commutes with the Hamiltonian of the harmonic oscillator. Needless to say, in general, one can combine every $H(\hat{q},\hat{p})$ with any $G(\hat{q},\hat{p})$. However, with particular $H(\hat{q},\hat{p})$ and $G(\hat{q},\hat{p})$, one defines some physical system for which, according to the standard interpretation, it is possible to describe changes in time and energy. By staying with the standard interpretation, one can say that if some quantum system evolves under the action of $H(\hat{q},\hat{p})$ until the moment, say, t_0, when it changes the energy, then one should apply $\hat{U}(i,j) \otimes e^{(1/i\hbar)\hat{t}\Delta E}$ on the state $|E_i\rangle \otimes |E_i\rangle$, which is the solution of the original Schrödinger equation for E_i, and get the other solution $|E_j\rangle \otimes |E_j\rangle$ of the same equation (here, $\hat{U}(i,j)$ is the unitary operator that rotates $|E_i\rangle$ into $|E_j\rangle$ in the first Hilbert space and ΔE is the energy difference among the given states). The moment t_0 at which this can happen depends on $G(\hat{q},\hat{p})$; that is, it has to belong to the spectrum of $G(\hat{q},\hat{p})$, just like E_i belongs to the spectrum of $H(\hat{q},\hat{p})$. In other words, $H(\hat{q},\hat{p})$ and $G(\hat{q},\hat{p})$ determine what the solutions of the original and the second Schrödinger equations are and everything related to this. In particular, on $H(\hat{q},\hat{p})$ and $G(\hat{q},\hat{p})$ depend in which amounts the energy of the system can be changed and in what time intervals this can happen if there is a sequence of time evolutions followed by the changes of the systems energy.

This leads us to the generalization of the original Schrödinger equation:

$$\left(c_s\hat{s} + c_t\hat{t}\right)|\Psi\rangle = F\left(\hat{q},\hat{p},\hat{s},\hat{t}\right)|\Psi\rangle, \quad |\Psi\rangle \in \mathscr{H}_q \otimes \mathscr{H}_t. \quad (10)$$

This equation superposes both types of the systems change. Obviously, the original and the second Schrödinger equation follow from (10) for the adequate choices of dimensional constants c_s, c_t, and $F(\hat{q},\hat{p},\hat{s},\hat{t})$.

3. Concluding Remarks

The proposed formalism, we believe, can be applied in the discussion of the situations when a quantum system is energy-driven by an external field, so its energy continuously changes, or in the cases with the instantaneous change of the energy in an interaction quench. With the generalized Schrödinger equation, one can analyze the response of the system to an imposed external time dependent field. That is, this equation is appropriate for investigations of the situations with time dependent Hamiltonians. However, such Hamiltonians shall be considered elsewhere [15]. On the other hand, the second Schrödinger equation is suitable for formal description of the systems response to instantaneous changes of the energy.

From the examples given above, it is obvious that there exist situations with commuting $H(\hat{q},\hat{p})$ and $F(\hat{q},\hat{p})$. Then, these two operators have common eigenstates in $\mathscr{H}_q$, say $|\psi_i(q)\rangle$, but there is no eigenvector in $\mathscr{H}_q \otimes \mathscr{H}_t$ that can simultaneously satisfy both Schrödinger equations (3) and (5). Namely, in $|q\rangle \otimes |t\rangle$ representation, the original Schrödinger equation would be solved with $\psi_i(q) \cdot e^{(1/i\hbar)E_i t}$, while $\psi_i(q) \cdot \delta(t - t_i)$ appear as the solutions of the second Schrödinger equation. A similar consideration stands for the generalized Schrödinger equation on the one hand and the original and the second Schrödinger equations on the other hand. That is, the solutions of the generalized Schrödinger equation are not simultaneously the solutions of the original and the second Schrödinger equation. In $|q\rangle \otimes |t\rangle$ representation, these differential equations have no common solutions. In other words, if (3), (5), and (10) are seen as the constraints in $\mathscr{H}_q \otimes \mathscr{H}_t$, these constraints have no intersection.

Finally, let us briefly comment on the measurement outcomes of the energy and time. Within the present formalism, both $\hat{s}$ and $\hat{t}$ have the whole R as spectrum, but only the values determined by $H(\hat{q},\hat{p})$ and $G(\hat{q},\hat{p})$ through Schrödinger equations can be attributed to the system under consideration. In other words, the physical system is defined by $H(\hat{q},\hat{p})$ and $G(\hat{q},\hat{p})$ and the Schrödinger equations select values of energy and time that are characteristic to that system. While other values are formally possible, they are unrelated to the system. Therefore, description of the measurement process should use orthogonal resolution of identity operator in a subspace determined by (3) or (5), not the identity operator in the whole $\mathscr{H}_q \otimes \mathscr{H}_t$.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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