Research Article

Semiquantum Chaos in Two GaAs Quantum Dots Coupled Linearly and Quadratically by Two Harmonic Potentials in Two Dimensions

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We analyze the phenomenon of semiquantum chaos in two GaAs quantum dots coupled linearly and quadratically by two harmonic potentials. We show how semiquantum dynamics should be derived via the Ehrenfest theorem. The extended Ehrenfest theorem in two dimensions is used to study the system. The numerical simulations reveal that, by varying the interdot distance and coupling parameters, the system can exhibit either periodic or quasi-periodic behavior and chaotic behavior.

1. Introduction

Classical mechanics deals with the fluctuations of macroscopic objects in phase space. It helps to drive and understand the matter laws fluctuations of an object. The historical evolution of dynamic systems evolves with the advent of nonlinear dynamics. Scientists like Newton, Euler, Lagrange, Laplace, and many others advocated the predictable behavior of integrable systems [1]. Those systems provide us later basics of studying systems that are nonintegrable. Nevertheless, we need to consider the manifestations of nonintegrable systems. That is the cause of apparition of chaos because an integrable system motion is either periodic or quasiperiodic. What properties does chaotic system satisfy? For a dynamic system to behave chaotically, it should fulfill three fundamental properties: boundedness, infinite recurrence, and sensitive dependence on initial conditions. For example, the authors in [2] studied chaos in semiconductor band-trap impact ionization, just to mention that.

However, many experimental observations did not explain the fluctuation of very small objects like atoms and molecules, that is, objects of size 10^{-8} m or smaller as

described by the wave function. Hence, it needs the help of quantum mechanics, where the scale is set by the Planck constant. The wave function is a complex value; it describes the behavior and gives the fluctuation laws of microscopic constituents of matter. In classical mechanics, time evolution can be described in terms of Hamilton's equation as well as in quantum mechanics. The latter describes how atoms are put together and why they are stable. The problem of chaos reflected in quantum physics can be traced back to the early days of quantum mechanics, when Poincaré worked on the three-body problem [3]. The concept of quantum chaos is very useful [4, 5]. It can be defined in numerous manners but chaotic quantum systems are characterized by their extreme sensitivity both to initial conditions and to parameters changes [6–8]. It is too earlier to make the concept of chaos for quantum systems and its experimental manifestations clear [9–12]. Nevertheless, there are three types of quantum chaos [13]: quantum chaology that refers to the study of the quantum signatures of classical chaos in the semiclassical or high quantum number regimes; semiquantum chaos, which refers rigorously to chaotic dynamics in semiquantum system; and genuine quantum chaos, which rigorously refers

to chaotic behavior in full quantum systems. The applications of quantum chaos are too many such as the best secure communication, the quantum cryptography, and the medical diagnosis.

Energy is optimized when objects are miniaturized. Particle moving in one-dimensional potentials has been studied by many authors. In 1998, Loss and DiVincenzo have proposed two laterally coupled quantum dots with one valence electron per dot; particular attention has been suggested toward quantum computing [14]. Then, in 2004, Fortuna and Porto [15] were interested in coupled quantum dot cells. They have studied the dynamic behavior by suitable variation of coupling parameter and initial conditions. Blum and Elze studied the semiquantum chaos in a double-well potential with the time dependence of variational principle [16]. Lekeufack et al. studied quantum chaology in a multiple-well adjustable potential heterostructure [17]. They investigated the signature of chaos in the semiquantum behavior of a classically regular triple-well heterostructure [18]. Pattanayak and Schieve also studied the semiquantal dynamics of fluctuations: ostensible quantum chaos [19, 20]. Besides, problems like independent Schrödinger's equation have been studied for different 1D potentials in phase space representation. For instance, there are free particle [21], linear potential [22], harmonic potential [23, 24], Morse potential [25], and so forth.

On the other hand, the quantum coupled dot potential is a bilayer system [26]. It is a useful game footing for understanding numerous quantum mechanical structures [27]. For example, a quadratic 2D potential is modeled in coupled semiconductor quantum dots as quantum gates by the author of [28] as $V(x, y) = (m\omega_0^2/2)[(1/4a^2)(x^2 - a^2)^2 + y^2]$, where ω_0 is the confining frequency, one for each dot, x and y are coordinates, m is the effective mass of electron, and a is the half distance between the center of the dots and the effective Bohr radius of a single isolated harmonic potential. An exchange coupling in semiconductor nanostructures is modeled by 2D quadratic potential [29] as $V(x, y) = (m\omega_0^2/2)[(x^2 - (R/2)^2)^2/R^2 + y^2]$. Here, the focus is on a linear coupled quantum dot to harmonic potential $V(x, y) = x^2(\alpha x^2 - \beta) + \kappa + y(\gamma y - \sigma x)$; the coupling parameter (σ) represents the shift of dots from the x -axis [30, 31]; with x and y being coordinates, κ is an interdot barrier; note that α is completely defined by ω_0 and the interdot distance; since β is set in such a way that the potential minimum could be zero, that is, $\beta = 2(\alpha\kappa)^{1/2}$ parameters on which the numerous variety of configurations are dependent, γ is a constant. The interdot distance is physically equivalent to the interdot barrier. Quantum dot has one degree of freedom as harmonic potential. We note that the two dots are exactly identical. We also study a quadratic coupled quantum dot to harmonic potential $V(x, y) = x^2(\alpha x^2 - \beta) + \kappa + y(\gamma y - \sigma x^2)$. Applications of these types of potential are found in study of many chemical reactions such as isomerization of malondialdehyde [32]. They induce several configurations according to the time evolution of quantum confinement of some rigid structures and so forth. The theoretical investigation of a linear and quadratic coupled quantum dot to harmonic potential is

important, since it allows both optimizing information and investigating fundamental physics.

To be more specific, in this paper, we aim at extending the Ehrenfest theorem in one dimension to two dimensions. It is possible to extend dynamic systems theory to partial differential equations. We may apply this approach to the dynamics of a particle moving in 2D potentials. The study of the semiquantal dynamics of a particle moving in a linear and quadratic coupled quantum dot to harmonic potential is of great interest. Classical effective Hamiltonian is used to describe the complete time evolution of the coupled classical and quantum vibrations, which is the expectation value of the quantum Hamiltonian.

The rest of the paper is organized as follows: Section 2 presents the methodology of the Ehrenfest theorem in two dimensions; in Section 3, we apply this methodology to a linear and quadratic coupled quantum dot to harmonic potential and present the numerical study of the semiquantum equations' motion. Finally, Section 4 gives the conclusion.

2. Methodology

We propose an extended method to construct extended effective Hamiltonian equation for two dimensions. The time-dependent variational principle (TDVP) formulation is very necessary to understand many equivalent methods such as the Gaussian trial wave function [16, 17, 19, 20]. Because its results are sometimes in good agreement with the experimental physics, the TDVP may be performed with the help of Dirac's variational principle. Variational parameters vanish at infinity and require the action $\Gamma[\Psi] = \int dt \langle \Psi(t) | i\hbar(\partial/\partial t) - H | \Psi(t) \rangle$. The wave function provides interesting results in both quantum chemistry and general physics. In ordinary quantum mechanics, a wave packet will be of the form $\Psi(x, y) = N(\pi\hbar)^{n/4} \exp\{[-1/2\hbar)((x - q_x)^2 + (y - q_y)^2) + (i/\hbar)(p_x + p_y)((x - q_x) + (y - q_y)) + i\lambda]\}$, if we restrict $|\Psi(t)\rangle$ to the family of coherent states [33, 34]. Here N, λ are the normalization and phase constants, respectively; the parameters p_x, p_y, q_x, q_y specify a point in the classical phase space of dimension $2n$. In this system, the Hamilton's equations for p_x, p_y, q_x, q_y , conserve the norm N and yield $\dot{\lambda} = (p_x + p_y)(\dot{q}_x + \dot{q}_y) - H(q_x, q_y, p_x, p_y)$, where H is the classical Hamiltonian. Therefore, $\lambda(t)$ is equal to the classical action $\Gamma = \int dt [(1/2)((\partial x/\partial t)^2 + (\partial y/\partial t)^2) - V(x, y)]$.

However, we will provide an alternative derivation of semiquantal dynamics via the Ehrenfest approach. The authors of [35, 36] said that, in classical mechanics or in quantum mechanics, the position and momentum are independent dynamical variables or operator. The configuration of the system in the $2N$ -dimensional $p - q$ is shown in phase space. The time evolution of a system can be described in terms of Hamilton equations $dq_j/dt = \partial H/\partial p_j$ and $dp_j/dt = -\partial H/\partial q_j$, where $j = 1, 2$; q_j and p_j are the position and momentum, respectively. In quantum mechanics, there are also several equivalent ways of describing the dynamics. Here, quantities like position and momentum play the roles of operators as well as variables [35, 36]. Let us focus on a

particle of unit mass moving in a two-dimensional time-independent bounded potential. The Hamiltonian on which a system is configurable is

$$\widehat{H} = \frac{\widehat{p}_x^2 + \widehat{p}_y^2}{2} + V(\widehat{q}_x, \widehat{q}_y), \quad (1)$$

where $\widehat{q}_x$, $\widehat{q}_y$, $\widehat{p}_x$, and $\widehat{p}_y$ are operators; they depend on time t through these coordinates. In two dimensions, (1) can be written as

$$\begin{aligned} \frac{d}{dt} \langle \widehat{q}_x \rangle &= \langle \widehat{p}_x \rangle, \\ \frac{d}{dt} \langle \widehat{p}_x \rangle &= - \left\langle \frac{\partial V(\widehat{q}_x, \widehat{q}_y)}{\partial \widehat{q}_x} \right\rangle, \\ \frac{d}{dt} \langle \widehat{q}_y \rangle &= \langle \widehat{p}_y \rangle, \\ \frac{d}{dt} \langle \widehat{p}_y \rangle &= - \left\langle \frac{\partial V(\widehat{q}_x, \widehat{q}_y)}{\partial \widehat{q}_y} \right\rangle, \end{aligned} \quad (2)$$

where $\langle \rangle$ indicates expectation values. Generally, the centroid does not go along the classical trajectory. We need to extend the above equations around the centroid using the 2D identity.

$$\langle F_{\widehat{q}_u}(\widehat{q}_u, \widehat{q}_v) \rangle = \frac{1}{n!} \langle \widehat{Q}_u^n \rangle F_{\widehat{q}_u}^{(n)}, \quad n \geq 0, \quad (3)$$

$$\langle F_{\widehat{q}_v}(\widehat{q}_u, \widehat{q}_v) \rangle = \frac{1}{n!} \langle \widehat{Q}_v^n \rangle F_{\widehat{q}_v}^{(n)}, \quad n \geq 0,$$

where $F_{\widehat{q}_u}^{(n)} = \partial^n F_{\widehat{q}_u} / \partial \widehat{q}_u^n |_{\langle \widehat{q}_u \rangle}$ and $\widehat{Q}_u = \widehat{q}_u - \langle \widehat{q}_u \rangle$; $F_{\widehat{q}_v}^{(n)} = \partial^n F_{\widehat{q}_v} / \partial \widehat{q}_v^n |_{\langle \widehat{q}_v \rangle}$, and $\widehat{Q}_v = \widehat{q}_v - \langle \widehat{q}_v \rangle$. Countably infinite number of equations could be generated by using these operator rules. The space is rendered finite by the belief that the wave packet is a squeezed coherent; hence, these relations are provided.

$$\begin{aligned} \langle \widehat{Q}_u^{2m} \rangle &= \frac{(2m)! \mu_u^m}{m! 2^m} \quad (\text{no summation}), \\ \langle \widehat{Q}_u^{2m+1} \rangle &= 0, \\ 4\mu_u \langle \widehat{P}_u^2 \rangle &= \hbar^2 + \alpha_u^2, \\ \langle \widehat{Q}_u \widehat{P}_u + \widehat{Q}_u \widehat{P}_u \rangle &= \alpha_u; \\ \langle \widehat{Q}_v^{2m} \rangle &= \frac{(2m)! \mu_v^m}{m! 2^m} \quad (\text{no summation}), \\ \langle \widehat{Q}_v^{2m+1} \rangle &= 0, \\ 4\mu_v \langle \widehat{P}_v^2 \rangle &= \hbar^2 + \alpha_v^2, \\ \langle \widehat{Q}_v \widehat{P}_v + \widehat{Q}_v \widehat{P}_v \rangle &= \alpha_v. \end{aligned} \quad (4)$$

The above equations are recognized as generalized Gaussian wave functions [33, 37–39]. After reducing the system to the dynamics of q_x , q_y , p_x , p_y , μ_x , μ_y , α_x , and α_y (where q_x , q_y , p_x , and p_y are written for $\langle \widehat{q}_x \rangle$, $\langle \widehat{q}_y \rangle$, $\langle \widehat{p}_x \rangle$, and $\langle \widehat{p}_y \rangle$), we introduce the change of variables $\mu_x = \rho_x^2$ and $\alpha_x = 2\rho_x \xi_x$ and $\mu_y = \rho_y^2$ and $\alpha_y = 2\rho_y \xi_y$. This gives the following new equations:

$$\frac{dq_x}{dt} = p_x, \quad (5a)$$

$$\frac{dq_y}{dt} = p_y, \quad (5b)$$

$$\frac{dp_x}{dt} = -\frac{\rho_x^{2m}}{m! 2^m} V_{q_x}^{(2m+1)}(q_x, q_y), \quad m = 0, 1, \dots \quad (5c)$$

$$\frac{dp_y}{dt} = -\frac{\rho_y^{2m}}{m! 2^m} V_{q_y}^{(2m+1)}(q_x, q_y), \quad m = 0, 1, \dots \quad (5d)$$

$$\frac{d\rho_x}{dt} = \xi_x, \quad (5e)$$

$$\frac{d\rho_y}{dt} = \xi_y, \quad (5f)$$

$$\frac{d\xi_x}{dt} = \frac{\hbar^2}{4\rho_x^3} - \frac{\rho_x^{2m-1}}{(m-1)! 2^{m-1}} V_{q_x}^{(2m)}(q_x, q_y), \quad (5g)$$

$$m = 1, 2, \dots$$

$$\frac{d\xi_y}{dt} = \frac{\hbar^2}{4\rho_y^3} - \frac{\rho_y^{2m-1}}{(m-1)! 2^{m-1}} V_{q_y}^{(2m)}(q_x, q_y), \quad (5h)$$

$$m = 1, 2, \dots$$

We observe that these new variables form an explicit canonically conjugate set. We note that the classical degrees of freedom are the “average” variables q_x , q_y , p_x , and p_y and the “fluctuation” variables ρ_x , ρ_y , ξ_x , and ξ_y , respectively. The extended Hamiltonian is

$$\begin{aligned} H_{\text{ext}} &= \frac{p_x^2 + p_y^2}{2} + \frac{\xi_x^2 + \xi_y^2}{2} \\ &\quad + V_{\text{ext}}(q_x, q_y, \rho_x, \rho_y), \\ V_{\text{ext}}(q_x, q_y, \rho_x, \rho_y) &= V(q_x, q_y) + \frac{\hbar^2}{8\rho_x^2} + \frac{\hbar^2}{8\rho_y^2} \\ &\quad + \frac{\rho_x^{2m}}{m! 2^m} V_{q_x}^{(2m)}(q_x, q_y) \\ &\quad + \frac{\rho_y^{2m}}{m! 2^m} V_{q_y}^{(2m)}(q_x, q_y), \end{aligned} \quad (6)$$

$$m = 1, 2, \dots,$$

where the subscript “ext” indicates the “extended” potential and Hamiltonian. This formulation is very useful and

FIGURE 1: The contour plots ((a) and (b)) and the mesh plots ((c) and (d)) of the potential with linear and quadratic coupling. The parameters are fixed: $\alpha = 0.2$, $\kappa = 3$, $\gamma = 0.5$, and $\sigma = 0.3$.

powerful because the gradient system is well explained by it, and the extended potential provides simple visualization of the configuration of the semiquantal space. We could get a qualitative feel for the semiquantal dynamics before starting the numerical analysis. First of all, the average and fluctuation variables are considered at the same footing and the phase space is dimensionally consistent: ρ_x and ρ_y have the dimension of lengths and ξ_x and ξ_y have the dimension of momentum.

3. Double Quantum Dot Potential Coupled Linearly and Quadratically to Harmonic Potential

The purpose of this section is to draw the basic set of equations of a particle unit mass moving in linear and quadratic

two GaAs quantum dots coupled to harmonic potential. Physical phenomena related to two GaAs double quantum dots are of great interest. Hence, quantum mechanisms concepts are defined in condensed matter physics [40]. The two-dimensional quantum-mechanical toy model in which a particle is moving in the upside down GaAs quantum dots for linear and quadratic coupling potential may be studied. First of all, the contour plots and the mesh plots of the 2D potential in the cases of linear and quadratic coupling are shown in Figure 1. From the latter, we observe that the coupling parameter σ is responsible for shift of dots from the x -axis. It stands that many physics phenomena can be observed in the two potentials dimensions. The potential with linear coupling is $V(q_x, q_y) = q_x^2(\alpha q_x^2 - 2(\alpha\kappa)^{1/2}) + \kappa + q_y(\gamma q_y - \sigma q_x)$ and the potential with quadratic coupling is $V(q_x, q_y) = q_x^2(\alpha q_x^2 - 2(\alpha\kappa)^{1/2}) + \kappa + q_y(\gamma q_y - \sigma q_x^2)$.

FIGURE 2: Phase portrait (a) and Poincaré sections (b) in the plane ξ_x, ρ_x with the initial conditions $q_{x0} = 1, q_{y0} = 1, p_{x0} = 3, p_{y0} = 1, \rho_{x0} = 1, \rho_{y0} = 1, \xi_{x0} = 1$, and $\xi_{y0} = 3$. We set the zero coupling parameters as $\kappa = 3$ and $\alpha = 0.1$.

3.1. Zero Coupling. We describe the zero coupling parameter GaAs quantum dots dynamics system with the Hamiltonian

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2} + \alpha \hat{q}_x^4 - 2(\alpha\kappa) \frac{1}{2} \hat{q}_x^2 + \kappa + \gamma \hat{q}_y^2; \quad (7)$$

we need to extend the Hamiltonian to

$$\begin{aligned} H = & \frac{p_x^2 + p_y^2}{2} + \frac{\xi_x^2 + \xi_y^2}{2} + \alpha (q_x^4 + 6\rho_x^2 q_x^2 + 3\rho_x^4) \\ & - 2(\alpha\kappa) \frac{1}{2} (\rho_x^2 + q_x^2) + \gamma (q_y^2 + \rho_y^2) \\ & + \frac{1}{8} \left(\frac{1}{\rho_x^2} + \frac{1}{\rho_y^2} \right) + \kappa. \end{aligned} \quad (8)$$

When the coupling parameter is equal to zero, the plots for either lower or higher interdot distance in those configurations display chaotic behavior. Figure 2 shows the phase portrait and Poincaré plots in the plane ξ_x, ρ_x . In order to determine the shape of trajectories of electrons, we find the periodicity of the system for various interdot barriers. It is almost the case for lower interdot barrier $\kappa = 3$, where irregular chaotic motion was easily observed and shown in Figure 2. Almost similar behavior is also seen for upper interdot distance (not shown).

We have to know the quantitative measure of the degree of chaos in a given region; then we plot the maximal Lyapunov exponent of (5a)–(5h) versus the parameter κ as shown in Figure 3. In Figure 3, the maximal Lyapunov exponent has

a positive value for any value of parameter κ . This means that the system described by (5a)–(5h) is chaotic.

3.2. Linear Coupling. The extended Hamiltonian for the dynamics of the linear coupled GaAs quantum dots system is

$$\begin{aligned} H = & \frac{p_x^2 + p_y^2}{2} + \frac{\xi_x^2 + \xi_y^2}{2} + \alpha (x^4 + 6\rho_x^2 q_x^2 + 3\rho_x^4) \\ & - 2(\alpha\kappa) \frac{1}{2} (\rho_x^2 + q_x^2) + \gamma (q_y^2 + \rho_y^2) - \sigma q_x q_y \\ & + \frac{1}{8} \left(\frac{1}{\rho_x^2} + \frac{1}{\rho_y^2} \right) + \kappa. \end{aligned} \quad (9)$$

The plots for lower interdot barrier and lower coupling parameter (not shown) display chaotic behavior for different chosen initial conditions closer to the minimum of the potential. Indeed, for the values of the control parameters κ and σ are increased to medium or upper values, we find chaotic trajectories again. So, at $\kappa = 200$, the system finds itself to be chaotic as shown in Figure 4. Although lower interdot barrier parameter induces chaos, raising interdot barrier κ or coupling parameter σ to higher value makes a bend appear in the plane (p_x, q_x) as shown in Figure 4.

The maximal Lyapunov exponent versus the coupling parameter σ is shown in Figure 5. We realize in Figure 5 that the maximal Lyapunov exponent $\lambda_{\max}$ has a positive value. It means that the system described by (5a)–(5h) is chaotic.

FIGURE 3: The maximal Lyapunov exponent in zero coupling case in (5a)–(5h). The parameter is fixed as $\alpha = 0.1$.

FIGURE 4: Phase portrait (a) and Poincaré sections (b) in the plane p_x, q_x with the initial conditions $q_{xo} = 1, q_{yo} = 1, p_{xo} = 3, p_{yo} = 7, \rho_{xo} = 1, \rho_{yo} = 1, \xi_{xo} = 4$, and $\xi_{yo} = 3$. We set the linear coupling parameters as $\sigma = 0.2, \kappa = 200$, and $\alpha = 0.1$.

3.3. *Quadratic Coupling.* The extended Hamiltonian is

$$\begin{aligned}
 H = & \frac{p_x^2 + p_y^2}{2} + \frac{\xi_x^2 + \xi_y^2}{2} + \alpha (q_x^4 + 6\rho_x^2 q_x^2 + 3\rho_x^4) \\
 & - 2(\alpha\kappa)^{1/2} (\rho_x^2 + q_x^2) + \gamma (q_y^2 + \rho_y^2) \\
 & - \sigma (q_x^2 q_y + q_y \rho_x^2) + \frac{1}{8} \left(\frac{1}{\rho_x^2} + \frac{1}{\rho_y^2} \right) + \kappa.
 \end{aligned} \tag{10}$$

For some specific value of the quadratic coupling parameter, the trajectories of the system exhibit chaotic behavior

as shown in Figure 6. It is interesting to note that our equations are coupled; classical and quantum interactions are linked. At $\hbar \rightarrow 0$ classical limit, only the first four equations remain, confirming that the fluctuation variables are responsible for quantum effects. As far as the coupling quadratic potential is concerned, it is the most useful and complicated. We may conclude by saying that more bends are observed by projecting on the planes (p_x, q_x) and (p_y, q_x) . Thus, it fascinates more and more spreading in each dot. Here again, the chaotic trajectories are very pronounced for higher interdot barrier (not shown). Further, chaotic motion is found from medium interdot distance $\kappa = 5$ as shown in Figure 6.

FIGURE 5: The maximal Lyapunov exponent in linear coupling case in (5a)–(5h). The parameter is fixed as $\alpha = 0.1$.

FIGURE 6: Phase portrait (a) and Poincaré sections (b) in the plane p_x, q_x with the initial conditions $q_{xo} = 2, q_{yo} = 1, p_{xo} = 1, p_{yo} = 1, \rho_{xo} = 1, \rho_{yo} = 2, \xi_{xo} = 1$, and $\xi_{yo} = 1$. We set the quadratic coupling parameters as $\sigma = 0.1, \kappa = 5$, and $\alpha = 0.1$.

The maximal Lyapunov exponent versus the coupling parameter σ is shown in Figure 7. From Figure 7, we note that the maximal Lyapunov exponent $\lambda_{\max}$ has a positive value. It means that the system described by (5a)–(5h) is chaotic.

4. Conclusion

In this paper, semiquantum chaos in two GaAs quantum dots coupled linearly and quadratically by two harmonic potentials was investigated using the Ehrenfest theorem. This work connected two distant concepts in physics: the quite popular double-dot potential in semiconductor structures

proposed by Loss and DiVincenzo for qubits in a quantum computer and quantum chaos. The existence of irregular chaotic motion in nonintegrable systems was pointed out. To address this situation, we used a variational method to extend the classical phase space in order to take into account the quantum fluctuations and derive a set of effective Hamiltonian equations in this extended space. By using this extended method, the existence of chaotic motion was shown in zero coupling, linear coupling, and quadratic coupling. Then, interdot distance parameter was used for controlling the system. It was discovered that the effects of interdot barrier parameter and coupling parameter induce irregular

FIGURE 7: The maximal Lyapunov exponent in quadratic coupling case in (5a)–(5h). The parameter is fixed as $\alpha = 0.1$.

behavior such as chaos. This result enables us to estimate the density of particles that have chaotic trajectories in the nonintegrable systems, which might be useful for best performing quantum computing efficiently estimating potential errors introduced by Loss and DiVincenzo in quantum computer. The effect of high temperature was neglected in this work and will be studied in future works. The study of the full quantum dynamics without classical counterpart will also be interesting for better monitoring of quantum chaos.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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