Research Article

Two Types of Solutions to a Class of (p, q) -Laplacian Systems with Critical Sobolev Exponents in $\mathbb{R}^N$

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We focus on the following elliptic system with critical Sobolev exponents: $-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + m(x)|u|^{p^*-2}u = \lambda|u|^{p^*-2}u + (1/\eta)G_u(u, v)$, $x \in \mathbb{R}^N$; $-\operatorname{div}(|\nabla v|^{q-2}\nabla v) + n(x)|v|^{q^*-2}v = \mu|v|^{q^*-2}v + (1/\eta)G_v(u, v)$, $x \in \mathbb{R}^N$; $u(x) > 0, v(x) > 0$, $x \in \mathbb{R}^N$, where $\mu, \lambda > 0$, $1 < p \leq q < N$, either $\eta \in (1, p)$ or $\eta \in (q, p^*)$, and critical Sobolev exponents $p^* = pN/(N-p)$ and $q^* = qN/(N-q)$. Conditions on potential functions $m(x), n(x)$ lead to no compact embedding. Relying on concentration-compactness principle, mountain pass lemma, and genus theory, the existence of solutions to the elliptic system with $\eta \in (q, p^*)$ or $\eta \in (1, p)$ will be established.

1. Introduction and Main Result

We aim at the existence of nontrivial weak solutions for the following elliptic system:

$$\begin{aligned} & -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + m(x)|u|^{p^*-2}u \\ & = \lambda|u|^{p^*-2}u + \frac{1}{\eta}G_u(u, v), \quad x \in \mathbb{R}^N, \\ & -\operatorname{div}(|\nabla v|^{q-2}\nabla v) + n(x)|v|^{q^*-2}v \\ & = \mu|v|^{q^*-2}v + \frac{1}{\eta}G_v(u, v), \quad x \in \mathbb{R}^N, \\ & u(x) > 0, \quad v(x) > 0, \quad x \in \mathbb{R}^N, \end{aligned} \quad (1)$$

where $\mu, \lambda > 0$, $1 < p \leq q < p^* < q^*$, and either $1 < \eta < p$ or $q < \eta < p^*$.

Discussion regarding critical growth has been dominated research in various branches of mathematical physics. Here, we quote a few literatures on critical Sobolev exponent and their applications; see Brezis and Nirenberg [1], Wu [2–4], and Garcia-Azorero et al. [5] and the references therein.

By applying variational arguments, Yin and Yang in [6] proved the existence of at least one or two positive solutions to the following system:

$$\begin{aligned} & -\Delta_p u + |u|^{p^*-2}u = f_{1\lambda_1}(x)|u|^{q^*-2}u \\ & \quad + \frac{2\alpha}{\alpha + \beta}g_\mu(x)|u|^{\alpha-2}u|v|^\beta, \\ & \quad x \in \Omega, \\ & -\Delta_q v + |v|^{q^*-2}v = f_{2\lambda_2}(x)|v|^{p^*-2}v \\ & \quad + \frac{2\beta}{\alpha + \beta}g_\mu(x)|u|^\alpha|v|^{\beta-2}v, \\ & \quad x \in \Omega, \end{aligned} \quad (2)$$

$$u = v = 0, \quad \text{on } \partial\Omega,$$

where $F_{i\lambda_i}(x) = \lambda_i f_{i+}(x) + f_{i-}(x)$ ($i = 1, 2$) are sign-changing functions.

Subsequently, at least one nontrivial solution was established in the work [7]:

$$\begin{aligned} -\Delta_p u &= f(x) |u|^{p^*-2} u + \lambda \frac{\partial F}{\partial u}(x, u, v), \quad \text{in } \mathbb{R}^N, \\ -\Delta_q v &= g(x) |v|^{q^*-2} v + \lambda \frac{\partial F}{\partial v}(x, u, v), \quad \text{in } \mathbb{R}^N, \end{aligned} \quad (3)$$

$$u(x), v(x) \longrightarrow 0, \quad \text{as } |x| \longrightarrow \infty$$

via mountain pass lemma. More results can be found in [7–10].

In papers cited above, the authors are fond of supposing that potential functions $m(x), n(x)$ satisfy such conditions:

- ($\mathcal{F}_1$) Functions $m(x), n(x) \in C(\mathbb{R}^N)$ and satisfy $m(x), n(x) \geq c_0 > 0$ with some constant c_0 . Moreover, for every $\kappa > 0$, $\text{meas}(\{x \in \mathbb{R}^N : m(x), n(x) \leq \kappa\}) < \infty$, where “meas” denotes the Lebesgue measure in $\mathbb{R}^N$.
- ($\mathcal{F}_2$) $m(x) \rightarrow +\infty, n(x) \rightarrow +\infty$, as $|x| \rightarrow \infty$.

We point out that one of the above conditions has been used in many papers to guarantee the compact embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{\nu}(\mathbb{R}^N)$ for each $\nu \in [p, p^*)$. Nevertheless, throughout the paper, we are going to weaken the conditions and give assumptions as follows:

- ($\mathcal{H}_1$) Let nonnegative function $G(u, v) \in C^1(\mathbb{R}^2)$ be homogeneous with $G(tu, tv) = t^\eta G(u, v)$, where $\forall (u, v) \in \mathbb{R}^2$ and either $\eta \in (1, p)$ or $\eta \in (q, p^*)$. Furthermore, there exists $\beta(x) > 0$ such that

$$\begin{aligned} 0 &\leq G(u, v), G_u(u, v)u, G_v(u, v)v \\ &\leq \beta(x) (|u|^\eta + |v|^\eta), \quad \forall (u, v) \in \mathbb{R}^2, \end{aligned} \quad (4)$$

and for all $(u, v), (\xi, \zeta) \in \mathbb{R}^2$

$$\begin{aligned} |G(u, v) - G(\xi, \zeta)| &\leq \beta(x) \\ &\cdot (|u|^{\eta-1} + |v|^{\eta-1} + |\xi|^{\eta-1} + |\zeta|^{\eta-1}) \\ &\cdot (|u - \xi| + |v - \zeta|). \end{aligned} \quad (5)$$

- ($\mathcal{H}_2$) Functions $m(x), n(x) \in C(\mathbb{R}^N)$ and satisfy $m(x), n(x) \geq c_0$ with some constant c_0 . In addition, for every $\kappa > 0$, $\text{meas}(\{x \in \mathbb{R}^N : m(x)/\beta(x), n(x)/\beta(x) \leq \kappa\}) < \infty$, where “meas” denotes the Lebesgue measure in $\mathbb{R}^N$.

- ($\mathcal{H}_3$) Function $\beta(x)$ is positive and $\beta(x) \in L_{\text{loc}}^\sigma(\mathbb{R}^N)$ with $\sigma = p^*/(p^* - \eta)$, $\eta \in (p, q^*)$. Moreover, $\limsup_{|x| \rightarrow \infty} \beta(x) < \infty$.

- ($\mathcal{H}_4$) Function $\beta(x)$ is positive and $\beta(x) \in L^\gamma(\mathbb{R}^N)$ with $\gamma = q^*/(q^* - \eta)$, $\eta \in (1, p)$.

Condition ($\mathcal{H}_2$) used in our paper is weaker than ($\mathcal{F}_1$) which leads to no compact embedding of $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\nu(\mathbb{R}^N)$. As far as the authors are aware, there is less literature dealing with such potential functions. Moreover, a more generalized function $G(u, v)$ appeared in the paper.

In order to state our main result, let us recall some Sobolev spaces and norms. Let X, Y be subspaces of $W^{1,p}(\mathbb{R}^N)$, $W^{1,q}(\mathbb{R}^N)$ with the norms

$$\begin{aligned} X &= \left\{ u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} m(x) |u|^p dx < \infty \right\} \\ \text{with } \|u\|_X &= \left(\int_{\mathbb{R}^N} (|\nabla u|^p + m(x) |u|^p) dx \right)^{1/p}, \\ Y &= \left\{ v \in W^{1,q}(\mathbb{R}^N) : \int_{\mathbb{R}^N} n(x) |v|^q dx < \infty \right\} \\ \text{with } \|v\|_Y &= \left(\int_{\mathbb{R}^N} (|\nabla v|^q + n(x) |v|^q) dx \right)^{1/q}, \\ \|u\|_t &= \left(\int_{\mathbb{R}^N} |u|^t dx \right)^{1/t}, \\ \|u\|_{t,m} &= \left(\int_{\mathbb{R}^N} m(x) |u(x)|^t dx \right)^{1/t}, \\ \|u\|_{t,n} &= \left(\int_{\mathbb{R}^N} n(x) |u(x)|^t dx \right)^{1/t}, \end{aligned} \quad (6)$$

$1 < t < N$.

Obviously, the norm of reflexive Banach space $E = X \times Y$ is given by

$$\|(u, v)\|_E = \|u\|_X + \|v\|_Y, \quad \forall (u, v) \in E. \quad (7)$$

Define

$$\begin{aligned} S_1 &= \inf_{u \in X \setminus \{0\}} \frac{\|u\|_X^p}{\left(\int_{\mathbb{R}^N} |u|^{p^*} dx \right)^{p/p^*}}, \\ S_2 &= \inf_{v \in Y \setminus \{0\}} \frac{\|v\|_Y^q}{\left(\int_{\mathbb{R}^N} |v|^{q^*} dx \right)^{q/q^*}}, \\ c^* &= \min \left\{ \frac{\lambda}{N} \left(\frac{S_1}{\lambda} \right)^{N/p}, \frac{\mu}{N} \left(\frac{S_2}{\mu} \right)^{N/q} \right\}. \end{aligned} \quad (8)$$

Corresponding functional of (1) defined in E is

$$\begin{aligned} J(u, v) &= \frac{1}{p} \|u\|_X^p + \frac{1}{q} \|v\|_Y^q - \frac{\lambda}{p^*} \int_{\mathbb{R}^N} |u|^{p^*} dx \\ &\quad - \frac{\mu}{q^*} \int_{\mathbb{R}^N} |v|^{q^*} dx - \frac{1}{\eta} \int_{\mathbb{R}^N} G(u, v) dx. \end{aligned} \quad (9)$$

The main results are enunciated as follows.

Theorem 1. Suppose ($\mathcal{H}_1$), ($\mathcal{H}_2$), and ($\mathcal{H}_3$) with $\eta \in (q, p^*)$ hold; then there exists $\Lambda_0 > 0$ such that if $0 < \lambda, \mu < \Lambda_0$, system (1) has at least a nontrivial solution.

Theorem 2. Suppose $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, and $(\mathcal{H}_4)$ with $\eta \in (1, p)$ hold; then there exists $\Lambda_1 > 0$ such that if $0 < \lambda, \mu < \Lambda_1$, problem (1) has infinitely many solutions $(u_n, v_n) \in E$ with $J(u_n, v_n) < 0$, $J(u_n, v_n) \rightarrow 0$, and $(u_n, v_n) \rightarrow 0$ as $n \rightarrow \infty$.

This work is structured as follows. Variational framework will be set up and some useful lemmas will be derived in Section 2. The proofs of Theorems 1 and 2 are established in Sections 3 and 4, respectively.

2. Preliminaries

The main purpose of this section is to present that functional J satisfies $(PS)_c$ condition. Besides, some useful technical lemmas will be listed.

Lemma 3. Assume that $(\mathcal{H}_1)$ – $(\mathcal{H}_4)$ hold; if $\{(u_n, v_n)\} \subset E$ is a $(PS)_c$ sequence, then sequence $\{(u_n, v_n)\}$ is bounded in E .

Proof. Let $J(u_n, v_n) \rightarrow c$, $J'(u_n, v_n) \rightarrow 0$ in E^* as $n \rightarrow \infty$.

Case 1. When $\eta \in (q, p^*)$, we can choose $q < \theta < \eta$ such that

$$\begin{aligned} c + 1 + \|(u_n, v_n)\|_E &\geq J(u_n, v_n) - \frac{1}{\theta} J'(u_n, v_n)(u_n, v_n) \\ &= \left(\frac{1}{p} - \frac{1}{\theta}\right) \|u_n\|_X^p + \left(\frac{1}{q} - \frac{1}{\theta}\right) \|v_n\|_Y^q \\ &\quad + \left(\frac{1}{\theta} - \frac{1}{p^*}\right) \int_{\mathbb{R}^N} \lambda |u_n|^{p^*} dx \\ &\quad + \left(\frac{1}{\theta} - \frac{1}{q^*}\right) \int_{\mathbb{R}^N} \mu |v_n|^{q^*} dx \\ &\quad + \left(\frac{1}{\theta} - \frac{1}{\eta}\right) \int_{\mathbb{R}^N} G(u_n, v_n) dx \\ &\geq \left(\frac{1}{p} - \frac{1}{\theta}\right) \|u_n\|_X^p + \left(\frac{1}{q} - \frac{1}{\theta}\right) \|v_n\|_Y^q \\ &\geq \left(\frac{1}{q} - \frac{1}{\theta}\right) \min \{ \|(u_n, v_n)\|_E^p, \|(u_n, v_n)\|_E^q \}. \end{aligned} \quad (10)$$

Case 2. When $\eta \in (1, p)$, we can choose $q < \theta < p^*$ such that

$$\begin{aligned} c + 1 + \|(u_n, v_n)\|_E &\geq J(u_n, v_n) - \frac{1}{\theta} J'(u_n, v_n)(u_n, v_n) \\ &= \left(\frac{1}{p} - \frac{1}{\theta}\right) \|u_n\|_X^p + \left(\frac{1}{q} - \frac{1}{\theta}\right) \|v_n\|_Y^q + \left(\frac{1}{\theta} - \frac{1}{p^*}\right) \\ &\quad \cdot \int_{\mathbb{R}^N} \lambda |u_n|^{p^*} dx + \left(\frac{1}{\theta} - \frac{1}{q^*}\right) \int_{\mathbb{R}^N} \mu |v_n|^{q^*} dx \\ &\quad - \left(\frac{1}{\eta} - \frac{1}{\theta}\right) \int_{\mathbb{R}^N} G(u_n, v_n) dx \geq \left(\frac{1}{p} - \frac{1}{\theta}\right) \|u_n\|_X^p \\ &\quad + \left(\frac{1}{q} - \frac{1}{\theta}\right) \|v_n\|_Y^q - \left(\frac{1}{\eta} - \frac{1}{\theta}\right) \int_{\mathbb{R}^N} \beta(x) (|u_n|^\eta \\ &\quad + |v_n|^\eta) dx \geq \left(\frac{1}{q} - \frac{1}{\theta}\right) \end{aligned}$$

$$\begin{aligned} &\cdot \min \{ \|(u_n, v_n)\|_E^p, \|(u_n, v_n)\|_E^q \} - \left(\frac{1}{\eta} - \frac{1}{\theta}\right) \|\beta(x)\|_\vartheta \\ &\cdot \|u_n\|_{p^*}^\eta - \left(\frac{1}{\eta} - \frac{1}{\theta}\right) \|\beta(x)\|_\gamma \|v_n\|_{q^*}^\eta \geq \left(\frac{1}{q} - \frac{1}{\theta}\right) \\ &\cdot \min \{ \|(u_n, v_n)\|_E^p, \|(u_n, v_n)\|_E^q \} - \left(\frac{1}{\eta} - \frac{1}{\theta}\right) \\ &\cdot S_1^{-\eta/p} \|\beta(x)\|_\vartheta \|u_n\|_X^\eta - \left(\frac{1}{\eta} - \frac{1}{\theta}\right) S_2^{-\eta/q} \|\beta(x)\|_\gamma \\ &\cdot \|v_n\|_Y^\eta \geq \left(\frac{1}{q} - \frac{1}{\theta}\right) \min \{ \|(u_n, v_n)\|_E^p, \|(u_n, v_n)\|_E^q \} \\ &- \max \left\{ \left(\frac{1}{\eta} - \frac{1}{\theta}\right) S_1^{-\eta/p} \|\beta(x)\|_\vartheta, \left(\frac{1}{\eta} - \frac{1}{\theta}\right) \right. \\ &\cdot S_2^{-\eta/q} \|\beta(x)\|_\gamma \left. \right\} \|(u_n, v_n)\|_E^\eta, \end{aligned} \quad (11)$$

where $\vartheta = p^*/(p^* - \eta) \geq \gamma = q^*/(q^* - \eta)$; the above inequalities imply the sequence $\{(u_n, v_n)\}$ is bounded in E and the proof is finished. $\square$

As the consequence of Lemma 3, there exist a constant $M > 0$, $(u, v) \in E$, and sequence $\{(u_n, v_n)\} \in E$ such that

$$\begin{aligned} \|(u_n, v_n)\|_E &\leq M, \quad \forall n \in \mathbb{N}, \quad \|(u, v)\|_E \leq M \\ (u_n, v_n) &\rightharpoonup (u, v) \text{ weakly in } E, \\ (u_n(x), v_n(x)) &\longrightarrow (u(x), v(x)) \text{ a.e. in } \mathbb{R}^N \\ (u_n, v_n) &\longrightarrow (u, v) \text{ strongly in } L_{\text{loc}}^s(\mathbb{R}^N) \\ &\quad \times L_{\text{loc}}^r(\mathbb{R}^N), \\ &\quad \forall s \in [1, p^*), \quad r \in [1, q^*). \end{aligned} \quad (12)$$

Lemma 4. Let $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, and $(\mathcal{H}_3)$ be satisfied with $p \leq \eta < p^*$. Suppose sequence $\{u_n\}$ is bounded in X ; then there exists $u \in X \cap L^\eta(\mathbb{R}^N, \beta)$ such that $u_n \rightarrow u$ strongly in $L^\eta(\mathbb{R}^N, \beta)$.

Proof. Since $\{u_n\}$ is bounded in X , there exist $u \in X$ and $M_1 > 0$ such that

$$\begin{aligned} \|u_n\| + \|u\| &\leq M_1; \\ u_n &\longrightarrow u \text{ strongly in } L_{\text{loc}}^s(\mathbb{R}^N), \\ &\quad \forall s \in [1, p^*). \end{aligned} \quad (13)$$

Thus, we have that $u_n \rightarrow u$ strongly in $L^p(B_i, \beta)$, where $B_i = \{x \in \mathbb{R}^N : |x| < i\}$ and $B_i^c = \{x \in \mathbb{R}^N : |x| \geq i\}$.

Step 1. We want to show that, for $\forall \epsilon > 0$, there exists $R > 1$ such that

$$\int_{B_R^c} \beta(x) |u_n|^p dx < \epsilon. \quad (14)$$

We can fix $\epsilon > 0$ and choose $t \in (1, N/(N-p))$ and a_0, κ , where

$$a_0 = \sup_{X \setminus \{0\}} \frac{\|u_n\|_{pt}^p}{\|u_n\|_X^p}, \quad (15)$$

$$\kappa > \frac{2M_1^p}{\epsilon} \geq \frac{2}{\epsilon} \sup_{n \geq 1} \|u_n\|_X^p.$$

Denote $A_i = \{x \in B_i : m(x)/\beta(x) \leq \kappa\}$; then $A_i \subset A_{i+1}$ for $i = 1, 2, \dots$, and $A_\infty = \{x \in \mathbb{R}^N : m(x)/\beta(x) \leq \kappa\} = \bigcup_{i=1}^\infty A_i$. What is more, there exist $i_0 \geq 1$ and $\beta_0 > 0$ such that $|\beta(x)| \leq \beta_0$ for $|x| \geq i_0$ by condition $(\mathcal{H}_3)$ and $\text{meas}(A_\infty)$ is bounded by condition $(\mathcal{H}_2)$; then, for every $\epsilon > 0$, there exists $i_1 \geq i_0$ such that for $i \geq i_1$

$$\begin{aligned} 0 &\leq \text{meas}(A_\infty) - \text{meas}(A_i) = \text{meas}(\Omega_i^c) \\ &\leq \left(\frac{\epsilon}{2a_0\beta_0M_1^p} \right)^{t/(t-1)}, \end{aligned} \quad (16)$$

where

$$\begin{aligned} \Omega_i &= \left\{ x \in B_i^c : \frac{m(x)}{\beta(x)} > \kappa \right\}, \\ \Omega_i^c &= \left\{ x \in B_i^c : \frac{m(x)}{\beta(x)} \leq \kappa \right\}. \end{aligned} \quad (17)$$

By (15) and $(\mathcal{H}_2)$, we observe that

$$\begin{aligned} \int_{\Omega_i} \beta(x) |u_n|^p dx &\leq \frac{1}{\kappa} \int_{\Omega_i} m(x) |u_n|^p dx \\ &\leq \frac{1}{\kappa} \int_{\mathbb{R}^N} m(x) |u_n|^p dx \leq \frac{1}{\kappa} \|u_n\|_X^p \leq \frac{\epsilon}{2}, \\ \int_{\Omega_i^c} \beta(x) |u_n|^p dx &\leq \left(\int_{\Omega_i^c} |u_n|^{pt} dx \right)^{1/t} \left(\int_{\Omega_i^c} |\beta(x)|^{t/(t-1)} dx \right)^{(t-1)/t} \\ &\leq a_0 \|u_n\|_X^p \|\beta(x)\|_{L^{t/(t-1)}(\Omega_i^c)} \\ &\leq a_0 \beta_0 \|u_n\|_X^p [\text{meas}(\Omega_i^c)]^{(t-1)/t} \leq \frac{\epsilon}{2}. \end{aligned} \quad (18)$$

Therefore, we get

$$\begin{aligned} \int_{B_i^c} \beta(x) |u_n|^p dx &= \int_{\Omega_i} \beta(x) |u_n|^p dx \\ &\quad + \int_{\Omega_i^c} \beta(x) |u_n|^p dx \leq \epsilon, \end{aligned} \quad (19)$$

and by Fatou's lemma

$$\int_{B_i^c} \beta(x) |u|^p dx \leq \liminf_{n \rightarrow \infty} \int_{B_i^c} \beta(x) |u_n|^p dx \leq \epsilon. \quad (20)$$

Thus, from (13) and (20), we can get that $u_n \rightarrow u$ strongly in $L^p(\mathbb{R}^N, \beta)$.

Step 2. For $p < \eta < p^*$, let $\eta = rp + (1-r)p^*$, $r \in (0, 1)$. By (19), we have that

$$\begin{aligned} &\int_{B_i^c} \beta(x) |u_n|^\eta dx \\ &\leq \int_{B_i^c} \beta(x)^r |u_n|^{rp} \beta(x)^{1-r} |u_n|^{(1-r)p^*} dx \\ &\leq \beta_0^{1-r} \|u_n\|_{L^p(B_i^c, \beta)}^{rp} \|u_n\|_{L^{p^*}(B_i^c)}^{(1-r)p^*} \\ &\leq \beta_0^{1-r} S_1^{-(1-r)p^*/p} \|u_n\|_X^{(1-r)p^*} \|u_n\|_{L^p(B_i^c, \beta)}^{rp} \\ &\leq \beta_0^{1-r} S_1^{-(1-r)p^*/p} M_1^{(1-r)p^*} \epsilon = C\epsilon. \end{aligned} \quad (21)$$

By Fatou's lemma, we can see that

$$\int_{B_i^c} \beta(x) |u|^\eta dx \leq \liminf_{n \rightarrow \infty} \int_{B_i^c} \beta(x) |u_n|^\eta dx \leq C\epsilon. \quad (22)$$

For each measure subset $\Omega \subset B_i$, we have that

$$\begin{aligned} \int_\Omega \beta(x) |u_n - u|^\eta dx &\leq \|\beta\|_{L^\sigma(\Omega)} \|u_n - u\|_{L^{p^*}(\Omega)}^\eta \\ &\leq \|\beta\|_{L^\sigma(\Omega)} \|u_n - u\|_{L^{p^*}(\mathbb{R}^N)}^\eta \\ &\leq S_1^{-\eta/p} (2M_1)^\eta \|\beta\|_{L^\sigma(\Omega)}. \end{aligned} \quad (23)$$

Thus, one can get that $\{\beta(x)|u_n - u|^\eta\}$ is uniformly integral and $\int_{B_i} \beta(x)|u_n - u|^\eta dx$ is bounded. Then, we have that

$$\lim_{n \rightarrow \infty} \int_{B_i} \beta(x) |u_n - u|^\eta dx = 0. \quad (24)$$

Therefore, from (22) and (24), we can obtain

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \beta(x) |u_n - u|^\eta dx = 0. \quad (25)$$

Similar argument can be applied to get that $v_n \rightarrow v$ strongly in $L^\eta(\mathbb{R}^N, \beta)$. Thus, one can easily get that $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} G(u_n, v_n) dx = \int_{\mathbb{R}^N} G(u, v) dx$. $\square$

Lemma 5. Assume that $(\mathcal{H}_1)$ and $(\mathcal{H}_4)$ hold with $\eta \in (1, p)$; if $\{(u_n, v_n)\}$ satisfies (12), then

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} G(u_n, v_n) dx = \int_{\mathbb{R}^N} G(u, v) dx. \quad (26)$$

Proof. By $(\mathcal{H}_1)$ and $(\mathcal{H}_4)$, we get that

$$\int_{B_R} \beta(x) |u_n|^\eta dx \rightarrow \int_{B_R} \beta(x) |u|^\eta dx, \quad n \rightarrow \infty, \quad (27)$$

where $B_R = \{x \in \mathbb{R}^N : |x| < R\}$ and $B_R^c = \mathbb{R}^N \setminus \overline{B_R}$ for any $R > 0$. Moreover, we obtain from Hölder inequality that

$$\begin{aligned} & \int_{B_R^c} |\beta(x)| |u_n|^\eta dx \\ & \leq \left(\int_{B_R^c} |\beta(x)|^{p^*/(p^*-\eta)} dx \right)^{(p^*-\eta)/p^*} \left(\int_{B_R^c} |u_n|^{p^*} dx \right)^{\eta/p^*} \\ & \leq S_1^{-\eta/p} \|\beta(x)\|_{L^\vartheta(B_R^c)} \|u_n\|_E^\eta \\ & \leq S_1^{-\eta/p} M^\eta \|\beta(x)\|_{L^\vartheta(B_R^c)} \longrightarrow 0, \quad \text{as } R \longrightarrow \infty, \end{aligned} \quad (28)$$

where $\vartheta = p^*/(p^* - \eta) \geq \gamma = q^*/(q^* - \eta)$. By Fatou's lemma, we have

$$\begin{aligned} \int_{B_R^c} |\beta(x)| |u|^\eta dx & \leq \liminf_{n \rightarrow \infty} \int_{B_R^c} |\beta(x)| |u_n|^\eta dx \\ & \leq M^\eta \|\beta(x)\|_{L^\vartheta(B_R^c)} \longrightarrow 0, \end{aligned} \quad (29)$$

as $R \longrightarrow \infty$.

Then, from (28) and (29), we can deduce $\int_{\mathbb{R}^N} \beta(x) |u_n|^\eta dx \rightarrow \int_{\mathbb{R}^N} \beta(x) |u|^\eta dx$. Similarly, we can get $\int_{\mathbb{R}^N} \beta(x) |v_n|^\eta dx \rightarrow \int_{\mathbb{R}^N} \beta(x) |v|^\eta dx$. Thus, the proof is completed. $\square$

Denote

$$\begin{aligned} P_n &= J'(u_n, v_n) (u_n - u, v_n - v) \\ &= \int_{\mathbb{R}^N} [|\nabla u_n|^{p-2} \nabla u_n \nabla (u_n - u) \\ &\quad + m(x) |u_n|^{p-2} u_n (u_n - u)] dx \\ &\quad + \int_{\mathbb{R}^N} [|\nabla v_n|^{q-2} \nabla v_n \nabla (v_n - v) \\ &\quad + n(x) |v_n|^{q-2} v_n (v_n - v)] dx \\ &\quad - \lambda \int_{\mathbb{R}^N} |u_n|^{p^*-2} u_n (u_n - u) dx - \mu \int_{\mathbb{R}^N} |v_n|^{q^*-2} \\ &\quad \cdot v_n (v_n - v) dx - \frac{1}{\eta} \\ &\quad \cdot \int_{\mathbb{R}^N} [G_u(u_n, v_n) (u_n - u) \\ &\quad + G_v(u_n, v_n) (v_n - v)] dx. \end{aligned} \quad (30)$$

Then the fact $J'(u_n, v_n) \rightarrow 0$ in E^* implies $P_n \rightarrow 0$ as $n \rightarrow \infty$. Similarly, the fact that $(u_n, v_n) \rightharpoonup (u, v)$ in E implies that $Q_n \rightarrow 0$, where

$$\begin{aligned} Q_n &= \int_{\mathbb{R}^N} [|\nabla u|^{p-2} \nabla u \nabla (u_n - u) \\ &\quad + m(x) |u|^{p-2} u (u_n - u)] dx \\ &\quad + \int_{\mathbb{R}^N} [|\nabla v|^{q-2} \nabla v \nabla (v_n - v) \\ &\quad + n(x) |v|^{q-2} v (v_n - v)] dx. \end{aligned} \quad (31)$$

Note that

$$\begin{aligned} P_n - Q_n &= T_n - \lambda \int_{\mathbb{R}^N} |u_n|^{p^*-2} u_n (u_n - u) dx \\ &\quad - \mu \int_{\mathbb{R}^N} |v_n|^{q^*-2} v_n (v_n - v) dx - \frac{1}{\eta} \\ &\quad \cdot \int_{\mathbb{R}^N} [G_u(u_n, v_n) (u_n - u) \\ &\quad + G_v(u_n, v_n) (v_n - v)] dx, \end{aligned} \quad (32)$$

where

$$\begin{aligned} T_n &= \int_{\mathbb{R}^N} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) \nabla (u_n - u) dx \\ &\quad + \int_{\mathbb{R}^N} m(x) (|u_n|^{p-2} u_n - |u|^{p-2} u) (u_n - u) dx \\ &\quad + \int_{\mathbb{R}^N} (|\nabla v_n|^{q-2} \nabla v_n - |\nabla v|^{q-2} \nabla v) \nabla (v_n - v) dx \\ &\quad + \int_{\mathbb{R}^N} n(x) (|v_n|^{q-2} v_n - |v|^{q-2} v) (v_n - v) dx. \end{aligned} \quad (33)$$

If $T_n \rightarrow 0$, this implies $\|u_n - u\|_X \rightarrow 0$ and $\|v_n - v\|_Y \rightarrow 0$ according to the following lemma.

Lemma 6 (see [11]). *There exists constant $C > 0$ such that, for all $x, y \in \mathbb{R}^N$,*

$$\begin{aligned} & (|x|^{p-2} x - |y|^{p-2} y, x - y) \\ & \geq C (|x| + |y|)^{p-2} |x - y|^2, \quad \text{if } 1 < p < 2, \\ & (|x|^{p-2} x - |y|^{p-2} y, x - y) \geq C |x - y|^p, \quad \text{if } p \geq 2, \end{aligned} \quad (34)$$

where $(\cdot, \cdot)$ denotes the usual scalar product in $\mathbb{R}^N$.

In the following, we will claim

$$\begin{aligned} & \int_{\mathbb{R}^N} \lambda |u_n|^{p^*-2} u_n (u_n - u) dx \longrightarrow 0, \\ & \int_{\mathbb{R}^N} \mu |v_n|^{q^*-2} v_n (v_n - v) dx \longrightarrow 0, \end{aligned} \quad (35)$$

as $n \longrightarrow \infty$.

Based on concentration-compactness principle [12] and (12), it follows that

$$\begin{aligned} & |\nabla u_n|^p \rightharpoonup \mu \geq |\nabla u|^p + \sum_{j \in \mathcal{J}} \mu_j \delta_{x_j}, \\ & |u_n|^{p^*} \rightharpoonup v = |u|^{p^*} + \sum_{j \in \mathcal{J}} v_j \delta_{x_j}, \\ & \mu_j \geq S_1 v_j^{p/p^*}, \end{aligned} \quad (36)$$

where δ_{x_j} is the Dirac measure at x_j and μ_j, ν_j are nonnegative bounded measures. Analogously, we get $|\nabla v_n|^q \rightarrow \bar{\mu}$ and $|v_n|^{q^*} \rightarrow \bar{\nu}$. Concentration at ∞ of $\{u_n\}$ is defined by

$$\begin{aligned}\mu_\infty &= \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |\nabla u_n|^p dx, \\ \nu_\infty &= \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |u_n|^{p^*} dx.\end{aligned}\quad (37)$$

Analogously, the concentration at ∞ of $\{v_n\}$ is given by $\bar{\mu}_\infty$ and $\bar{\nu}_\infty$.

Step 1. For $j \in \mathcal{J}$, we define $\phi_j \in C_0^\infty(\mathbb{R}^N)$ as follows:

$$\phi_j(x) = \begin{cases} 1, & \text{if } x \in B(x_j, \epsilon), \\ 0, & \text{if } x \in \mathbb{R}^N \setminus B(x_j, 2\epsilon), \\ |\nabla \phi_j| \leq C, & \text{the other.} \end{cases} \quad (38)$$

One can see that $\{u_n \phi_j\}$ is bounded in X ; we obtain $\lim_{n \rightarrow \infty} \langle J'(u_n, v_n), (u_n \phi_j, 0) \rangle = 0$. Namely,

$$\begin{aligned}& - \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p-2} u_n \nabla u_n \nabla \phi_j dx \\ &= \lim_{n \rightarrow \infty} \left[\int_{\mathbb{R}^N} (|\nabla u_n|^p \phi_j + m(x) |u_n|^p \phi_j) dx \right. \\ & \quad \left. - \lambda \int_{\mathbb{R}^N} |u_n|^{p^*} \phi_j dx - \frac{1}{\eta} \int_{\mathbb{R}^N} \frac{\partial G}{\partial u}(u_n, v_n) u_n \phi_j dx \right].\end{aligned}\quad (39)$$

The fact that $u_n \rightarrow u$ produces $\int_{\mathbb{R}^N} (\partial G / \partial u)(u_n, v_n) u_n \phi_j dx \rightarrow \int_{\mathbb{R}^N} (\partial G / \partial u)(u, v) u \phi_j dx$. In the light of Lions' principle, we acquire

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u_n|^p \phi_j dx &\rightarrow \int_{\mathbb{R}^N} \phi_j d\mu, \\ \int_{\mathbb{R}^N} |u_n|^{p^*} \phi_j dx &\rightarrow \int_{\mathbb{R}^N} \phi_j d\nu.\end{aligned}\quad (40)$$

Moreover

$$\begin{aligned}& \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \left| \int_{\mathbb{R}^N} |\nabla u_n|^{p-2} u_n \nabla u_n \nabla \phi_j dx \right| \\ & \leq \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \left(\int_{\mathbb{R}^N} |\nabla u_n|^p dx \right)^{(p-1)/p} \\ & \cdot \left(\int_{\mathbb{R}^N} |u_n \nabla \phi_j|^p dx \right)^{1/p} \leq M^{p-1} \\ & \cdot \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \left(\int_{\mathbb{R}^N} |u_n \nabla \phi_j|^p dx \right)^{1/p} \leq M^{p-1} \\ & \cdot \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \left(\int_{\mathbb{B}_{2\epsilon}(x_j)} |u_n|^{p^*} dx \right)^{1/p^*}\end{aligned}$$

$$\begin{aligned}& \cdot \left(\int_{\mathbb{B}_{2\epsilon}(x_j)} |\nabla \phi_j|^N dx \right)^{1/N} \leq C \\ & \cdot \lim_{\epsilon \rightarrow 0} \left(\int_{\mathbb{B}_{2\epsilon}(x_j)} |u|^{p^*} dx \right)^{1/p^*} = 0.\end{aligned}\quad (41)$$

At the same time, we have

$$\begin{aligned}& \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{\partial G}{\partial u}(u_n, v_n) u_n \phi_j dx \\ & \leq \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\mathbb{B}_{2\epsilon}(x_j)} \beta(x) (|u_n|^\eta + |v_n|^\eta) \phi_j dx \\ & \rightarrow 0, \\ & \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^p \phi_j dx \rightarrow 0.\end{aligned}\quad (42)$$

From the aforementioned equations, it follows that

$$0 = \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \langle J'(u_n, v_n), (u_n \phi_j, 0) \rangle = \mu_j - \lambda \nu_j; \quad (43)$$

combining with (36), we obtain

$$\begin{aligned}& \nu_j = 0 \\ & \text{or } \nu_j \geq \left(\frac{S_1}{\lambda} \right)^{N/P}.\end{aligned}\quad (44)$$

Step 2. By choosing a suitable cut-off function $\varphi(x) \in C^\infty(\mathbb{R}^N, [0, 1])$ such that

$$\varphi(x) = \begin{cases} 0, & \text{if } |x| \leq 1, \\ 1, & \text{if } |x| \geq 2, \end{cases} \quad (45)$$

we set $\varphi_R(x) = \varphi(x/R)$; then $\{\varphi_R u_n\}$ is bounded in X and $\lim_{R \rightarrow \infty} \langle J'(u_n, v_n), (u_n \varphi_R, 0) \rangle = 0$.

In a similar manner, we gain

$$\begin{aligned}& - \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p-2} u_n \nabla u_n \nabla \varphi_R dx \\ &= \lim_{n \rightarrow \infty} \left[\int_{\mathbb{R}^N} (|\nabla u_n|^p \varphi_R + m(x) |u_n|^p \varphi_R) dx \right. \\ & \quad \left. - \lambda \int_{\mathbb{R}^N} |u_n|^{p^*} \varphi_R dx \right. \\ & \quad \left. - \frac{1}{\eta} \int_{\mathbb{R}^N} \frac{\partial G}{\partial u}(u_n, v_n) u_n \varphi_R dx \right].\end{aligned}\quad (46)$$

Note that

$$\begin{aligned}
& \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \int_{\mathbb{R}^N} |\nabla u_n|^{p-2} |u_n| |\nabla u_n| |\nabla \varphi_R| dx \right| \\
& \leq \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \left(\int_{\mathbb{R}^N} |\nabla u_n|^p dx \right)^{(p-1)/p} \\
& \quad \cdot \left(\int_{\mathbb{R}^N} |u_n \nabla \varphi_R|^p dx \right)^{1/p} \leq M^{p-1} \\
& \quad \cdot \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \left(\int_{\mathbb{R}^N} |u_n \nabla \varphi_R|^p dx \right)^{1/p} \leq M^{p-1} \\
& \quad \cdot \lim_{R \rightarrow \infty} \left[\left(\int_{R \leq |x| \leq 2R} |u|^{p^*} dx \right)^{1/p^*} \right. \\
& \quad \cdot \left. \left(\int_{R \leq |x| \leq 2R} |\nabla \varphi_R|^N dx \right)^{1/N} \right] \leq C' \\
& \quad \cdot \lim_{R \rightarrow \infty} \left(\int_{R \leq |x| \leq 2R} |u|^{p^*} dx \right)^{1/p^*} = 0,
\end{aligned} \tag{47}$$

which deduces

$$\begin{aligned}
v_\infty &= 0 \\
\text{or } v_\infty &\geq \left(\frac{S_1}{\lambda} \right)^{N/P}.
\end{aligned} \tag{48}$$

Next we will prove $v_j \geq (S_p/\lambda)^{N/P}$ and $v_\infty \geq (S_p/\lambda)^{N/P}$ are impossible.

Case 1. If we fix $\eta \in (q, p^*)$, there exists j_0 such that $v_{j_0} \geq (S_p/\lambda)^{N/P}$; then one can get

$$\begin{aligned}
& J(u_n, v_n) - J'(u_n, v_n) \left(\frac{u_n}{p}, \frac{v_n}{q} \right) \\
& = \left(\frac{1}{p} - \frac{1}{p^*} \right) \int_{\mathbb{R}^N} \lambda |u_n|^{p^*} dx \\
& \quad + \left(\frac{1}{q} - \frac{1}{q^*} \right) \int_{\mathbb{R}^N} \mu |v_n|^{q^*} dx \\
& \quad + \frac{1}{\eta} \int_{\mathbb{R}^N} \left[\frac{1}{p} G_u u_n + \frac{1}{q} G_v v_n + G(u_n, v_n) \right] dx \\
& \geq \frac{\lambda}{N} \int_{\mathbb{R}^N} |u_n|^{p^*} dx + \frac{\mu}{N} \int_{\mathbb{R}^N} |v_n|^{q^*} dx.
\end{aligned} \tag{49}$$

Let $n \rightarrow \infty$; we gain

$$\begin{aligned}
c &\geq \frac{\lambda}{N} \int_{\mathbb{R}^N} |u|^{p^*} dx + \frac{\lambda}{N} \sum_{j \in \mathcal{J}} v_j + \frac{\mu}{N} \int_{\mathbb{R}^N} |v|^{q^*} dx \\
&\quad + \frac{\mu}{N} \sum_{j \in \mathcal{J}} \bar{v}_j \geq \frac{\lambda}{N} \sum_{j \in \mathcal{J}} v_j + \frac{\mu}{N} \sum_{j \in \mathcal{J}} \bar{v}_j \\
&\geq \frac{\lambda}{N} \left(\frac{S_1}{\lambda} \right)^{N/P} + \frac{\mu}{N} \left(\frac{S_2}{\mu} \right)^{N/q},
\end{aligned} \tag{50}$$

which gives rise to a contradiction, so $v_j = 0$ for all $j \in \mathcal{J}$. With the similar argument, we can get $\mu_\infty = v_\infty = 0$.

Case 2. If we fix $\eta \in (1, p)$, one can see

$$\begin{aligned}
0 > c &= \lim_{n \rightarrow \infty} \left[J(u_n, v_n) - J'(u_n, v_n) \left(\frac{u_n}{p^*}, \frac{v_n}{q^*} \right) \right] \\
&= \left(\frac{1}{p} - \frac{1}{p^*} \right) \|u_n\|_X^p + \left(\frac{1}{q} - \frac{1}{q^*} \right) \|v_n\|_Y^q \\
&\quad - \left(\frac{1}{\eta} - \frac{1}{q^*} \right) \int_{\mathbb{R}^N} G(u_n, v_n) dx \geq \frac{1}{N} \\
&\quad \cdot \min \{ \| (u_n, v_n) \|_E^p, \| (u_n, v_n) \|_E^q \} - \left(\frac{1}{\eta} - \frac{1}{q^*} \right) \\
&\quad \cdot \int_{\mathbb{R}^N} \beta(x) (|u_n|^\eta + |v_n|^\eta) dx \geq \frac{1}{N} \\
&\quad \cdot \min \{ \| (u_n, v_n) \|_E^p, \| (u_n, v_n) \|_E^q \} - \left(\frac{1}{\eta} - \frac{1}{q^*} \right) \\
&\quad \cdot \left(\| \beta(x) \|_9 S_1^{-\eta/p} \|u_n\|_X^\eta + \| \beta(x) \|_Y S_2^{-\eta/q} \|v_n\|_Y^\eta \right) \\
&\geq \frac{1}{N} \min \{ \| (u_n, v_n) \|_E^p, \| (u_n, v_n) \|_E^q \} - \left(\frac{1}{\eta} - \frac{1}{q^*} \right) \\
&\quad \cdot \max \{ \| \beta(x) \|_9 S_1^{-\eta/p}, \| \beta(x) \|_Y S_2^{-\eta/q} \} \| (u_n, v_n) \|_E^\eta;
\end{aligned} \tag{51}$$

if $\min \{ \| (u_n, v_n) \|_E^p, \| (u_n, v_n) \|_E^q \} = \| (u_n, v_n) \|_E^p$ we can obtain

$$\begin{aligned}
\| (u_n, v_n) \|_E &\leq \left(N \left(\frac{1}{\eta} - \frac{1}{q^*} \right) \right. \\
&\quad \cdot \max \{ \| \beta(x) \|_9 S_1^{-\eta/p}, \| \beta(x) \|_Y S_2^{-\eta/q} \} \Big)^{1/(p-\eta)};
\end{aligned} \tag{52}$$

otherwise,

$$\begin{aligned}
\| (u_n, v_n) \|_E &\leq \left(N \left(\frac{1}{\eta} - \frac{1}{q^*} \right) \right. \\
&\quad \cdot \max \{ \| \beta(x) \|_9 S_1^{-\eta/p}, \| \beta(x) \|_Y S_2^{-\eta/q} \} \Big)^{1/(q-\eta)};
\end{aligned} \tag{53}$$

moreover,

$$\begin{aligned}
0 > c &= \lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \left[J(u_n, v_n) \right. \\
&\quad \left. - J'(u_n, v_n) \left(\frac{u_n}{p^*} \varphi_R, \frac{v_n}{q^*} \varphi_R \right) \right] \geq \left(\frac{1}{p} - \frac{1}{p^*} \right) \mu_\infty \\
&\quad + \left(\frac{1}{q} - \frac{1}{q^*} \right) \bar{\mu}_\infty - \left(\frac{1}{\eta} - \frac{1}{q^*} \right) \max \{ \| \beta(x) \|_9 S_1^{-\eta/p}, \\
&\quad \| \beta(x) \|_Y S_2^{-\eta/q} \} \| (u_n, v_n) \|_E^\eta = \frac{1}{N} (\mu_\infty + \bar{\mu}_\infty) - \left(\left(\frac{1}{\eta} \right. \right. \\
&\quad \left. \left. - \frac{1}{q^*} \right) \max \{ \| \beta(x) \|_9 S_1^{-\eta/p}, \| \beta(x) \|_Y S_2^{-\eta/q} \} \right)^{p/(p-\eta)}
\end{aligned}$$

$$\begin{aligned}
\cdot N^{\eta/(p-\eta)} &\geq \frac{1}{N} \left(\left(\frac{S_1}{\lambda} \right)^{N/p} + \left(\frac{S_2}{\mu} \right)^{N/q} \right) - \left(\left(\frac{1}{\eta} \right. \right. \\
&\quad \left. \left. - \frac{1}{q^*} \right) \max \{ \|\beta(x)\|_9 S_1^{-\eta/p}, \|\beta(x)\|_7 S_2^{-\eta/q} \} \right)^{p/(p-\eta)} \\
&\cdot N^{\eta/(p-\eta)}.
\end{aligned} \tag{54}$$

We can easily get that there exists $\Lambda_0 > 0$ being small such that when $0 < \lambda, \mu < \Lambda_0$, the right side is positive, which gives rise to a contradiction.

Up to now, we have shown that

$$\begin{aligned}
\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^{p^*} &= \int_{\mathbb{R}^N} |u|^{p^*} \\
\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |v_n|^{q^*} &= \int_{\mathbb{R}^N} |v|^{q^*}.
\end{aligned} \tag{55}$$

In view of (12) and the Brezis-Lieb Lemma [13], we acquire

$$\begin{aligned}
\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n - u|^{p^*} dx &= 0, \\
\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |v_n - v|^{q^*} dx &= 0.
\end{aligned} \tag{56}$$

As a result of the above, it yields $\|(u_n - u, v_n - v)\|_E \rightarrow 0$, which implies the functional of (1) fulfils $(PS)_c$ condition.

3. Proof of Theorem 1

The purpose of this section is to give proof of Theorem 1 which mainly relies on mountain pass lemma as follows.

Lemma 7 (see [14]). *Let X be a real Banach space, $J \in C^1(E, \mathbb{R})$, $J(0) = 0$. If*

- $(\mathcal{P}_1)$ *there exist constants $r, b > 0$ such that $J(u) \geq b$ for all $u \in X$ with $\|u\|_X = r$,*
- $(\mathcal{P}_2)$ *there is $e, \|e\|_X > r$, such that $J(e) \leq 0$,*
- $(\mathcal{P}_3)$ *the functional $J(u)$ satisfies PS condition: every sequence $\{u_n\} \subset X$, such that $J(u_n) \rightarrow c$ for some c , and $J'(u_n) \rightarrow 0$ in X^* as $n \rightarrow \infty$, is relatively compact in X ,*

then J possesses a critical value $c \geq b$ which can be characterized as

$$c = \inf_{g \in \Gamma} \max_{t \in [0,1]} J(g(t)), \tag{57}$$

where

$$\Gamma = \{g \in C([0,1], X) \mid g(0) = 0, g(1) = e\}. \tag{58}$$

In the following, we want to show that

$$c < c^* = \min \left\{ \frac{\lambda}{N} \left(\frac{S_1}{\lambda} \right)^{N/p}, \frac{\mu}{N} \left(\frac{S_2}{\mu} \right)^{N/q} \right\}. \tag{59}$$

We suppose $c^* = (\lambda/N)(S_1/\lambda)^{N/p}$ and we can get the other case with similar arguments. Let $\omega = u/\|u\|_{p^*} \in X \setminus \{0\}$; then we get the following lemma.

Lemma 8. *Under hypotheses $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, and $(\mathcal{H}_3)$ with $\eta \in (q, p^*)$, there exists $\Lambda_0 > 0$ such that if $0 < \lambda < \Lambda_0$, we have $0 < \sup_{t \geq 0} J(t\omega, 0) < c^*$.*

Proof. For

$$\begin{aligned}
J(t\omega, 0) &= \frac{t^p}{p} \|\omega\|_X^p - \frac{t^{p^*}}{p^*} \int_{\mathbb{R}^N} \lambda |\omega|^{p^*} dx \\
&\quad - \frac{1}{\eta} \int_{\mathbb{R}^N} G(t\omega, 0) dx \\
&= \frac{t^p}{p} \|\omega\|_X^p - \frac{t^{p^*}}{p^*} \int_{\mathbb{R}^N} \lambda |\omega|^{p^*} dx \\
&\quad - \frac{t^\eta}{\eta} \int_{\mathbb{R}^N} G(\omega, 0) dx
\end{aligned} \tag{60}$$

with $p < \eta < p^*$, we can easily get $J(t\omega, 0) \rightarrow -\infty$ as $t \rightarrow \infty$, $J(t_0\omega, 0) < 0$ with t_0 large enough, and $J(t\omega, 0) \geq r > 0$ with t sufficiently small. So $0 < \sup_{t \geq 0} J(t\omega, 0) = J(t_\lambda\omega, 0)$ for some $t_\lambda > 0$; namely,

$$\begin{aligned}
J(t_\lambda\omega, 0) &= \frac{t_\lambda^p}{p} \|\omega\|_X^p - \frac{t_\lambda^{p^*}}{p^*} \int_{\mathbb{R}^N} \lambda |\omega|^{p^*} dx \\
&\quad - \frac{t_\lambda^\eta}{\eta} \int_{\mathbb{R}^N} G(\omega, 0) dx.
\end{aligned} \tag{61}$$

We can have $J'(t_\lambda\omega, 0)(t_\lambda\omega, 0) = 0$; that is,

$$\begin{aligned}
t_\lambda^p \|\omega\|_X^p - t_\lambda^{p^*} \int_{\mathbb{R}^N} \lambda |\omega|^{p^*} dx \\
- \frac{1}{\eta} \int_{\mathbb{R}^N} G_u(t_\lambda\omega, 0)(t_\lambda\omega, 0) dx = 0.
\end{aligned} \tag{62}$$

By direct computation $t_\lambda^p \|\omega\|_X^p \geq t_\lambda^{p^*} \int_{\mathbb{R}^N} \lambda |\omega|^{p^*} dx$, we can get $t_\lambda \leq (S_1/\lambda)^{(N-p)/p^2}$.

Meanwhile

$$\begin{aligned}
c &\leq \max_{t \geq 0} J(t\omega, 0) = \max_{t \geq 0} \left\{ \frac{t^p}{p} \|\omega\|_X^p \right. \\
&\quad \left. - \frac{t^{p^*}}{p^*} \int_{\mathbb{R}^N} \lambda |\omega|^{p^*} dx - \frac{t^\eta}{\eta} \int_{\mathbb{R}^N} G(\omega, 0) dx \right\} \\
&\leq \max_{t \geq 0} \left\{ \frac{t^p}{p} \|\omega\|_X^p - \frac{t^{p^*}}{p^*} \int_{\mathbb{R}^N} \lambda |\omega|^{p^*} dx \right\}.
\end{aligned} \tag{63}$$

Denote $F(t) = (t^p/p)\|\omega\|_X^p - (t^{p^*}/p^*) \int_{\mathbb{R}^N} \lambda |\omega|^{p^*} dx$, $t > 0$; it can achieve its maximum at

$$\begin{aligned}
t &= \left(\frac{\|\omega\|_X^p}{\int_{\mathbb{R}^N} \lambda |\omega|^{p^*} dx} \right)^{(N-p)/p^2} \\
&= \left(\frac{S_1}{\lambda} \right)^{(N-p)/p^2};
\end{aligned} \tag{64}$$

$$\begin{aligned}
\sup_{t>0} F(t) &= F\left(\frac{S_1}{\lambda}\right) \\
&= \frac{\|\omega\|_X^p}{p} \left(\left(\frac{S_1}{\lambda}\right)^{(N-p)/p^2} \right)^p \\
&\quad - \frac{\int_{\mathbb{R}^N} \lambda |\omega|^{p^*} dx}{p^*} \left(\left(\frac{S_1}{\lambda}\right)^{(N-p)/p^2} \right)^{p^*} \\
&= \frac{\|\omega\|_X^p}{p} \left(\frac{S_1}{\lambda}\right)^{(N-p)/p} \\
&\quad - \frac{\int_{\mathbb{R}^N} \lambda |\omega|^{p^*} dx}{p^*} \left(\frac{S_1}{\lambda}\right)^{N/p} \\
&= \frac{\lambda}{N} \left(\frac{S_1}{\lambda}\right)^{N/p} = c^*
\end{aligned} \tag{65}$$

thus we get $0 < \sup_{t \geq 0} J(t\omega) < c^*$. $\square$

Proof of Theorem 1. Obviously, functional $J(u, v)$ satisfies conditions $(\mathcal{P}_1)$ – $(\mathcal{P}_3)$ of the Lemma 7 we can define $c = \inf_{g \in \Gamma} \max_{t \in [0,1]} J(g(t))$, where Γ denotes the class of all continuous paths $g(t)$ joining the origin to $t_0\omega$. Thus, mountain pass lemma can be applied to conclude that $J(u, v)$ has a critical point with corresponding critical value c . So, system (1) has at least one nontrivial solution. $\square$

4. Proof of Theorem 2

In this section, we want to apply genus theory to get the main conclusion of Theorem 2. Let X be a Banach space and Σ be the class of closed and symmetric subsets with respect to the origin of $X - \{0\}$. For $Z \in \Sigma$, genus $\gamma(Z)$ is given by

$$\begin{aligned}
\gamma(Z) &= \min \{k \in \mathbb{N} : \exists \varphi \in C(Z, \mathbb{R}^k \setminus \{0\}), \varphi(x) \\
&= -\varphi(-x)\}.
\end{aligned} \tag{66}$$

Since $\beta(x) \in L^{\gamma}(\mathbb{R}^N)$ with $\gamma = q^*/(q^* - \eta)$, $\eta \in (1, p)$, thus we can have

$$\begin{aligned}
\int_{\mathbb{R}^N} G(u, v) dx &\leq \int_{\mathbb{R}^N} \beta(x) (|u|^{\eta} + |v|^{\eta}) dx \\
&\leq \|\beta(x)\|_{\vartheta} \|u\|_{p^*}^{\eta} + \|\beta(x)\|_{\gamma} \|v\|_{q^*}^{\eta} \\
&\leq \|\beta(x)\|_{\vartheta} S_1^{-\eta/p} \|u\|_X^{\eta} + \|\beta(x)\|_{\gamma} S_2^{-\eta/q} \|v\|_Y^{\eta} \\
&\leq \max \{ \|\beta(x)\|_{\vartheta} S_1^{-\eta/p}, \|\beta(x)\|_{\gamma} S_2^{-\eta/q} \} \|(u, v)\|_E^{\eta},
\end{aligned} \tag{67}$$

where $\vartheta = p^*/(p^* - \eta) \geq \gamma$.

Moreover,

$$\begin{aligned}
\int_{\mathbb{R}^N} \left(\frac{\lambda}{p^*} |u|^{p^*} + \frac{\mu}{q^*} |v|^{q^*} \right) dx &\leq \frac{\lambda}{p^*} S_1^{-p^*/p} \|u\|_X^{p^*} \\
&\quad + \frac{\mu}{q^*} S_2^{-q^*/q} \|v\|_Y^{q^*} \leq \max \left\{ \frac{\lambda}{p^*} S_1^{-p^*/p}, \frac{\mu}{q^*} S_2^{-q^*/q} \right\} \\
&\quad \cdot \max \{ \|(u, v)\|_E^{p^*}, \|(u, v)\|_E^{q^*} \}.
\end{aligned} \tag{68}$$

The corresponding functional $J(u, v)$ of (1) turns into

$$\begin{aligned}
J(u, v) &= \frac{1}{p} \|u\|_X^p + \frac{1}{q} \|v\|_Y^q - \frac{\lambda}{p^*} \int_{\mathbb{R}^N} |u|^{p^*} dx - \frac{\mu}{q^*} \\
&\quad \cdot \int_{\mathbb{R}^N} |v|^{q^*} dx - \frac{1}{\eta} \int_{\mathbb{R}^N} G(u, v) dx \geq \frac{1}{q} \\
&\quad \cdot \min \{ \|(u, v)\|_E^p, \|(u, v)\|_E^q \} \\
&\quad - \max \left\{ \frac{\lambda}{p^*} S_1^{-p^*/p}, \frac{\mu}{q^*} S_2^{-q^*/q} \right\} \\
&\quad \cdot \max \{ \|(u, v)\|_E^{p^*}, \|(u, v)\|_E^{q^*} \} \\
&\quad - \frac{1}{\eta} \max \{ \|\beta(x)\|_{\vartheta} S_1^{-\eta/p}, \|\beta(x)\|_{\gamma} S_2^{-\eta/q} \} \|(u, v)\|_E^{\eta} \\
&= \frac{1}{q} \min \{ \|(u, v)\|_E^p, \|(u, v)\|_E^q \} - b_1 \\
&\quad \cdot \max \{ \|(u, v)\|_E^{p^*}, \|(u, v)\|_E^{q^*} \} - b_2 \|(u, v)\|_E^{\eta},
\end{aligned} \tag{69}$$

where $b_1 = \max\{(\lambda/p^*)S_1^{-p^*/p}, (\mu/q^*)S_2^{-q^*/q}\}$ and $b_2 = (1/\eta) \max\{\|\beta(x)\|_{\vartheta} S_1^{-\eta/p}, \|\beta(x)\|_{\gamma} S_2^{-\eta/q}\}$. Define

$$h(x) = \frac{1}{q} \min \{x^p, x^q\} - b_1 \max \{x^{p^*}, x^{q^*}\} - b_2 x^{\eta}; \tag{70}$$

we can observe that there exists $\Lambda_1 > 0$ being small such that if $0 < \lambda, \mu < \Lambda_1$, there exists $0 < x_0 < x_1$ such that $h(x)$ becomes $-, 0, +, 0, -$ if $x \in (0, x_0), x_0, (x_0, x_1), x_1, (x_1, +\infty)$, respectively.

We set up the truncated functional [15]

$$\begin{aligned}
F(u, v) &= \frac{1}{p} \|u\|_X^p + \frac{1}{q} \|v\|_Y^q \\
&\quad - \psi(u, v) \left(\int_{\mathbb{R}^N} \frac{\lambda}{p^*} |u|^{p^*} + \frac{\mu}{q^*} |v|^{q^*} dx \right) \\
&\quad - \frac{1}{\eta} \int_{\mathbb{R}^N} G(u, v) dx,
\end{aligned} \tag{71}$$

where $\psi(u, v) = \psi(z) = \omega(\|z\|_E)$ and $\omega : \mathbb{R}^+ \rightarrow [0, 1]$ is a nonincreasing C^∞ function such that $\omega(x) = 1$ for $x \leq x_0$ and $\omega(x) = 0$ for $x \geq x_1$. Obviously, $F(u, v)$ is even and $F(u, v)$ satisfies $(PS)_c$ with $c < 0$.

Lemma 9. Assume $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, and $(\mathcal{H}_4)$ with $\eta \in (1, p)$ hold. Given $n \in \mathbb{N}$, there exists $\epsilon = \epsilon(n) < 0$ such that

$$\gamma(\{(u, v) \in E : F(u, v) \leq \epsilon\}) \geq n. \tag{72}$$

Proof. Fix n , and suppose E_n is n -dimensional subspace of E . We can take $(u_n, v_n) \in E_n$, with $\|(u_n, v_n)\|_E = 1$. Thus, for $0 < \rho < x_0$, we have

$$\begin{aligned} F(\rho u_n, \rho v_n) &= J(\rho u_n, \rho v_n) \\ &\leq \frac{1}{\rho} \rho^p \\ &\quad - \min \left\{ \frac{\lambda}{p^*}, \frac{\mu}{q^*} \right\} \rho^{q^*} \left(\int_{\mathbb{R}^N} |u|^{p^*} + |v|^{q^*} dx \right) \\ &\quad - \frac{1}{\eta} \rho^\eta \int_{\mathbb{R}^N} G(u, v) dx; \end{aligned} \quad (73)$$

we can choose $\epsilon < 0$ and $\varsigma < x_0$ small enough such that $F(\varsigma u, \varsigma v) \leq \epsilon$. Let $S_\varsigma = \{(u, v) \in E, \|(u, v)\|_E = \varsigma\}$; then, $S_\varsigma \cap E_n \subset \gamma(\{u \in E : F(u, v) \leq \epsilon\})$. Hence, $\gamma(\{(u, v) \in E : F(u, v) \leq \epsilon\}) \geq \gamma(S_\varsigma \cap E_n) = n$. $\square$

Lemma 10. Denote $\Sigma_k = \{Z \in \Sigma : \gamma(Z) \geq k\}$ and set

$$c_k = \inf_{Z \in \Sigma_k} \sup_{(u, v) \in Z} F(u, v), \quad (74)$$

$$B_c = \{(u, v) \in E : F^I(u, v) = 0, F(u, v) = c\};$$

then, if $c = c_k = c_{k+1} = \dots = c_{k+r}$, $\gamma(B_c) \geq r + 1$

Proof. For convenience, we denote $F^c = \{(u, v) \in E : F(u, v) \leq c\}$. Suppose $c = c_k = c_{k+1} = \dots = c_{k+r}$; we observe that $c < 0$ and B_c is a compact set. If $\gamma(B_c) \leq r$, there exists a symmetric and closed set $W_1, B_c \subset W_1$ such that $\gamma(W_1) \leq r$. According to deformation lemma, one can get an odd homeomorphism $\varrho : E \rightarrow E$ such that

$$\varrho(F^{c+\delta} \setminus W_1) \subset F^{c-\delta}, \quad f \text{ or } \delta > 0. \quad (75)$$

Based on the definition $c = c_{k+r}$, there exists $W_2 \in \Sigma_{k+r}$ such that $\sup_{(u, v) \in W_2} F(u, v) < c + \delta$ and

$$\varrho(W_2 \setminus W_1) \subset \rho(F^{c+\delta} \setminus W_1) \subset F^{c-\delta}. \quad (76)$$

Hence we gain

$$\begin{aligned} \gamma(\overline{W_2 \setminus W_1}) &\geq \gamma(W_2) - \gamma(W_1) \geq k, \\ \gamma(\varrho(\overline{W_2 \setminus W_1})) &\geq \gamma(\overline{W_2 \setminus W_1}) \geq k. \end{aligned} \quad (77)$$

Therefore, $\varrho(\overline{W_2 \setminus W_1}) \in \Sigma_k$ and it is in contradiction to (76). Lemma 10 and Kajikiya's [16] symmetric mountain pass lemma give the proof of Theorem 2. $\square$

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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