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Research Article

Adaptive Finite-Time Mixed Interlayer Synchronization of Two-Layer Complex Networks with Time-Varying Coupling Delay

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This paper is concerned with two-layer complex networks with unidirectional interlayer couplings, where the drive and response layer have time-varying coupling delay and different topological structures. An adaptive control scheme is proposed to investigate finite-time mixed interlayer synchronization (FMIS) of two-layer networks. Based on the Lyapunov stability theory, a criterion for realizing FMIS is derived. In addition, several sufficient conditions for realizing mixed interlayer synchronization are given. Finally, some numerical simulations are presented to verify the correctness and effectiveness of theoretical results. Meanwhile, the proposed adaptive control strategy is demonstrated to be nonfragile with the noise perturbation.

1. Introduction

As an important and typical dynamic behavior of complex networks, synchronization has been extensively investigated in many fields, such as physics, mathematics, information science, biology, and sociology. In the literatures, many works have primarily focused on synchronization within a network having no connection with other networks. However, in real-world situations, many systems are often composed of some interacting networks. For example, a transportation system consists of road network, railway network, and air network. A communication system is composed of some subnetworks depending on phone, email, QQ, Wechat, etc. Thus, multiplex networks, proposed by Mucha et al. [1], would be more appropriate for describing systems in the real-world than traditional (single-layer) complex networks. In the past few years, many efforts have been made to investigate various problems of multiplex networks, such as topological structure, dynamic behavior [2], synchronizability [3, 4], spectral property [5], diffusion process [6], and synchronization [7–10].

Interlayer synchronization is also called counterpart synchronization [11] which describes how the nodes in one network behave coherently with the corresponding ones in other connected networks. This concept can be regarded as the development of synchronization for coupled drive-response systems [12]. In a drive-response chaotic system, the response network is driven by signals from the drive network, but the latter is not influenced by the former. These two coupled single-layer networks, whose topological structures may be different, can be viewed as a two-layer network. For example, a two-layer network with unidirectional interlayer couplings is shown in Figure 1, where the layer X and the layer Y represent the drive layer and the response layer, respectively. Synchronization between the layer X and the layer Y exists extensively in the real world. This kind of interlayer synchronization can be also understood as outer synchronization [13] between two coupled networks. In the past decade, it has gained considerable attention—see [11, 13–20] and the references therein. Some control schemes have been proposed to realize all kinds of outer synchronization such as complete outer synchronization [13–17], inverse outer synchronization [18], generalized outer synchronization [19], and finite-time outer synchronization [21]. As a special case of generalized outer synchronization, mixed outer synchronization (MOS) was first proposed in [22]. Wang et al. [22] studied MOS between two complex networks with the same topological structure and time-varying coupling delay by

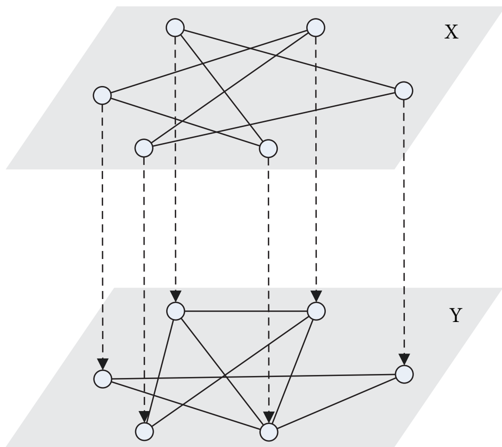

Figure 1: A two-layer network with unidirectional interlayer couplings.

designing robust linear feedback controllers. Later, Zheng and Shao [23] reported MOS between two complex networks with the same topological structure and output couplings via impulsive hybrid control. Sheng et al. [24] considered MOS between two complex networks with time-varying delay coupling and nondelay coupling by using pinning feedback control and impulsive control.

Finite-time mixed outer synchronization (FMOS) [25] between two complex networks is a recently developed MOS. Essentially, FMOS can be regarded as finite-time mixed interlayer synchronization (FMIS). In the state of FMIS, different state variables of the corresponding nodes can attain finite-time synchronization, finite-time antisynchronization, and even finite-time amplitude death simultaneously. FMIS is a kind of finite-time synchronization, in which the synchronization error remains within a prescribed range in a fixed time interval for a given range of initial error. Notice that finite-time synchronization is defined in a fixed finite-time interval. Hence, it has attracted increasing attention [26–31]. In [25], He et al. discussed FMOS between two complex networks with time-varying coupling delay and the same topological structures by designing a simple and robust linear state feedback controller. However, for most two-layer networks in real-world situations, the nodes are not always identical and topological structures of two layers are not always the same. In this paper, we consider FMIS of two-layer networks with time-varying coupling delay and different topological structures by designing an adaptive control scheme. Based on the Lyapunov stability theory, a sufficient condition for realizing FMIS of two-layer networks is derived. In addition, a criterion for realizing MIS is obtained. Finally, some numerical simulations are provided to verify the effectiveness of our theoretical results. Moreover, FMIS is analyzed when the two-layer networks are disturbed by the additive noise.

The rest of this paper is organized as follows. In Section 2, we formulate the two-layer network model and introduce some preliminaries. In Section 3, some sufficient conditions for realizing FMIS or MIS of two-layer networks are given under the proposed adaptive controllers. In Section 4, some numerical simulations are presented. Conclusions are finally drawn in Section 5.

2. Model and Preliminaries

Consider a dynamical two-layer complex network (See Figure 1) which is composed of the drive layer X and the response layer Y. Assume that each layer consists of N nodes. The drive layer X and the response layer Y are described as follows:

$$\begin{aligned}
\dot{x}_i(t) &= A_d^i x_i(t) + A_n^i x_i(t) + f_i(x_i(t)) \\
&\quad + \sum_{j=1}^{N} a_{ij} G\left(x_j(t-\tau(t))\right), \quad i = 1, 2, \cdots, N, \\
\dot{y}_i(t) &= A_d^i y_i(t) + A_n^i y_i(t) + f_i(y_i(t)) \\
&\quad + \sum_{j=1}^{N} b_{ij} G\left(y_j(t-\tau(t))\right) + u_i, \\
&\qquad i = 1, 2, \cdots, N,
\end{aligned} \tag{1}$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in})^T \in \mathbb{R}^n$ and $y_i = (y_{i1}, y_{i2}, \cdots, y_{in})^T \in \mathbb{R}^n$ are the state vectors, A_d and A_n are the diagonal and nondiagonal matrices representing the linear part of the i-th node dynamics, $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a smooth nonlinear vector function such that $Hf_i(x) = f_i(Hx)$, where H is a scaling matrix defined as $H = \mathrm{diag}(h_1, h_2, \cdots, h_n)$ ($h_i \in \{-1, 0, 1\}$). $G(x_i(t)) = (g(x_{i1}(t), g(x_{i2}(t)), \cdots, g(x_{in}(t)))^T$ is a nonlinear vector-valued function which denotes the inner connection of each node. $A = (a_{ij})_{N\times N} \in \mathbb{R}^{N\times N}$ is the weight configuration matrix of the drive layer X, where a_{ij} is defined as follows: if there is a connection from node i to node j ($j \neq i$), $a_{ij} \neq 0$; otherwise, $a_{ij} = 0$. The diagonal elements of matrix A are defined as $a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}$, $i = 1, 2, \cdots, N$. $B = (a_{ij})_{N\times N} \in \mathbb{R}^{N\times N}$ denotes the weight configuration matrix of the response layer Y and has the same meanings as that of A. $\tau(t) \geq 0$ is the time-varying coupling delay. u_i is a controller for node i to be designed later.

Now, we introduce some assumptions, definitions, and lemmas, which are necessary to the proofs of main results.

Assumption 1. The nonlinear function $f_i(x)$ ($i = 1, 2, \cdots, N$) satisfies the Lipschitz condition; i.e., there exists a positive constant α_i such that

$$\left\| f_i(x) - f_i(y) \right\| \leq \alpha_i \left\| x - y \right\|, \quad \forall x, y \in \mathbf{R}^n. \tag{2}$$

Assumption 2. The nonlinear function $G(x)$ satisfies $HG(x) = G(Hx)$ and the following inequality:

$$\left\| G(x) - G(y) \right\| \leq \sqrt{L} \left\| x - y \right\|, \quad \forall x, y \in \mathbf{R}^n, \tag{3}$$

where L is a known positive constant.

Assumption 3. The time-varying coupling delay $\tau(t)$ is a differential function with

$$0 \le \dot{\tau}(t) \le \varepsilon < 1, \quad 0 \le \tau(t) \le \overline{\tau}. \tag{4}$$

Notice that this assumption is obviously satisfied when the delay $\tau(t)$ is a constant.

For $i = 1, 2, \cdots, N$, let $e_i(t) = y_i(t) - Hx_i(t)$ be the synchronization error of node i.

Definition 4. Suppose that q_1, q_2, and T are three positive constants. Let $q_1 \le q_2$. The two-layer network (1) is said to achieve finite-time mixed interlayer synchronization (FMIS) with respect to (q_1, q_2, T, H) if there exists a controller $u_i(t)$ $(i = 1, 2, \cdots, N)$ for node i such that

$$\sup_{-\overline{\tau} \le v \le 0} \|e(v)\|^2 \le q_1, \tag{5}$$

and one has

$$\|e(t)\|^2 \le q_2, \quad \forall t \in [0, T], \tag{6}$$

where $e(t) = (\|e_1(t)\|, \|e_2(t)\|, \ldots, \|e_N(t)\|)^T$.

Remark 5. In view of Definition 4, when $H = I$ and $H = -I$, FMIS is reduced to the known notions: finite-time outer synchronization and finite-time outer antisynchronization, respectively.

Definition 6. The two-layer network (1) is said to achieve mixed interlayer synchronization (MIS) with respect to the scaling matrix H if there exists a controller $u_i(t)$ $(i = 1, 2, \cdots, N)$ for node i such that

$$\lim_{t \to \infty} \|e_i(t)\| = 0. \tag{7}$$

Lemma 7. *Let $x, y \in R^n$ and $P \in R^{n\times n}$. One has the following inequality:*

$$2x^T P^T y \le \eta x^T P^T P x + \frac{1}{\eta} y^T y \tag{8}$$

for any scalar $\eta > 0$.

3. Main Results

In this section, we will give some sufficient conditions for realizing FMIS or MIS of two-layer networks based on adaptive control. The adaptive controllers are designed as follows:

$$\begin{aligned} u_i(t) = {} & HA_n^i x_i(t) - A_n^i y_i(t) + \sum_{j=1}^{N} m_{ij} G\left(y_j(t - \tau(t))\right) \\ & - d_i(t) e_i(t), \quad i = 1, 2, \ldots, N. \end{aligned} \tag{9}$$

where

$$\begin{aligned} \dot{d}_i &= k_i e_i^T(t) e_i(t), \\ \dot{m}_{ij} &= -e_i^T(t) G\left(y_j(t - \tau(t))\right). \end{aligned} \tag{10}$$

Here, k_i $(i = 1, 2, \ldots, N)$ is a positive constant.

The error system can be described as

$$\begin{aligned} \dot{e}_i(t) = {} & A_d^i e_i(t) + f_i(y_i(t)) - Hf_i(x_i(t)) \\ & + \sum_{j=1}^{N} b_{ij} G\left(y_j(t - \tau(t))\right) \\ & - \sum_{j=1}^{N} a_{ij} HG\left(x_j(t - \tau(t))\right) \\ & + \sum_{j=1}^{N} m_{ij} G\left(y_j(t - \tau(t))\right) - d_i(t) e_i(t), \\ & i = 1, 2, \ldots, N. \end{aligned} \tag{11}$$

Theorem 8. *Suppose that Assumptions 1–3 hold. Let q_1, q_2, and T be three positive constants, and let $q_1 \le q_2$. FMIS of the two-layer network (1) can be realized with respect to (q_1, q_2, T, H) under the adaptive controllers (9) if there exist a diagonal matrix $D^* = \operatorname{diag}(d_1^*, d_2^*, \cdots, d_N^*)$ and a scalar $\theta > 0$ such that*

$$\left(\alpha + \xi + \frac{L}{2(1-\varepsilon)}\right) I_N + \frac{1}{2} AA^T - D^* - \theta I_N < 0, \tag{12}$$

$$e^{\theta T} q_1 \left(1 + \frac{L\overline{\tau}}{1-\varepsilon}\right) + e^{\theta T} \mu \le q_2. \tag{13}$$

Here, $\alpha = \max\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, $\xi = \max\{\lambda_{max}(A_d^1), \lambda_{max}(A_d^2), \ldots, \lambda_{max}(A_d^N)\}$, and μ is a positive constant which depends on $m_{ij}(0)$ $(i, j = 1, 2, \cdots, N)$ and $d_i(0)$ $(i = 1, 2, \cdots, N)$.

Proof. Consider the following Lyapunov function:

$$\begin{aligned} V(t) = {} & \frac{1}{2}\sum_{i=1}^{N} e_i^T(t) e_i(t) + \frac{1}{2}\sum_{i=1}^{N} \frac{1}{k_i} (d_i - d_i^*)^2 \\ & + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \left(b_{ij} + m_{ij} - a_{ij}\right)^2 \\ & + \frac{L}{2(1-\varepsilon)} \int_{t-\tau(t)}^{t} \sum_{i=1}^{N} e_i^T(s) e_i(s)\, ds, \end{aligned} \tag{14}$$

where d_i^* is a undetermined positive constant.

The derivative of $V(t)$ along (11) is

$$\begin{aligned} \dot{V}(t) = {} & \sum_{i=1}^{N} e_i^T \dot{e}_i + \sum_{i=1}^{N} (d_i - d_i^*) \dot{d}_i \\ & + \sum_{i=1}^{N}\sum_{j=1}^{N} \left(b_{ij} + m_{ij} - a_{ij}\right) \dot{m}_{ij} + \frac{L}{2(1-\varepsilon)} \\ & \cdot \sum_{i=1}^{N} e_i^T(t) e_i(t) - \frac{L(1 - \dot{\tau}(t))}{2(1-\varepsilon)} \\ & \cdot \sum_{i=1}^{N} e_i^T(t - \tau(t)) e_i(t - \tau(t)) = \sum_{i=1}^{N} e_i^T A_d^i e_i + \sum_{i=1}^{N} e_i^T \end{aligned}$$

$$
\begin{aligned}
&\cdot\left(f_i\left(y_i(t)\right)-Hf_i\left(x_i(t)\right)\right)+\sum_{i=1}^{N}\sum_{j=1}^{N}b_{ij}e_i^T(t)\\
&\cdot G\left(y_j(t-\tau(t))\right)-\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}e_i^THG\left(x_j(t-\tau(t))\right)\\
&+\sum_{i=1}^{N}\sum_{j=1}^{N}m_{ij}e_i^TG\left(y_j(t-\tau(t))\right)-\sum_{i=1}^{N}d_i(t)e_i^T(t)\\
&\cdot e_i(t)+\sum_{i=1}^{N}\left(d_i-d_i^*\right)e_i^T(t)e_i(t)\\
&-\sum_{i=1}^{N}\sum_{j=1}^{N}\left(b_{ij}+m_{ij}-a_{ij}\right)e_i^T(t)G\left(y_j(t-\tau(t))\right)\\
&+\frac{L}{2(1-\varepsilon)}\sum_{i=1}^{N}e_i^T(t)e_i(t)-\frac{L(1-\dot{\tau}(t))}{2(1-\varepsilon)}\\
&\cdot\sum_{i=1}^{N}e_i^T(t-\tau(t))e_i(t-\tau(t))=\sum_{i=1}^{N}e_i^TA_d^ie_i\\
&+\sum_{i=1}^{N}e_i^T\left[f_i\left(y_i(t)\right)-f_i\left(Hx_i(t)\right)\right]+\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}e_i^T\\
&\cdot\left[G\left(y_j(t-\tau(t))\right)-HG\left(x_j(t-\tau(t))\right)\right]\\
&-\sum_{i=1}^{N}d_i^*e_i^Te_i+\frac{L}{2(1-\varepsilon)}\sum_{i=1}^{N}e_i^T(t)e_i(t)\\
&-\frac{L(1-\dot{\tau}(t))}{2(1-\varepsilon)}\sum_{i=1}^{N}e_i^T(t-\tau(t))e_i(t-\tau(t)).
\end{aligned}
\tag{15}
$$

Let $z_j(t-\tau(t))=Hx_j(t-\tau(t))$. Then we get

$$
\begin{aligned}
\dot{V}(t)={}&\sum_{i=1}^{N}e_i^TA_d^ie_i+\sum_{i=1}^{N}e_i^T\left[f_i\left(y_i(\mathrm{t})\right)-f_i\left(Hx_i(t)\right)\right]\\
&+\sum_{m=1}^{n}\widetilde{e}_m^T(t)A(G(\widetilde{y}_m(t-\tau(t))\\
&-G\left(\widetilde{z}_m(t-\tau(t))\right))-\sum_{i=1}^{N}d_i^*e_i^Te_i\\
&+\frac{L}{2(1-\varepsilon)}\sum_{i=1}^{N}e_i^T(t)e_i(t)\\
&-\frac{L(1-\dot{\tau}(t))}{2(1-\varepsilon)}\sum_{i=1}^{N}e_i^T(t-\tau(t))e_i(t-\tau(t)),
\end{aligned}
\tag{16}
$$

where $\widetilde{y}_m(t)=(y_{1m}(t),y_{2m}(t),\cdots,y_{Nm}(t))^T$, $\widetilde{z}_m(t)=(z_{1m}(t),z_{2m}(t),\cdots,z_{Nm}(t))^T$, and $\widetilde{e}_m(t)=(e_{1m}(t),e_{2m}(t),\cdots,e_{Nm}(t))^T$.

From Lemma 7, we derive

$$
\begin{aligned}
&\sum_{m=1}^{n}\widetilde{e}_m^T(t)A(G(\widetilde{y}_m(t-\tau(t))-G\left(\widetilde{z}_m(t-\tau(t))\right)).\\
&\leq\frac{1}{2}\sum_{m=1}^{n}\widetilde{e}_m^T(t)AA^T\widetilde{e}_m(t)\\
&\quad+\frac{1}{2}\sum_{m=1}^{n}\left\|G(\widetilde{y}_m(t-\tau(t))-G\left(\widetilde{z}_m(t-\tau(t))\right)\right\|^2.
\end{aligned}
\tag{17}
$$

Together with Assumptions 1 and 2, it follows that

$$
\begin{aligned}
\dot{V}(t)\leq{}&(\alpha+\xi)\sum_{m=1}^{n}\widetilde{e}_m^T(t)\widetilde{e}_m(t)\\
&+\frac{1}{2}\sum_{m=1}^{n}\widetilde{e}_m^T(t)\left(AA^T\right)\widetilde{e}_m(t)\\
&+\frac{L}{2}\sum_{m=1}^{n}\widetilde{e}_m^T(t-\tau(t))\widetilde{e}_m(t-\tau(t))-\sum_{i=1}^{N}d_i^*e_i^Te_i\\
&+\frac{L}{2(1-\varepsilon)}\sum_{m=1}^{n}\widetilde{e}_m^T(t)\widetilde{e}_m(t)\\
&-\frac{L(1-\dot{\tau}(t))}{2(1-\varepsilon)}\sum_{m=1}^{n}\widetilde{e}_m^T(t-\tau(t))\widetilde{e}_m(t-\tau(t)).
\end{aligned}
\tag{18}
$$

Here $\alpha=\max\{\alpha_1,\alpha_2,\cdots,\alpha_n\}$ and $\xi=\max\{\lambda_{max}(A_d^1),\lambda_{max}(A_d^2),\ldots,\lambda_{max}(A_d^N)\}$.

Making use of Assumption 3, we have

$$
\frac{L}{2}\left(1-\frac{1-\dot{\tau}(t)}{1-\varepsilon}\right)\leq 0. \tag{19}
$$

Furthermore, we obtain

$$
\begin{aligned}
\dot{V}(t)\leq{}&\sum_{m=1}^{n}\widetilde{e}_m^T(t)\\
&\cdot\left[\left(\alpha+\xi+\frac{L}{2(1-\varepsilon)}\right)I_N+\frac{1}{2}AA^T-D^*\right]\\
&\cdot\widetilde{e}_m(t),
\end{aligned}
\tag{20}
$$

where $D^*=\mathrm{diag}(d_1^*,d_2^*,\cdots,d_N^*)$. Together with (12), we have

$$
\dot{V}(t)\leq\theta\sum_{i=1}^{N}e_i^T(t)e_i(t)\leq 2\theta V(t). \tag{21}
$$

Multiplying (21) by $e^{-\theta t}$, we obtain $(d/dt)(e^{\theta t}V(t))<0$. Integrating from 0 to t $(t\in[0,T])$, it follows that

$$
e^{-\theta t}V(t)<V(0). \tag{22}
$$

By (14), we get

$$\begin{aligned} V(0) &= \frac{1}{2}e^T(0)e(0) + \frac{1}{2}\sum_{i=1}^{N}\frac{1}{k_i}(d_i(0) - d_i^*)^2 \\ &\quad + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(b_{ij} + m_{ij}(0) - a_{ij}\right)^2 \\ &\quad + \frac{L}{2(1-\varepsilon)}\int_{-\tau(t)}^{0}\sum_{i=1}^{N}e_i^T(s)e_i(s)\,ds \\ &\le \frac{1}{2}\|e(0)\|^2 + \frac{L\overline{\tau}}{2(1-\varepsilon)}\sup_{-\overline{\tau}\le v\le 0}\|e(v)\|^2 \\ &\quad + \frac{1}{2}\sum_{i=1}^{N}\frac{1}{k_i}(d_i(0) - d_i^*)^2 \\ &\quad + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(b_{ij} + m_{ij}(0) - a_{ij}\right)^2 \\ &\le \frac{1}{2}q_1\left(1 + \frac{L\overline{\tau}}{1-\varepsilon}\right) + \frac{1}{2}\mu, \end{aligned} \tag{23}$$

where $\mu = \sum_{i=1}^{N}(1/k_i)(d_i(0) - d_i^*)^2 + \sum_{i=1}^{N}\sum_{j=1}^{N}(b_{ij} + m_{ij}(0) - a_{ij})^2$. Combining this with (13), we have

$$\begin{aligned} \|e(t)\|^2 &\le 2V(t) \le 2e^{\theta T}V(0) \\ &\le e^{\theta T}q_1\left(1 + \frac{L\overline{\tau}}{1-\varepsilon}\right) + e^{\theta T}\mu < q_2. \end{aligned} \tag{24}$$

The proof of Theorem 8 is finished. □

Corollary 9. *Let Assumptions 1–3 be satisfied. Suppose that the configuration matrix of the drive layer is the same as that of the response layer, i.e., $A = B$. Let q_1, q_2, and T be three positive constants and let $q_1 \le q_2$. Under the simplified adaptive controllers*

$$\begin{aligned} u_i(t) &= HA_n^i x_i(t) - A_n^i y_i(t) - d_i(t)e_i(t), \\ \dot{d}_i &= k_i e_i^T(t)e_i(t), \end{aligned} \tag{25}$$

where k_i $(i = 1, 2, \ldots, N)$ is any positive constant, FMIS of the two-layer network (1) can be realized with respect to (q_1, q_2, T, H) if there exist a diagonal matrix $D^* = \mathrm{diag}(d_1^*, d_2^*, \cdots, d_N^*)$ and a scalar $\theta > 0$ such that

$$\begin{aligned} &\left(\alpha + \xi + \frac{L}{2(1-\varepsilon)}\right)I_N + \frac{1}{2}AA^T - D^* - \theta I_N < 0, \\ &e^{\theta T}q_1\left(1 + \frac{L\overline{\tau}}{1-\varepsilon}\right) + e^{\theta T}\mu \le q_2. \end{aligned} \tag{26}$$

Here, $\alpha = \max\{\alpha_1, \alpha_2, \cdots, \alpha_N\}$, $\xi = \max\{\lambda_{max}(A_d^1), \lambda_{max}(A_d^2), \ldots, \lambda_{max}(A_d^N)\}$, and μ is a positive constant which depends on $d_i(0)$ $(i = 1, 2, \cdots, N)$.

Theorem 10. *Suppose that Assumptions 1–3 hold. MIS of two-layer network (1) can be realized with the adaptive controllers (9).*

In fact, choose (14) as the Lyapunov function and use the calculation of $\dot{V}(t)$ in the proof of Theorem 8; we have (20). When d_i^* is sufficiently large, we get

$$\left(\alpha + \xi + \frac{L}{2(1-\varepsilon)}\right)I_N + \frac{1}{2}AA^T - D^* < 0. \tag{27}$$

This implies $\dot{V}(t) \le 0$. Let

$$E = \left\{(\mathbf{e}, \mathbf{d}, M) \in \mathbb{R}^{nN} \times \mathbb{R}^N \times \mathbb{R}^{N\times N} : \dot{V} = 0\right\}, \tag{28}$$

where $\mathbf{e} = (e_1^T(t), e_2^T(t), \ldots, e_N^T(t))^T$, $\mathbf{d} = (d_1, d_2, \cdots, d_N)^T$, and $M = (m_{ij})_{N\times N}$. The largest invariant set is as follows:

$$\begin{aligned} S = \Bigg\{&(\mathbf{e}, \mathbf{d}, M) \in \mathbb{R}^{nN} \times \mathbb{R}^N \times \mathbb{R}^{N\times N} : \mathbf{e} = 0,\ \dot{\mathbf{d}} \\ &= 0,\ \dot{M} = 0,\ \sum_{j=1}^{N}\left(b_{ij} + m_{ij} - a_{ij}\right) = 0\Bigg\}. \end{aligned} \tag{29}$$

Therefore, by the LaSalle invariance principle, starting from arbitrary initial values, the trajectory asymptotically tends to the largest invariant set S, which implies $\lim_{t\to\infty}e_i(t) = 0$ $(i = 1, 2, \cdots, N)$. Thus, MIS in two-layer network (1) can be realized with the adaptive controllers (9).

Remark 11. When the MIS of two-layer network (1) is realized, i.e., $e_i(t) \longrightarrow 0$ $(i = 1, 2, \cdots, N)$, it can be further obtained that $\dot{d}_i \longrightarrow 0$ and $\dot{m}_{ij} \longrightarrow 0$ $(i, j = 1, 2, \cdots, N)$. This implies that $d_i(t)$ and $m_{ij}(t)$ will tend to be constant when $t \longrightarrow \infty$.

Remark 12. For the two-layer network (1), the nodes in each layer may be nonidentical and topological structures of two layers may be different. For each node, the inner connecting function is nonlinear. In addition, two configuration matrices A and B are not assumed to be symmetric or irreducible. Consequently, Theorems 8 and 10 can be widely applied in practice.

Corollary 13. *Suppose that Assumptions 1–3 hold. MIS of two-layer network (1) can be realized with the following adaptive controllers:*

$$\begin{aligned} u_i(t) &= HA_n^i x_i(t) - A_n^i y_i(t) \\ &\quad + \sum_{j=1}^{N}m_{ij}G\left(y_j(t - \tau(t))\right) - d_i(t)e_i(t), \\ &\qquad i = 1, 2, \ldots, N, \end{aligned} \tag{30}$$

where $\dot{d}_i = k_i e_i^T(t)e_i(t)$ and k_i is any positive constant. For $i, j = 1, 2, \ldots, N$, $m_{ij}(t)$ is defined as follows:

(i) *$\dot{m}_{ij} = -e_i^T(t)G(y_j(t - \tau(t)))$, when $a_{ij} \ne b_{ij}$.*

(ii) $m_{ij} = 0$, *when* $a_{ij} = b_{ij}$.

Corollary 14. *Suppose that Assumptions 1–3 hold. If the configuration matrix of the drive layer is the same as that of the response layer, i.e., $A = B$, then MIS of two-layer network (1) can be realized with the following simplified adaptive controllers:*

$$u_i(t) = HA_n^i x_i(t) - A_n^i y_i(t) - d_i(t) e_i(t), \quad i = 1, 2, \ldots, N, \tag{31}$$

where $\dot{d}_i = k_i e_i^T(t) e_i(t)$ and k_i is any positive constant.

4. Numerical Simulations

In this section, some numerical examples are presented to illustrate the effectiveness of results in Section 3. Each layer of two-layer network (1) is made of 10 nodes. The dynamics of each node is taken as the modified Chua's circuit system, which is described as

$$\begin{aligned} \dot{x}_1 &= p\left(x_2 - \frac{2x_1^3 - x_1}{7}\right), \\ \dot{x}_2 &= x_1 - x_2 + x_3, \\ \dot{x}_3 &= -qx_2, \end{aligned} \tag{32}$$

where $> 0, q > 0$. This system is rewritten into $\dot{x}(t) = A_d x(t) + A_n x(t) + f(x(t))$, where

$$\begin{aligned} x &= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \\ A_d &= \begin{pmatrix} \frac{p}{7} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ A_n &= \begin{pmatrix} 0 & p & 0 \\ 1 & 0 & 1 \\ 0 & -q & 0 \end{pmatrix}, \\ f(x) &= \begin{pmatrix} -\frac{2px_1^3}{7} \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{33}$$

Hence, $\xi = \lambda_{max}(A_d) = p/7$. Let $p = 10$ and $q = 100/7$. In view of the calculation in [22], it follows that

$$\|f(x) - f(y)\| \le \frac{27p}{14} \|x - y\|. \tag{34}$$

This means that Assumption 1 is satisfied with $\alpha_i = 27p/14$ $(i = 1, 2, \cdots, 10)$. Thus, $\alpha = 27p/14$. For the nonlinear function $G(x) = (g(x_1), g(x_2), \ldots, g(x_n))$, $g(x_i)$ $(i = 1, 2, \ldots, n)$ is chosen as $g(x_i) = \sin x_i + x_i$. It can be easily derived that $\|G(x) - G(y)\| \le 2\|x - y\|$, which implies $L = 4$ in Assumption 2.

Let $\tau(t) = 0.1 - 0.1e^{-t}$. Obviously, it follows that $\varepsilon = 0.2$ and $\overline{\tau} = 0.1$. The scaling matrix is given by $H = \mathrm{diag}(1, 0, -1)$. Then the error variables are written as

$$\begin{aligned} e_{i1}(t) &= y_{i1}(t) - x_{i1}(t), \\ e_{i2}(t) &= y_{i2}(t), \\ e_{i3}(t) &= y_{i3}(t) + x_{i3}(t). \end{aligned} \tag{35}$$

Moreover, it is easily verified that $f(Hx) = Hf(x)$ and $G(Hx) = HG(x)$.

The coupled two-layer network is described as follows:

$$\begin{aligned} \dot{x}_i(t) &= A_d x_i(t) + A_n x_i(t) + f(x_i(t)) \\ &\quad + \sum_{j=1}^{10} a_{ij} G(x_j(t - \tau(t))), \\ &\qquad i = 1, 2, \cdots, 10, \\ \dot{y}_i(t) &= A_d y_i(t) + A_n y_i(t) + f(y_i(t)) \\ &\quad + \sum_{j=1}^{10} b_{ij} G(y_j(t - \tau(t))) + u_i, \\ &\qquad i = 1, 2, \cdots, 10, \end{aligned} \tag{36}$$

where

$$\begin{aligned} u_i(t) &= HA_n x_i(t) - A_n y_i(t) \\ &\quad + \sum_{j=1}^{10} m_{ij} G(y_j(t - \tau(t))) - d_i(t) e_i(t) \end{aligned} \tag{37}$$

with $\dot{d}_i = k_i e_i^T(t) e_i(t)$, and $\dot{m}_{ij} = -e_i^T(t) G(y_j(t - \tau(t)))$. Here, k_i $(i = 1, 2, \ldots, 10)$ is any positive constant.

Let K_1 and K_2 be two matrices defined as follows:

$$K_1 = \begin{pmatrix} -9 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & -7 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 2 \\ 1 & 1 & -10 & 1 & 1 & 0 & 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & -6 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & -8 & 0 & 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 & 1 & -7 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & -11 & 1 & 3 & 1 \\ 3 & 1 & 0 & 1 & 1 & 0 & 1 & -9 & 1 & 1 \\ 1 & 1 & 2 & 1 & 3 & 1 & 1 & 1 & -12 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 1 & 3 & 1 & -10 \end{pmatrix} \tag{38}$$

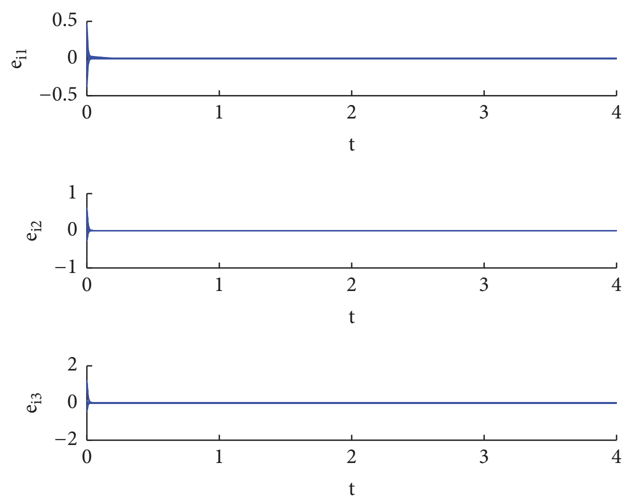

FIGURE 2: Finite-time mixed interlayer synchronization errors e_i $(i = 1, 2, \ldots, 10)$ with $A \neq B$.

and

$$K_2 = \begin{pmatrix} -10 & 2 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 1 \\ 1 & -9 & 0 & 1 & 1 & 3 & 1 & 0 & 1 & 1 \\ 0 & 1 & -7 & 1 & 1 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & -9 & 1 & 1 & 0 & 1 & 3 & 1 \\ 1 & 0 & 1 & 2 & -8 & 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & -7 & 1 & 0 & 1 & 1 \\ 1 & 0 & 3 & 1 & 1 & 0 & -10 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 & 1 & 1 & 1 & -9 & 1 & 1 \\ 1 & 1 & 1 & 1 & 3 & 1 & 1 & 1 & -11 & 1 \\ 3 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & -12 \end{pmatrix}. \tag{39}$$

4.1. FMIS of Two-Layer Networks with respect to (q_1, q_2, T, H). In the subsection, we consider FMIS of the two-layer network (36) under two cases: $A \neq B$ and $A = B$.

Example 15 ($A \neq B$). Suppose that $A = K_1$ and $B = K_2$. Let $q_1 = 1.5$, $q_2 = 2.4$, and $T = 4$. In view of inequality (12) in Theorem 8, we can get $\theta = 0.01$ and $d_i^* = 128$ $(i = 1, 2, \cdots, N)$. For $v \in [-0.1, 0]$, we arbitrarily take $x_i(v)$ and $y_i(v)$ in the interval $[-1, 1]$ for $i = 1, 2, \cdots, N$. It follows that $\sup_{-\overline{\tau} \leq v \leq 0} \|e(v)\|^2 \leq q_1$. According to inequality (13), we choose k_i, $d_i(0)$ and $m_{ij}(0)$ as $k_i = 10$, $d_i(0) = 128$ and $m_{ij}(0) = b_{ij} - a_{ij}$ for $i, j = 1, 2, \ldots, 10$. Under the adaptive controllers (37), Figure 2 shows that the FMIS of two-layer network (36) is realized with respect to (q_1, q_2, T, H).

As is well-known, the dynamics of networks is always affected by all kinds of noises from the environment. In the following, the FMIS will be further investigated when the network (36) is disturbed by additive noise. Here, two cases are analyzed by numerical simulations. Let $\phi = (0.5(0.25\sin(t) + 0.5\cos(5t)), 0.1\sin(2\pi t), 0.3e^{-t})^T$ and

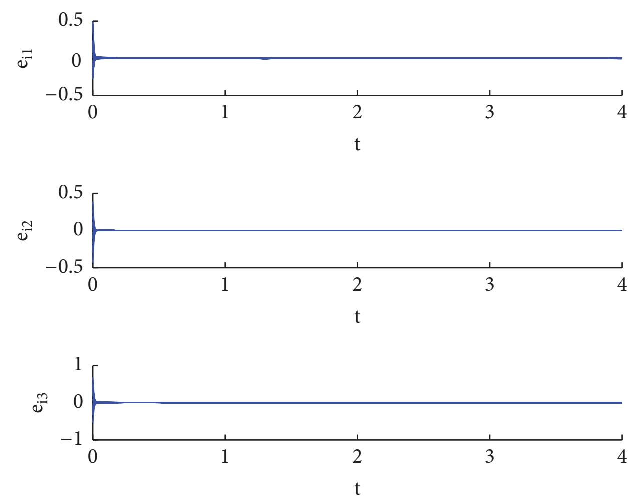

FIGURE 3: Finite-time mixed interlayer synchronization errors e_i $(i = 1, 2, \ldots, 10)$ for Case 1.

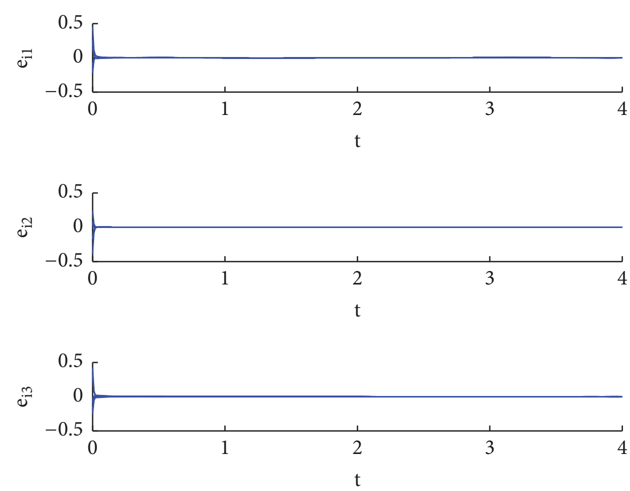

FIGURE 4: Finite-time mixed interlayer synchronization errors e_i $(i = 1, 2, \ldots, 10)$ for Case 2.

$\varphi(t) = (0.3\cos(2t), 0.2e^{-3t}, 0.1(\cos(t) - \sin(0.1t)))^T$ be two additive noise functions.

Case 1. $\phi(t)$ is added to the right hand of two equations in (36). This means that every node in (36) is affected by the same noise.

Case 2. $\phi(t)$ and $\varphi(t)$ are added to the right hand of two equations in (36), respectively. This indicates that the nodes in the drive layer and the ones in the response layer are affected by different noise.

For these two cases, the steps of above numerical simulation are repeated with the adaptive controllers (37). The synchronization errors for two cases are shown in Figures 3 and 4. Hence, the proposed scheme is demonstrated to be nonfragile with the noise perturbation.

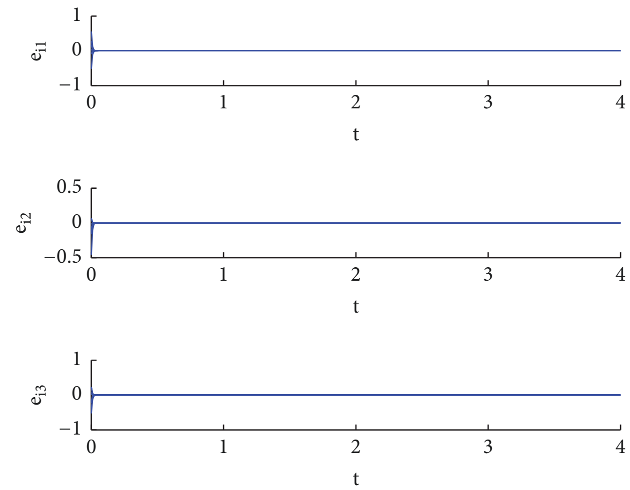

FIGURE 5: Finite-time mixed interlayer synchronization errors e_i $(i = 1, 2, \ldots, 10)$ with $A = B$.

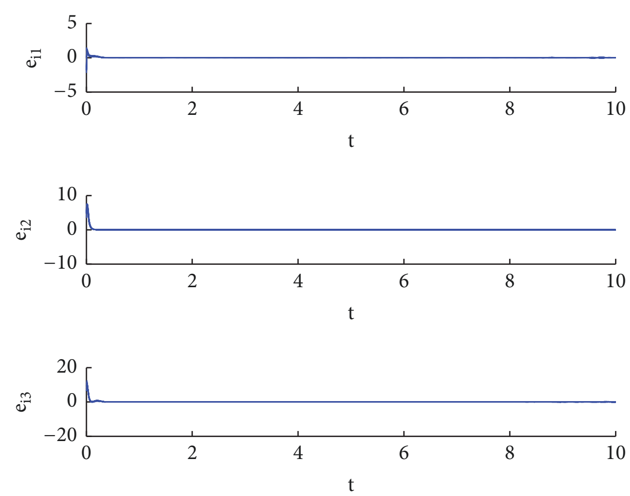

FIGURE 6: Mixed synchronization errors e_i $(i = 1, 2, \ldots, 10)$ with $A \neq B$.

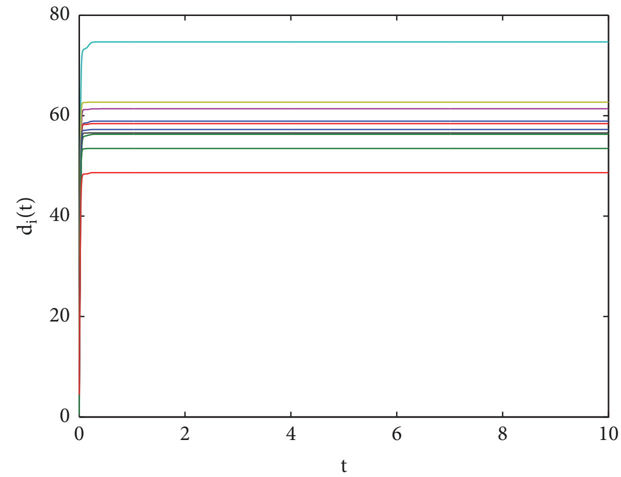

FIGURE 7: Evolution of feedback gains $d_i(t)$ with $k_i = 10$, $i = 1, 2, \ldots, 10$.

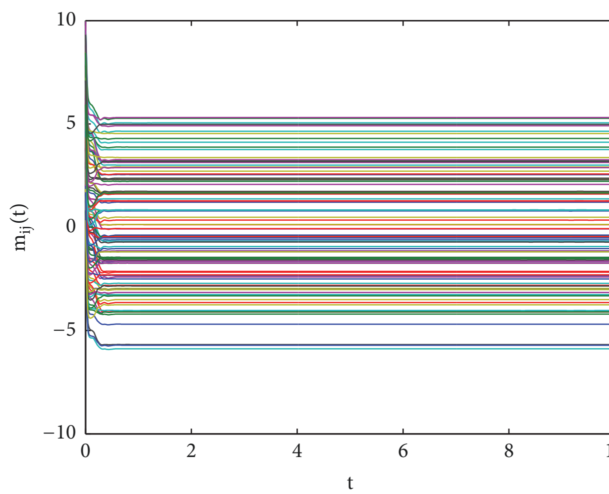

FIGURE 8: Evolution of adaptive parameters $m_{ij}(t)$, $i, j = 1, 2, \ldots, 10$.

Example 16 ($A = B$). Suppose that $A = B = K_1$. Let $q_1 = 1.7$, $q_2 = 2.7$, and $T = 4$. According to Corollary 9, we can get the values of θ and d_i^* $(i = 1, 2, \ldots, 10)$. For $\upsilon \in [-0.1, 0]$, we arbitrarily take $x_i(\upsilon)$ and $y_i(\upsilon)$ in the interval $[-1, 1]$ for $i = 1, 2, \cdots, N$. In view of the calculation, we can take the other parameters and initial values which are the same as those in Example 15. Under the simplified adaptive controllers in Corollary 9, the synchronization errors are shown in Figure 5. The FMIS of two-layer network (36) is realized with respect to (q_1, q_2, T, H).

4.2. MIS of Two-Layer Networks. In the subsection, MIS of the two-layer network (36) is considered under two cases: $A \neq B$ and $A = B$. Let $x_i(0)$, $y_i(0)$, $d_i(0)$, and $m_{ij}(0)$ be arbitrarily given in the interval $[0, 10]$ for $i, j = 1, 2, \ldots, 10$.

Let $A = K_1$ and $B = K_2$. The synchronization error is shown in Figure 6, which demonstrates the effectiveness of Theorem 10. Figures 7 and 8 present the evolution of $d_i(t)$ and $m_{ij}(t)$, respectively. It is obvious that $d_i(t)$ and $m_{ij}(t)$ $(i, j = 1, 2, \ldots, 10)$ tend to be constant.

Suppose that $A = B = K_1$. Figure 9 presents the synchronization error with the simplified adaptive controllers in Corollary 14. Hence, the MIS in two-layer network (36) can be realized.

5. Conclusions

In this paper, we have investigated finite-time mixed interlayer synchronization (FMIS) of two-layer complex network with unidirectional interlayer couplings. For the two-layer network, the nodes in each layer are nonidentical and two single-layer networks have time-varying coupling delay and different topological structures. We have proposed an adaptive control scheme to realize FMIS of two-layer networks. Based on the Lyapunov stability theory, a sufficient condition for realizing FMIS has been derived. In addition, some criteria for realizing MIS have been obtained. Finally, some

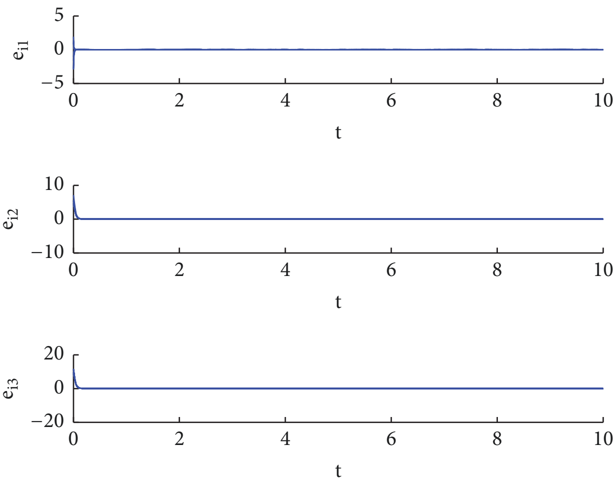

FIGURE 9: Mixed synchronization errors e_i $(i = 1, 2, \ldots, 10)$ with $A = B$.

numerical simulations have been presented to verify the correctness and effectiveness of theoretical results. Moreover, FMIS has been analyzed when the two-layer networks are disturbed by the additive noise. Two numerical simulations show that the proposed adaptive control strategy is nonfragile with the noise perturbation.

Abbreviations

FMIS: Finite-time mixed interlayer synchronization
MOS: Mixed outer synchronization
FMOS: Finite-time mixed outer synchronization
MIS: Mixed interlayer synchronization.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

Acknowledgments

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