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Research Article

Duality Identities for Moduli Functions of Generalized Melvin Solutions Related to Classical Lie Algebras of Rank 4

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We consider generalized Melvin-like solutions associated with nonexceptional Lie algebras of rank 4 (namely, A_4, B_4, C_4, and D_4) corresponding to certain internal symmetries of the solutions. The system under consideration is a static cylindrically symmetric gravitational configuration in D dimensions in presence of four Abelian 2-forms and four scalar fields. The solution is governed by four moduli functions $H_s(z)$ ($s = 1, \ldots, 4$) of squared radial coordinate $z = \rho^2$ obeying four differential equations of the Toda chain type. These functions turn out to be polynomials of powers $(n_1, n_2, n_3, n_4) = (4, 6, 6, 4), (8, 14, 18, 10), (7, 12, 15, 16), (6, 10, 6, 6)$ for Lie algebras A_4, B_4, C_4, and D_4, respectively. The asymptotic behaviour for the polynomials at large distances is governed by some integer-valued 4×4 matrix ν connected in a certain way with the inverse Cartan matrix of the Lie algebra and (in A_4 case) the matrix representing a generator of the $\mathbb{Z}_2$-group of symmetry of the Dynkin diagram. The symmetry properties and duality identities for polynomials are obtained, as well as asymptotic relations for solutions at large distances. We also calculate 2-form flux integrals over 2-dimensional discs and corresponding Wilson loop factors over their boundaries.

1. Introduction

In this paper, we investigate properties of multidimensional generalization of Melvin's solution [1], which was presented earlier in [2]. Originally, model from [2] contains metric, n Abelian 2-forms and $l \geq n$ scalar fields. Here we consider a special solutions with $n = l = 4$, governed by a 4×4 Cartan matrix (A_{ij}) for simple nonexceptional Lie algebras of rank 4: A_4, B_4, C_4, and D_4. The solutions from [2] are special case of the so-called generalized fluxbrane solutions from [3].

It is well known that the original Melvin's solution in four dimensions describes the gravitational field of a magnetic flux tube. The multidimensional analog of such a flux tube, supported by a certain configuration of form fields, is referred to as a fluxbrane (a "thickened brane" of magnetic flux). The appearance of fluxbrane solutions was originally motivated by superstring/brane models and M-theory. For generalizations of the Melvin solution and fluxbrane solutions see [4–21] and references therein.

In [3] there were considered the generalized fluxbrane solutions which are described in terms of moduli functions $H_s(z) > 0$ defined on the interval $(0, +\infty)$, where $z = \rho^2$ and ρ is a radial coordinate. Functions $H_s(z)$ obey n nonlinear differential master equations of Toda-like type governed by some matrix $(A_{ss'})$, and the following boundary conditions are imposed: $H_s(+0) = 1$, $s = 1, \ldots, n$.

Here, as in [2], we assume that the matrix $(A_{ss'})$ is a Cartan matrix for some simple finite-dimensional Lie algebra $\mathcal{G}$ of rank n ($A_{ss} = 2$ for all s). A conjecture was suggested in [3] that in this case the solutions to master equations with the above boundary conditions are polynomials of the form:

$$H_s(z) = 1 + \sum_{k=1}^{n_s} P_s^{(k)} z^k, \tag{1}$$

where $P_s^{(k)}$ are constants. Here $P_s^{(n_s)} \neq 0$ and

$$n_s = 2\sum_{s'=1}^{n} A^{ss'}, \tag{2}$$

where we denote $(A^{ss'}) = (A_{ss'})^{-1}$. Integers n_s are components of the twice dual Weyl vector in the basis of simple (co)roots [22].

Therefore, the functions H_s (which may be called "fluxbrane polynomials") define a special solution to open Toda chain equations [23, 24] corresponding to simple finite-dimensional Lie algebra $\mathcal{G}$ [25]. In [2, 26] a program (in Maple) for calculation of these polynomials for classical series of Lie algebras (A-, B-, C-, and D-series) was suggested. It was pointed out in [3] that the conjecture on polynomial structure of $H_s(z)$ is valid for Lie algebras of A- and C- series.

One of the goals of this paper is to study interesting geometric properties of the solution considered for case of nonexceptional Lie algebras of rank 4. In particular, we prove some symmetry properties, as well as the so-called duality relations of fluxbrane polynomials which establishes a behaviour of the solutions under the inversion transformation $\rho \longrightarrow 1/\rho$, which makes the model in tune with T-duality in string models and also can be mathematically understood in terms of the groups of symmetry of Dynkin diagrams for the corresponding Lie algebras. In our case these groups of symmetry are either identical ones (for Lie algebras B_4 and C_4) or isomorphic to the group $\mathbb{Z}_2$ (for Lie algebra A_4) or isomorphic to the group S_3 which is the group of permutation of 3 elements (for Lie algebra D_4). These duality identities may be used in deriving a $1/\rho$-expansion for solutions at large distances ρ. The corresponding asymptotic behaviour of the solutions is studied.

The analogous analysis was performed recently for the case of rank-2 Lie algebras: A_2, $B_2 = C_2$, G_2 in [27], and for rank-3 algebras A_3, B_3, and C_3 in [28]. Also, in [29] the conjecture from [3] was verified for the Lie algebra E_6 and certain duality relations for six E_6-polynomials were found.

The paper is organized as follows. In Section 2 we present a generalized Melvin solutions from [2] for the case of four scalar fields and four 2-forms. In Section 3 we deal with the solutions for the Lie algebras A_4, B_4, C_4, and D_4. We find symmetry properties and duality relations for polynomials and present asymptotic relations for the solutions. We also calculate 2-form flux integrals $\Phi^s(R) = \int_{D_R} F^s$ and corresponding Wilson loop factors, where F^s are 2-forms and D_R is 2-dimensional disc of radius R. The flux integrals converge, i.e., have finite limits for $R = +\infty$ [30].

2. The Setup and Generalized Melvin Solutions

Let us consider the following product manifold:

$$M = (0, +\infty) \times M_1 \times M_2, \tag{3}$$

where $M_1 = S^1$ and M_2 is a $(D-2)$-dimensional Ricci-flat manifold.

On this manifold, we define the action

$$S = \int d^D x \sqrt{|g|} \left\{ R[g] - \delta_{ab} g^{MN} \partial_M \varphi^a \partial_N \varphi^b - \frac{1}{2} \sum_{s=1}^{4} \exp\left[2\vec{\lambda}_s \vec{\varphi}\right] (F^s)^2 \right\}, \tag{4}$$

where $g = g_{MN}(x) dx^M \otimes dx^N$ is a metric on M, $\vec{\varphi} = (\varphi^a) \in \mathbb{R}^4$ is vector of scalar fields, $F^s = dA^s = (1/2) F^s_{MN} dx^M \wedge dx^N$ is a 2-form, and $\vec{\lambda}_s = (\lambda_s^a) \in \mathbb{R}^4$ is dilatonic coupling vector, $s = 1, \ldots, 4$; $a = 1, \ldots, 4$. Here we use the notations $|g| \equiv |\det(g_{MN})|$; $(F^s)^2 \equiv F^s_{M_1 M_2} F^s_{N_1 N_2} g^{M_1 N_1} g^{M_2 N_2}$.

There is a family of exact cylindrically symmetric solutions to the field equations corresponding for the action (4) and depending on the radial coordinate ρ. The solution has the form [2]

$$g = \left(\prod_{s=1}^{4} H_s^{2h_s/(D-2)} \right) \cdot \left\{ d\rho \otimes d\rho + \left(\prod_{s=1}^{4} H_s^{-2h_s} \right) \rho^2 d\phi \otimes d\phi + g^2 \right\}, \tag{5}$$

$$\exp(\varphi^a) = \prod_{s=1}^{4} H_s^{h_s \lambda_s^a}, \tag{6}$$

$$F^s = q_s \left(\prod_{l=1}^{4} H_l^{-A_{sl}} \right) \rho d\rho \wedge d\phi, \tag{7}$$

$s, a = 1, \ldots, 4$, where $g^1 = d\phi \otimes d\phi$ is a metric on $M_1 = S^1$ and g^2 is a Ricci-flat metric of signature $(-, +, \ldots, +)$ on M_2. Here $q_s \neq 0$ are integration constants ($q_s = -Q_s$ in notations of [2]).

For further convenience, let us denote $z = \rho^2$. As it was shown in earlier works, the functions $H_s(z) > 0$ obey the set of master equations

$$\frac{d}{dz}\left(\frac{z}{H_s} \frac{d}{dz} H_s \right) = P_s \prod_{l=1}^{4} H_l^{-A_{sl}}, \tag{8}$$

with the boundary conditions

$$H_s(+0) = 1, \tag{9}$$

where

$$P_s = \frac{1}{4} K_s q_s^2, \tag{10}$$

$s = 1, \ldots, 4$. The boundary condition (9) guarantees the absence of a conic singularity (for the metric (5)) for $\rho = +0$.

There are some relations for the parameters h_s:

$$h_s = K_s^{-1}, \quad K_s = B_{ss} > 0, \tag{11}$$

where

$$B_{sl} \equiv 1 + \frac{1}{2-D} + \vec{\lambda}_s \vec{\lambda}_l, \tag{12}$$

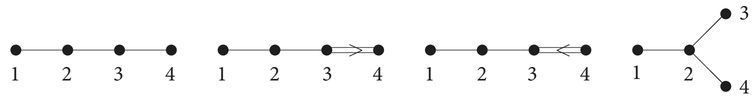

Figure 1: The Dynkin diagrams for the Lie algebras A_4, B_4, C_4, and D_4, respectively.

$s, l = 1, \ldots, 4$. In these relations, we have denoted

$$(A_{sl}) = \left(\frac{2B_{sl}}{B_{ll}}\right). \tag{13}$$

The latter matrix is the so-called "quasi-Cartan" matrix. One can prove that if (A_{sl}) is a Cartan matrix for a certain simple Lie algebra $\mathcal{G}$ of rank 4 then there exists a set of vectors $\vec{\lambda}_1, \ldots, \vec{\lambda}_4$ obeying (13). See also Remark 1 in the next section.

The solution considered can be understood as a special case of the fluxbrane solutions from [3, 19].

Therefore, here we investigate a multidimensional generalization of Melvin's solution [1] for the case of four scalar fields and four 2-forms. Note that the original Melvin's solution without scalar field would correspond to $D = 4$, one (electromagnetic) 2-form, $M_1 = S^1$ $(0 < \phi < 2\pi)$, $M_2 = \mathbb{R}^2$, and $g^2 = -dt \otimes dt + dx \otimes dx$.

3. Solutions Related to Simple Classical Rank-4 Lie Algebras

In this section we consider the solutions associated with the simple nonexceptional Lie algebras $\mathcal{G}$ of rank 4. This means than the matrix $A = (A_{sl})$ coincides with one of the Cartan matrices

$$(A_{ss'}) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix}, \tag{14}$$

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$$

for $\mathcal{G} = A_4, B_4, C_4, D_4$, respectively.

Each of these matrices can be graphically described by drawing the Dynkin diagrams pictured on Figure 1 for these four Lie algebras.

Using (11)-(13) we can get

$$K_s = \frac{D-3}{D-2} + \vec{\lambda}_s^2, \tag{15}$$

where $h_s = K_s^{-1}$ and

$$\vec{\lambda}_s \vec{\lambda}_l = \frac{1}{2} K_l A_{sl} - \frac{D-3}{D-2} \equiv G_{sl}, \tag{16}$$

$s, l = 1, 2, 3, 4$; (15) is a special case of (16).

From (11) and (13) it also follows that

$$\frac{h_s}{h_l} = \frac{K_l}{K_s} = \frac{B_{ll}}{B_{ss}} = \frac{B_{ls}}{B_{ss}} \frac{B_{ll}}{B_{sl}} = \frac{A_{ls}}{A_{sl}} \tag{17}$$

for any $s \neq l$ obeying $A_{sl}, A_{ls} \neq 0$. This implies

$$\begin{aligned} K_1 &= K_2 = K_3 = K, \\ K_4 &= K, \frac{1}{2}K, 2K, K, \end{aligned} \tag{18}$$

or

$$\begin{aligned} h_1 &= h_2 = h_3 = h, \\ h_4 &= h, 2h, \frac{1}{2}h, h \end{aligned} \tag{19}$$

$(h = K^{-1})$ for $\mathcal{G} = A_4, B_4, C_4, D_4$, respectively.

Remark 1. For large enough K_1 in (18) there exist vectors $\vec{\lambda}_s$ obeying (16) (and hence (15)). Indeed, the matrix (G_{sl}) is positive definite if $K_1 > K_*$, where K_* is some positive number. Hence there exists a matrix Λ, such that $\Lambda^T \Lambda = G$. We put $(\Lambda_{as}) = (\lambda_s^a)$ and get the set of vectors obeying (16).

Polynomials. According to the polynomial conjecture, the set of moduli functions $H_1(z), \ldots, H_4(z)$, obeying (8) and (9) with the Cartan matrix $A = (A_{sl})$ from (14) are polynomials with powers $(n_1, n_2, n_3, n_4) = (4, 6, 6, 4)$, $(8, 14, 18, 10)$, $(7, 12, 15, 16)$, $(6, 10, 6, 6)$ calculated by using (2) for Lie algebras A_4, B_4, C_4, and D_4, respectively.

One can prove this conjecture by solving the system of nonlinear algebraic equations for the coefficients of these polynomials following from master equations (8). Below we present a list of the polynomials obtained by using appropriate MATHEMATICA algorithm. For convenience, we use the rescaled variables (as in [25]):

$$p_s = \frac{P_s}{n_s}. \tag{20}$$

A_4-*Case.* For the Lie algebra $A_4 \cong sl(5)$ we have

$$H_1 = 1 + 4p_1 z + 6p_1 p_2 z^2 + 4p_1 p_2 p_3 z^3 + p_1 p_2 p_3 p_4 z^4, \tag{21}$$

$$\begin{aligned} H_2 = {} & 1 + 6p_2 z + \left(6p_1 p_2 + 9p_2 p_3\right) z^2 \\ & + \left(16p_1 p_2 p_3 + 4p_2 p_3 p_4\right) z^3 \\ & + \left(6p_1 p_2^2 p_3 + 9p_1 p_2 p_3 p_4\right) z^4 \\ & + 6p_1 p_2^2 p_3 p_4 z^5 + p_1 p_2^2 p_3^2 p_4 z^6, \end{aligned} \tag{22}$$

$$\begin{aligned} H_3 = {} & 1 + 6p_3 z + \left(9p_2 p_3 + 6p_3 p_4\right) z^2 \\ & + \left(4p_1 p_2 p_3 + 16p_2 p_3 p_4\right) z^3 \\ & + \left(9p_1 p_2 p_3 p_4 + 6p_2 p_3^2 p_4\right) z^4 \\ & + 6p_1 p_2 p_3^2 p_4 z^5 + p_1 p_2^2 p_3^2 p_4 z^6, \end{aligned} \tag{23}$$

$$H_4 = 1 + 4p_4 z + 6p_3 p_4 z^2 + 4p_2 p_3 p_4 z^3 + p_1 p_2 p_3 p_4 z^4. \tag{24}$$

B_4-*Case.* For the Lie algebra $B_4 \cong so(9)$ the fluxbrane polynomials are

$$\begin{aligned} H_1 = {} & 1 + 8p_1 z + 28p_1 p_2 z^2 + 56p_1 p_2 p_3 z^3 \\ & + 70p_1 p_2 p_3 p_4 z^4 + 56p_1 p_2 p_3 p_4^2 z^5 \\ & + 28p_1 p_2 p_3^2 p_4^2 z^6 + 8p_1 p_2^2 p_3^2 p_4^2 z^7 + p_1^2 p_2^2 p_3^2 p_4^2 z^8, \end{aligned} \tag{25}$$

$$\begin{aligned} H_2 = {} & 1 + 14p_2 z + \left(28p_1 p_2 + 63p_2 p_3\right) z^2 \\ & + \left(224p_1 p_2 p_3 + 140p_2 p_3 p_4\right) z^3 + \left(196p_1 p_2^2 p_3 \right. \\ & \left. + 630p_1 p_2 p_3 p_4 + 175p_2 p_3 p_4^2\right) z^4 + \left(980p_1 p_2^2 p_3 p_4 \right. \\ & \left. + 896p_1 p_2 p_3 p_4^2 + 126p_2 p_3^2 p_4^2\right) z^5 + \left(490p_1 p_2^2 p_3^2 p_4 \right. \\ & \left. + 1764p_1 p_2^2 p_3 p_4^2 + 700p_1 p_2 p_3^2 p_4^2 + 49p_2^2 p_3^2 p_4^2\right) z^6 \\ & + 3432p_1 p_2^2 p_3^2 p_4^2 z^7 + \left(49p_1^2 p_2^2 p_3^2 p_4^2 \right. \\ & + 700p_1 p_2^3 p_3^2 p_4^2 + 1764p_1 p_2^2 p_3^3 p_4^2 \\ & \left. + 490p_1 p_2^2 p_3^2 p_4^3\right) z^8 + \left(126p_1^2 p_2^3 p_3^2 p_4^2 \right. \\ & \left. + 896p_1 p_2^3 p_3^3 p_4^2 + 980p_1 p_2^2 p_3^3 p_4^3\right) z^9 \\ & + \left(175p_1^2 p_2^3 p_3^3 p_4^2 + 630p_1 p_2^3 p_3^3 p_4^3 \right. \\ & \left. + 196p_1 p_2^2 p_3^3 p_4^4\right) z^{10} + \left(140p_1^2 p_2^3 p_3^3 p_4^3 \right. \\ & \left. + 224p_1 p_2^3 p_3^3 p_4^4\right) z^{11} + \left(63p_1^2 p_2^3 p_3^3 p_4^4 \right. \\ & \left. + 28p_1 p_2^3 p_3^4 p_4^4\right) z^{12} + 14p_1^2 p_2^3 p_3^4 p_4^4 z^{13} \\ & + p_1^2 p_2^4 p_3^4 p_4^4 z^{14}, \end{aligned} \tag{26}$$

$$\begin{aligned} H_3 = {} & 1 + 18p_3 z + \left(63p_2 p_3 + 90p_3 p_4\right) z^2 \\ & + \left(56p_1 p_2 p_3 + 560p_2 p_3 p_4 + 200p_3 p_4^2\right) z^3 \\ & + \left(630p_1 p_2 p_3 p_4 + 630p_2 p_3^2 p_4 + 1575p_2 p_3 p_4^2 \right. \\ & \left. + 225p_3^2 p_4^2\right) z^4 + \left(1260p_1 p_2 p_3^2 p_4 + 2016p_1 p_2 p_3 p_4^2 \right. \\ & \left. + 5292p_2 p_3^2 p_4^2\right) z^5 + \left(490p_1 p_2^2 p_3^2 p_4 \right. \\ & + 9996p_1 p_2 p_3^2 p_4^2 + 1225p_2^2 p_3^2 p_4^2 + 5103p_2 p_3^3 p_4^2 \\ & \left. + 1750p_2 p_3^2 p_4^3\right) z^6 + \left(5616p_1 p_2^2 p_3^2 p_4^2 \right. \\ & + 12600p_1 p_2 p_3^3 p_4^2 + 3528p_2^2 p_3^3 p_4^2 \\ & \left. + 5040p_1 p_2 p_3^2 p_4^3 + 5040p_2 p_3^3 p_4^3\right) z^7 \\ & + \left(441p_1^2 p_2^2 p_3^2 p_4^2 + 17172p_1 p_2^2 p_3^3 p_4^2 \right. \\ & + 4410p_1 p_2^2 p_3^2 p_4^3 + 15750p_1 p_2 p_3^3 p_4^3 \\ & \left. + 4410p_2^2 p_3^3 p_4^3 + 1575p_2 p_3^3 p_4^4\right) z^8 \\ & + \left(2450p_1^2 p_2^2 p_3^3 p_4^2 + 5600p_1 p_2^3 p_3^3 p_4^2 \right. \\ & + 32520p_1 p_2^2 p_3^3 p_4^3 + 5600p_1 p_2 p_3^3 p_4^4 \\ & \left. + 2450p_2^2 p_3^3 p_4^4\right) z^9 + \left(1575p_1^2 p_2^3 p_3^3 p_4^2 \right. \\ & + 4410p_1^2 p_2^2 p_3^3 p_4^3 + 15750p_1 p_2^3 p_3^3 p_4^3 \\ & + 4410p_1 p_2^2 p_3^4 p_4^3 + 17172p_1 p_2^2 p_3^3 p_4^4 \\ & \left. + 441p_2^2 p_3^4 p_4^4\right) z^{10} + \left(5040p_1^2 p_2^3 p_3^3 p_4^3 \right. \\ & + 5040p_1 p_2^3 p_3^4 p_4^3 + 3528p_1^2 p_2^2 p_3^3 p_4^4 \\ & \left. + 12600p_1 p_2^3 p_3^3 p_4^4 + 5616p_1 p_2^2 p_3^4 p_4^4\right) z^{11} \\ & + \left(1750p_1^2 p_2^3 p_3^4 p_4^3 + 5103p_1^2 p_2^3 p_3^3 p_4^4 \right. \\ & + 1225p_1^2 p_2^2 p_3^4 p_4^4 + 9996p_1 p_2^3 p_3^4 p_4^4 \\ & \left. + 490p_1 p_2^2 p_3^4 p_4^5\right) z^{12} + \left(5292p_1^2 p_2^3 p_3^4 p_4^4 \right. \\ & \left. + 2016p_1 p_2^3 p_3^5 p_4^4 + 1260p_1 p_2^3 p_3^4 p_4^5\right) z^{13} \\ & + \left(225p_1^2 p_2^4 p_3^4 p_4^4 + 1575p_1^2 p_2^3 p_3^5 p_4^4 \right. \\ & \left. + 630p_1^2 p_2^3 p_3^4 p_4^5 + 630p_1 p_2^3 p_3^5 p_4^5\right) z^{14} \\ & + \left(200p_1^2 p_2^4 p_3^5 p_4^4 + 560p_1^2 p_2^3 p_3^5 p_4^5 + 56p_1 p_2^3 p_3^5 p_4^6\right) \end{aligned}$$

$$\begin{aligned}
&\cdot z^{15} + \left(90 p_1^2 p_2^4 p_3^5 p_4^5 + 63 p_1^2 p_2^3 p_3^5 p_4^6\right) z^{16} \\
&+ 18 p_1^2 p_2^4 p_3^5 p_4^6 z^{17} + p_1^2 p_2^4 p_3^6 p_4^6 z^{18},
\end{aligned} \tag{27}$$

$$\begin{aligned}
H_4 &= 1 + 10 p_4 z + 45 p_3 p_4 z^2 + \left(70 p_2 p_3 p_4 + 50 p_3 p_4^2\right) \\
&\cdot z^3 + \left(35 p_1 p_2 p_3 p_4 + 175 p_2 p_3 p_4^2\right) z^4 \\
&+ \left(126 p_1 p_2 p_3 p_4^2 + 126 p_2 p_3^2 p_4^2\right) z^5 \\
&+ \left(175 p_1 p_2 p_3^2 p_4^2 + 35 p_2 p_3^2 p_4^3\right) z^6 + \left(50 p_1 p_2^2 p_3^2 p_4^2\right. \\
&\left.+ 70 p_1 p_2 p_3^2 p_4^3\right) z^7 + 45 p_1 p_2^2 p_3^2 p_4^3 z^8 \\
&+ 10 p_1 p_2^2 p_3^3 p_4^3 z^9 + p_1 p_2^2 p_3^3 p_4^4 z^{10}.
\end{aligned} \tag{28}$$

C_4-*Case.* For the Lie algebra $C_4 \cong sp(6)$ we get the following polynomials:

$$\begin{aligned}
H_1 &= 1 + 7 p_1 z + 21 p_1 p_2 z^2 + 35 p_1 p_2 p_3 z^3 \\
&+ 35 p_1 p_2 p_3 p_4 z^4 + 21 p_1 p_2 p_3^2 p_4 z^5 + 7 p_1 p_2^2 p_3^2 p_4 z^6 \\
&+ p_1^2 p_2^2 p_3^2 p_4 z^7,
\end{aligned} \tag{29}$$

$$\begin{aligned}
H_2 &= 1 + 12 p_2 z + \left(21 p_1 p_2 + 45 p_2 p_3\right) z^2 \\
&+ \left(140 p_1 p_2 p_3 + 80 p_2 p_3 p_4\right) z^3 + \left(105 p_1 p_2^2 p_3\right. \\
&\left.+ 315 p_1 p_2 p_3 p_4 + 75 p_2 p_3^2 p_4\right) z^4 + \left(420 p_1 p_2^2 p_3 p_4\right. \\
&\left.+ 336 p_1 p_2 p_3^2 p_4 + 36 p_2^2 p_3^2 p_4\right) z^5 + 924 p_1 p_2^2 p_3^2 p_4 z^6 \\
&+ \left(36 p_1^2 p_2^2 p_3^2 p_4 + 336 p_1 p_2^3 p_3^2 p_4 + 420 p_1 p_2^2 p_3^3 p_4\right) \\
&\cdot z^7 + \left(75 p_1^2 p_2^3 p_3^2 p_4 + 315 p_1 p_2^3 p_3^3 p_4\right. \\
&\left.+ 105 p_1 p_2^2 p_3^3 p_4^2\right) z^8 + \left(80 p_1^2 p_2^3 p_3^3 p_4\right. \\
&\left.+ 140 p_1 p_2^3 p_3^3 p_4^2\right) z^9 + \left(45 p_1^2 p_2^3 p_3^3 p_4^2\right. \\
&\left.+ 21 p_1 p_2^3 p_3^4 p_4^2\right) z^{10} + 12 p_1^2 p_2^3 p_3^4 p_4^2 z^{11} \\
&+ p_1^2 p_2^4 p_3^4 p_4^2 z^{12},
\end{aligned} \tag{30}$$

$$\begin{aligned}
H_3 &= 1 + 15 p_3 z + \left(45 p_2 p_3 + 60 p_3 p_4\right) z^2 \\
&+ \left(35 p_1 p_2 p_3 + 320 p_2 p_3 p_4 + 100 p_3^2 p_4\right) z^3 \\
&+ \left(315 p_1 p_2 p_3 p_4 + 1050 p_2 p_3^2 p_4\right) z^4 \\
&+ \left(1302 p_1 p_2 p_3^2 p_4 + 576 p_2^2 p_3^2 p_4 + 1125 p_2 p_3^3 p_4\right) z^5 \\
&+ \left(1050 p_1 p_2^2 p_3^2 p_4 + 2240 p_1 p_2 p_3^3 p_4\right. \\
&\left.+ 1215 p_2^2 p_3^3 p_4 + 500 p_2 p_3^3 p_4^2\right) z^6 + \left(225 p_1^2 p_2^2 p_3^2 p_4\right. \\
&\left.+ 3990 p_1 p_2^2 p_3^3 p_4 + 1260 p_1 p_2 p_3^3 p_4^2 + 960 p_2^2 p_3^3 p_4^2\right) \\
&\cdot z^7 + \left(960 p_1^2 p_2^2 p_3^3 p_4 + 1260 p_1 p_2^3 p_3^3 p_4\right. \\
&\left.+ 3990 p_1 p_2^2 p_3^3 p_4^2 + 225 p_2^2 p_3^4 p_4^2\right) z^8 \\
&+ \left(500 p_1^2 p_2^3 p_3^3 p_4 + 1215 p_1^2 p_2^2 p_3^3 p_4^2\right. \\
&\left.+ 2240 p_1 p_2^3 p_3^3 p_4^2 + 1050 p_1 p_2^2 p_3^4 p_4^2\right) z^9 \\
&+ \left(1125 p_1^2 p_2^3 p_3^3 p_4^2 + 576 p_1^2 p_2^2 p_3^4 p_4^2\right. \\
&\left.+ 1302 p_1 p_2^3 p_3^4 p_4^2\right) z^{10} + \left(1050 p_1^2 p_2^3 p_3^4 p_4^2\right. \\
&\left.+ 315 p_1 p_2^3 p_3^5 p_4^2\right) z^{11} + \left(100 p_1^2 p_2^4 p_3^4 p_4^2\right. \\
&\left.+ 320 p_1^2 p_2^3 p_3^5 p_4^2 + 35 p_1 p_2^3 p_3^5 p_4^3\right) z^{12} \\
&+ \left(60 p_1^2 p_2^4 p_3^5 p_4^2 + 45 p_1^2 p_2^3 p_3^5 p_4^3\right) z^{13} \\
&+ 15 p_1^2 p_2^4 p_3^5 p_4^3 z^{14} + p_1^2 p_2^4 p_3^6 p_4^3 z^{15},
\end{aligned} \tag{31}$$

$$\begin{aligned}
H_4 &= 1 + 16 p_4 z + 120 p_3 p_4 z^2 + \left(160 p_2 p_3 p_4\right. \\
&\left.+ 400 p_3^2 p_4\right) z^3 + \left(70 p_1 p_2 p_3 p_4 + 1350 p_2 p_3^2 p_4\right. \\
&\left.+ 400 p_3^2 p_4^2\right) z^4 + \left(672 p_1 p_2 p_3^2 p_4 + 1296 p_2^2 p_3^2 p_4\right. \\
&\left.+ 2400 p_2 p_3^2 p_4^2\right) z^5 + \left(1400 p_1 p_2^2 p_3^2 p_4\right. \\
&\left.+ 1512 p_1 p_2 p_3^2 p_4^2 + 4096 p_2^2 p_3^2 p_4^2 + 1000 p_2 p_3^3 p_4^2\right) \\
&\cdot z^6 + \left(400 p_1^2 p_2^2 p_3^2 p_4 + 5600 p_1 p_2^2 p_3^2 p_4^2\right. \\
&\left.+ 1120 p_1 p_2 p_3^3 p_4^2 + 4320 p_2^2 p_3^3 p_4^2\right) z^7 \\
&+ \left(2025 p_1^2 p_2^2 p_3^2 p_4^2 + 8820 p_1 p_2^2 p_3^3 p_4^2\right. \\
&\left.+ 2025 p_2^2 p_3^4 p_4^2\right) z^8 + \left(4320 p_1^2 p_2^2 p_3^3 p_4^2\right. \\
&\left.+ 1120 p_1 p_2^3 p_3^3 p_4^2 + 5600 p_1 p_2^2 p_3^4 p_4^2 + 400 p_2^2 p_3^4 p_4^3\right) \\
&\cdot z^9 + \left(1000 p_1^2 p_2^3 p_3^3 p_4^2 + 4096 p_1^2 p_2^2 p_3^4 p_4^2\right. \\
&\left.+ 1512 p_1 p_2^3 p_3^4 p_4^2 + 1400 p_1 p_2^2 p_3^4 p_4^3\right) z^{10} \\
&+ \left(2400 p_1^2 p_2^3 p_3^4 p_4^2 + 1296 p_1^2 p_2^2 p_3^4 p_4^3\right. \\
&\left.+ 672 p_1 p_2^3 p_3^4 p_4^3\right) z^{11} + \left(400 p_1^2 p_2^4 p_3^4 p_4^2\right. \\
&\left.+ 1350 p_1^2 p_2^3 p_3^4 p_4^3 + 70 p_1 p_2^3 p_3^5 p_4^3\right) z^{12} \\
&+ \left(400 p_1^2 p_2^4 p_3^4 p_4^3 + 160 p_1^2 p_2^3 p_3^5 p_4^3\right) z^{13}
\end{aligned}$$

$$+ 120 p_1^2 p_2^4 p_3^5 p_4^3 z^{14} + 16 p_1^2 p_2^4 p_3^6 p_4^3 z^{15} + p_1^2 p_2^4 p_3^6 p_4^4 z^{16}. \tag{32}$$

D_4-Case. For the Lie algebra $D_4 \cong so(8)$ we obtain the polynomials

$$\begin{aligned} H_1 = {} & 1 + 6p_1 z + 15 p_1 p_2 z^2 + (10 p_1 p_2 p_3 \\ & + 10 p_1 p_2 p_4) z^3 + 15 p_1 p_2 p_3 p_4 z^4 + 6 p_1 p_2^2 p_3 p_4 z^5 \\ & + p_1^2 p_2^2 p_3 p_4 z^6, \end{aligned} \tag{33}$$

$$\begin{aligned} H_2 = {} & 1 + 10 p_2 z + (15 p_1 p_2 + 15 p_2 p_3 + 15 p_2 p_4) z^2 \\ & + (40 p_1 p_2 p_3 + 40 p_1 p_2 p_4 + 40 p_2 p_3 p_4) z^3 \\ & + \left(25 p_1 p_2^2 p_3 + 25 p_1 p_2^2 p_4 + 135 p_1 p_2 p_3 p_4 \right. \\ & \left. + 25 p_2^2 p_3 p_4\right) z^4 + 252 p_1 p_2^2 p_3 p_4 z^5 + \left(25 p_1^2 p_2^2 p_3 p_4 \right. \\ & \left. + 135 p_1 p_2^3 p_3 p_4 + 25 p_1 p_2^2 p_3^2 p_4 + 25 p_1 p_2^2 p_3 p_4^2\right) z^6 \\ & + \left(40 p_1^2 p_2^3 p_3 p_4 + 40 p_1 p_2^3 p_3^2 p_4 + 40 p_1 p_2^3 p_3 p_4^2\right) z^7 \\ & + \left(15 p_1^2 p_2^3 p_3^2 p_4 + 15 p_1^2 p_2^3 p_3 p_4^2 + 15 p_1 p_2^3 p_3^2 p_4^2\right) z^8 \\ & + 10 p_1^2 p_2^3 p_3^2 p_4^2 z^9 + p_1^2 p_2^4 p_3^2 p_4^2 z^{10}, \end{aligned} \tag{34}$$

$$\begin{aligned} H_3 = {} & 1 + 6 p_3 z + 15 p_2 p_3 z^2 + (10 p_1 p_2 p_3 \\ & + 10 p_2 p_3 p_4) z^3 + 15 p_1 p_2 p_3 p_4 z^4 + 6 p_1 p_2^2 p_3 p_4 z^5 \\ & + p_1 p_2^2 p_3^2 p_4 z^6, \end{aligned} \tag{35}$$

$$\begin{aligned} H_4 = {} & 1 + 6 p_4 z + 15 p_2 p_4 z^2 + (10 p_1 p_2 p_4 \\ & + 10 p_2 p_3 p_4) z^3 + 15 p_1 p_2 p_3 p_4 z^4 + 6 p_1 p_2^2 p_3 p_4 z^5 \\ & + p_1 p_2^2 p_3 p_4^2 z^6. \end{aligned} \tag{36}$$

Let us denote

$$H_s \equiv H_s(z) = H_s(z, (p_i)), \quad (p_i) \equiv (p_1, p_2, p_3, p_4). \tag{37}$$

One can easily write down the asymptotic behaviour of the polynomials obtained:

$$H_s = H_s(z, (p_i)) \sim \left(\prod_{l=1}^{4} (p_l)^{\nu^{sl}} \right) z^{n_s} \equiv H_s^{as}(z, (p_i)), \quad \text{as } z \longrightarrow \infty, \tag{38}$$

where we introduced the integer-valued matrix $\nu = (\nu^{sl})$ having the form

$$\nu = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 4 & 4 & 4 \\ 2 & 4 & 6 & 6 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 2 & 1 \\ 2 & 4 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 2 & 4 & 6 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix} \tag{39}$$

for Lie algebras A_4, B_4, C_4, D_4, respectively. In these four cases there is a simple property

$$\sum_{l=1}^{4} \nu^{sl} = n_s, \quad s = 1, 2, 3, 4. \tag{40}$$

Note that for Lie algebras B_4, C_4, and D_4 we have

$$\begin{aligned} \nu &= 2A^{-1}, \\ \mathcal{G} &= B_4, C_4, D_4 \end{aligned} \tag{41}$$

where A^{-1} is inverse Cartan matrix, whereas in the A_4-case the matrix ν is related to the inverse Cartan matrix as follows:

$$\begin{aligned} \nu &= A^{-1}(I + P), \\ \mathcal{G} &= A_4. \end{aligned} \tag{42}$$

Here I is 4×4 identity matrix and

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{43}$$

is a permutation matrix corresponding to the permutation $\sigma \in S_4$ (S_4 is symmetric group)

$$\sigma : (1, 2, 3, 4) \longmapsto (4, 3, 2, 1), \tag{44}$$

by the following relation $P = (P^i_j) = (\delta^i_{\sigma(j)})$. Here σ is the generator of the group $G = \{\sigma, \mathrm{id}\}$ which is the group of symmetry of the Dynkin diagram for A_4. G is isomorphic to the group $\mathbb{Z}_2$.

In case of D_4 the group of symmetry of the Dynkin diagram G' is isomorphic to the symmetric group S_3 acting on the set of three vertices $\{1, 3, 4\}$ of the Dynkin diagram via their permutations. The existence of the above symmetry groups $G \cong \mathbb{Z}_2$ and $G' \cong S_3$ implies certain identity properties for the fluxbrane polynomials $H_s(z)$.

Let us denote $\widehat{p}_i = p_{\sigma(i)}$ for the A_4 case and $\widehat{p}_i = p_i$ for B_4, C_4, and D_4 cases ($i = 1, 2, 3, 4$). We call the ordered set $(\widehat{p}_i)$ as *dual* one to the ordered set (p_i). It corresponds to the action (trivial or nontrivial) of the group $\mathbb{Z}_2$ on vertices of the Dynkin diagrams for above algebras.

Then we obtain the following identities which were directly verified by using MATHEMATICA algorithms.

Symmetry Relations

Proposition 2. *The fluxbrane polynomials obey for all p_i and $z > 0$ the identities*

$$\begin{aligned} H_{\sigma(s)}\left(z,\left(p_i\right)\right) &= H_s\left(z,\left(\hat{p}_i\right)\right) \quad \text{for } A_4 \text{ case}, \\ H_{\sigma'(s)}\left(z,\left(p_i\right)\right) &= H_s(z,\left(p_{\sigma'(i)}\right) \quad \text{for } D_4 \text{ case}, \end{aligned} \tag{45}$$

for any $\sigma' \in S_3$, $s = 1, \ldots, 4$. We call relations (45) as symmetry ones.

Duality Relations

Proposition 3. *The fluxbrane polynomials corresponding to Lie algebras A_4, B_4, C_4, and D_4 obey for all $p_i > 0$ and $z > 0$ the identities*

$$H_s\left(z,\left(p_i\right)\right) = H_s^{as}\left(z,\left(p_i\right)\right) H_s\left(z^{-1},\left(\hat{p}_i^{-1}\right)\right), \tag{46}$$

$s = 1, 2, 3, 4.$

We call relations (46) as duality ones.

Fluxes. Here we deal with an oriented 2-dimensional manifold $M_R = (0, R) \times S^1$, $R > 0$. One can define the flux integrals over this manifold:

$$\Phi^s(R) = \int_{M_R} F^s = 2\pi \int_0^R d\rho \rho \mathscr{B}^s, \tag{47}$$

where we denoted

$$\mathscr{B}^s = q_s \prod_{l=1}^{4} H_l^{-A_{sl}}. \tag{48}$$

It can be easily understood that total flux integrals $\Phi^s = \Phi^s(+\infty)$ are convergent. Indeed, due to polynomial assumption (1) we have

$$H_s \sim C_s \rho^{2n_s}, \quad C_s = \prod_{l=1}^{4} \left(p_l\right)^{\nu^{sl}}, \tag{49}$$

as $\rho \longrightarrow +\infty$. From (48), (49), and the equality $\sum_1^n A_{sl} n_l = 2$, following from (2), we get

$$\mathscr{B}^s \sim q_s C^s \rho^{-4}, \quad C^s = \prod_{l=1}^{4} p_l^{-(A\nu)_s{}^l}, \tag{50}$$

and hence the integral (47) is convergent for any $s = 1, 2, 3, 4$.

Using (42) and (50) we have for the A_4-case

$$C^s = \prod_{l=1}^{4} p_l^{-(I+P)_s{}^l} = \prod_{l=1}^{4} p_l^{-\delta_s^l - \delta_{\sigma(s)}^l} = p_s^{-1} p_{\sigma(s)}^{-1}. \tag{51}$$

Similarly, due to (41) and (50) we get for Lie algebras B_4, C_4, and D_4

$$C^s = p_s^{-2}. \tag{52}$$

After that, we can calculate the fluxes $\Phi^s(R)$. Using the master equations (8) one can write

$$\begin{aligned} \int_0^R d\rho \rho \mathscr{B}^s &= q_s P_s^{-1} \frac{1}{2} \int_0^{R^2} dz \frac{d}{dz}\left(\frac{z}{H_s} \frac{d}{dz} H_s\right) \\ &= \frac{1}{2} q_s P_s^{-1} \frac{R^2 H_s'\left(R^2\right)}{H_s\left(R^2\right)}, \end{aligned} \tag{53}$$

where $H_s' = dH_s/dz$. Thus, using (47) we easily obtain

$$\Phi^s(R) = 4\pi q_s^{-1} h_s \frac{R^2 H_s'\left(R^2\right)}{H_s\left(R^2\right)}. \tag{54}$$

Note that the manifold $M_* = (0, +\infty) \times S^1$ is isomorphic to the manifold $\mathbb{R}_*^2 = \mathbb{R}^2 \setminus \{0\}$. Therefore, one can understand the family of solutions under consideration as defined on the manifold $\mathbb{R}_*^2 \times M_2$, where coordinates ρ, ϕ are polar coordinates in a domain of $\mathbb{R}_*^2$: $x = \rho \cos\phi$ and $y = \rho \sin\phi$, where x, y are standard coordinates of $\mathbb{R}^2$. It was shown in [30] that there exist forms A^s globally defined on $\mathbb{R}^2$ and obeying the relation $F^s = dA^s$.

Now let us we consider a 2-dimensional oriented manifold (disk) $D_R = \{(x, y) : x^2 + y^2 \leq R^2\}$. Its boundary $\partial D_R \equiv C_R = \{(x, y) : x^2 + y^2 = R^2\}$ is a circle of radius R, i.e., 1-dimensional oriented manifold with the orientation inherited from that of D_R obeying the relation $\int_{C_R} d\phi = 2\pi$.

The Stokes theorem yields in this case

$$\Phi^s(R) = \int_{M_R} F^s = \int_{D_R} dA^s = \int_{C_R} A^s. \tag{55}$$

According to the definition of Abelian Wilson loop (factor), we have

$$W^s\left(C_R\right) = \exp\left(i \int_{C_R} A^s\right) = \exp\left(i\Phi^s(R)\right). \tag{56}$$

Relations (1) and (54) imply (see (10))

$$\Phi^s = \Phi^s(+\infty) = 4\pi n_s q_s^{-1} h_s, \tag{57}$$

$s = 1, 2, 3, 4$. Any (total) flux Φ^s depends upon one integration constant $q_s \neq 0$, while the integrand form F^s depends upon all constants: q_1, q_2, q_3, q_4. As a consequence, we obtain finite limits

$$\lim_{R \longrightarrow +\infty} W^s\left(C_R\right) = \exp\left(i\Phi^s\right). \tag{58}$$

In the A_4-case we have

$$\left(q_1\Phi^1, q_2\Phi^2, q_3\Phi^3, q_4\Phi^4\right) = 4\pi h(4, 6, 6, 4), \tag{59}$$

where $h_1 = h_2 = h_3 = h_4 = h$.

In the B_4-case we find

$$\begin{aligned}&\left(q_1\Phi^1, q_2\Phi^2, q_3\Phi^3, q_4\Phi^4\right)\\&\quad = 4\pi\left(8h_1, 14h_2, 18h_3, 10h_4\right) = 4\pi h\,(8, 14, 18, 20)\,,\end{aligned} \tag{60}$$

where $h_1 = h_2 = h_3 = h$, $h_4 = 2h$.

In the C_4-case we obtain

$$\begin{aligned}&\left(q_1\Phi^1, q_2\Phi^2, q_3\Phi^3, q_4\Phi^4\right)\\&\quad = 4\pi\left(7h_1, 12h_2, 15h_3, 16h_4\right) = 4\pi h\,(7, 12, 15, 8)\,,\end{aligned} \tag{61}$$

where $h_1 = h_2 = h_3 = h$, $h_4 = (1/2)h$.

In the D_4-case we are led to relations

$$\left(q_1\Phi^1, q_2\Phi^2, q_3\Phi^3, q_4\Phi^4\right) = 4\pi h\,(6, 10, 6, 6)\,, \tag{62}$$

where $h_1 = h_2 = h_3 = h_4 = h$. (In all examples relations (19) are used.)

Note that, for $D = 4$ and $g^2 = -dt \otimes dt + dx \otimes dx$, q_s coincides with the value of the x-component of the s-th magnetic field on the axis of symmetry, $s = 1, 2, 3, 4$.

Asymptotic Relations. Here we can write down the asymptotic relations for the solution under consideration as $\rho \longrightarrow +\infty$:

$$\begin{aligned}g_{as} = {}&\left(\prod_{l=1}^{4} p_l^{a_l}\right)^{2/(D-2)} \rho^{2A}\left\{d\rho\otimes d\rho\right.\\&\left.+\left(\prod_{l=1}^{4} p_l^{a_l}\right)^{-2}\rho^{2-2A(D-2)}d\phi\otimes d\phi + g^2\right\},\end{aligned} \tag{63}$$

$$\varphi_{as}^a = \sum_{s=1}^{4} h_s\lambda_s^a\left(\sum_{l=1}^{4}\nu^{sl}\ln p_l + 2n_s\ln\rho\right), \tag{64}$$

$$F_{as}^s = q_s p_s^{-1} p_{\theta(s)}^{-1}\rho^{-3}d\rho\wedge d\phi, \tag{65}$$

$a, s = 1, 2, 3, 4$, where

$$\begin{aligned}a_l &= \sum_{s=1}^{4} h_s\nu^{sl},\\ A &= 2\,(D-2)^{-1}\sum_{s=1}^{4} n_s h_s,\end{aligned} \tag{66}$$

and in (65) we put $\theta = \sigma$ for $\mathcal{G} = A_4$ and $\theta = \mathrm{id}$ for $\mathcal{G} = B_4, C_4, D_4$. In derivation of asymptotic relations, (40), (49), and (51) were used. We note that for $\mathcal{G} = B_4, C_4, D_4$ the asymptotic value of form F_{as}^s depends upon q_s, $s = 1, 2, 3, 4$, while in the A_4-case F_{as}^s depends upon q_1 and q_4 for $s = 1, 4$, and F_{as}^s depends upon q_2, q_3 for $s = 2, 3$.

We note also that by putting $q_1 = 0$ we get the Melvin-type solutions corresponding to Lie algebras A_3, B_3, C_3, and A_3, respectively, which were analyzed in [28]. (The case of the rank 2 Lie algebra G_2 [27] may be obtained for the D_4 case when $q_1 = q_3 = q_4$.)

Dilatonic Black Holes. Relations (constraints) on dilatonic coupling vectors (12), (13) appear also for dilatonic black hole (DBH) solutions which are defined on the manifold

$$M = \left(R_0, +\infty\right)\times\left(M_0 = S^2\right)\times\left(M_1 = \mathbb{R}\right)\times M_2, \tag{67}$$

where $R_0 = 2\mu > 0$ and M_2 is a Ricci-flat manifold. These DBH solutions on M from (67) for the model under consideration may be extracted from general black brane solutions; see [21, 25, 31]. They read

$$\begin{aligned}g = {}&\left(\prod_{s=1}^{4}\mathbf{H}_s^{2h_s/(D-2)}\right)\left\{f^{-1}dR\otimes dR + R^2 g^0\right.\\&\left.-\left(\prod_{s=1}^{4}\mathbf{H}^{-2h_s}\right)f dt\otimes dt + g^2\right\},\end{aligned} \tag{68}$$

$$\exp\left(\varphi^a\right) = \prod_{s=1}^{4}\mathbf{H}^{h_s\lambda_s^a}, \tag{69}$$

$$F^s = -Q_s R^{-2}\left(\prod_{l=1}^{4}\mathbf{H}_l^{-A_{sl}}\right)dR\wedge dt, \tag{70}$$

$s, a = 1, 2, 3, 4$, where $f = 1 - 2\mu R^{-1}$, g^0 is the standard metric on $M_0 = S^2$, and g^2 is a Ricci-flat metric of signature $(+, \ldots, +)$ on M_2. Here $Q_s \neq 0$ are integration constants (charges).

The functions $\mathbf{H}_s = \mathbf{H}_s(R) > 0$ obey the master equations

$$R^2\frac{d}{dR}\left(f\frac{R^2}{\mathbf{H}_s}\frac{d}{dR}\mathbf{H}_s\right) = B_s\prod_{l=1}^{4}\mathbf{H}_l^{-A_{sl}}, \tag{71}$$

with the following boundary conditions on the horizon and at infinity imposed:

$$\begin{aligned}\mathbf{H}_s\left(R_0 + 0\right) &= \mathbf{H}_{s0} > 0,\\ \mathbf{H}_s\left(+\infty\right) &= 1,\end{aligned} \tag{72}$$

where

$$B_s = -K_s Q_s^2, \tag{73}$$

$s = 1, 2, 3, 4$. Here relations (11) are also valid. For Lie algebras of rank 4 the functions $\mathbf{H}_s$ are polynomials with respect to R^{-1}, which may be obtained (at least for small enough q_s) from fluxbrane polynomials $H_s(z)$ presented in this paper. See [25].

4. Conclusions

In this paper, the generalized multidimensional family of Melvin-type solutions was considered corresponding to simple nonexceptional finite-dimensional Lie algebras of rank 4: $\mathcal{G} = A_4, B_4, C_4, D_4$. Each solution of that family is governed by a set of 4 fluxbrane polynomials $H_s(z)$, $s = 1, 2, 3, 4$. These polynomials define special solutions to open Toda chain equations corresponding to the Lie algebra $\mathcal{G}$.

The polynomials $H_s(z)$ depend also upon parameters q_s, which coincides for $D = 4$ (up to a sign) with the values of colored magnetic fields on the axis of symmetry.

We have found the symmetry relations and the duality identities for polynomials. These identities may be used in deriving $1/\rho$-expansion for solutions at large distances ρ, e.g., for asymptotic relations which are presented in the paper.

There were also calculated two-dimensional flux integrals $\Phi^s(R) = \int_{D_R} F^s$ $(s = 1, 2, 3, 4)$ over a disc D_R of radius R and a corresponding Wilson loop factors $W^s(C_R)$ over a circle C_R. It turns out that each total flux $\Phi^s(\infty)$ depends only upon one corresponding parameter q_s, whereas the integrand F^s depends on all parameters q_s, $s = 1, 2, 3, 4$.

The case of exceptional Lie algebra F_4 will be considered in a separate publication.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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