Hindawi
Advances in Mathematical Physics
Volume 2018, Article ID 8540674, 15 pages
https://doi.org/10.1155/2018/8540674

Research Article

Rota-Baxter Leibniz Algebras and Their Constructions

Liangyun Zhang, Linhan Li, and Huihui Zheng

College of Science, Nanjing Agricultural University, Nanjing, Jiangsu 210095, China

Correspondence should be addressed to Liangyun Zhang; zlyun@njau.edu.cn

Received 17 August 2018; Revised 20 October 2018; Accepted 25 October 2018; Published 2 December 2018

Academic Editor: Francesco Toppan

In this paper, we introduce the concept of Rota-Baxter Leibniz algebras and explore two characterizations of Rota-Baxter Leibniz algebras. And we construct a number of Rota-Baxter Leibniz algebras from Leibniz algebras and associative algebras and discover some Rota-Baxter Leibniz algebras from augmented algebra, bialgebra, and weak Hopf algebra. In the end, we give all Rota-Baxter operators of weight 0 and −1 on solvable and nilpotent Leibniz algebras of dimension ≤3, respectively.

1. Introduction

The Leibniz algebra [1] was mentioned by Bloh at the first time, which was called a D-algebra in 1965. Later, Loday improved and named it as Leibniz algebra. In Loday's work, he was mainly interested in the properties of the corresponding homology theory on "group level" ("Leibniz K-Theory").

Leibniz algebras are a well-established algebraic structure generalizing Lie algebras with their own structure and homology theory. Moreover, they have much more applications in homological algebra, noncommutative geometry, physics, and so on (see [1–9]).

The Baxter algebra was firstly found in the work [3] of Baxter in 1960, which was used to solve the problem of probability [10]. A Baxter algebra is an associative algebra A with a linear operator P on A that satisfies the Baxter identity

$$P(x)P(y) = P(P(x)y + xP(y)) \tag{1}$$

for all $x, y \in A$.

In the 1960s, Rota began a study of Rota-Baxter algebras from an algebraic and combinatorial perspective in connection with hypergeometric functions, incidence algebras, and symmetric functions and obtained some interesting results (see [11–13]). A Rota-Baxter algebra is an associative algebra A with a linear operator P on A that satisfies the Rota-Baxter identity

$$P(x)P(y) = P(P(x)y + xP(y) + \lambda xy) \tag{2}$$

for all $x, y \in A$, where λ (called the weight) is a fixed element in the base ring of the algebra A.

In recent years, many scholars such as Andrews, Guo, and Bai et al. found and established the relations between Rota-Baxter algebras and Hopf algebras, Lie algebras, shuffle products, and dendriform algebras. Rota-Baxter algebras have been more and more important and have attracted much attention nowadays (see [11, 14–22]).

In this paper, our main aims are to introduce the concept of Rota-Baxter Leibniz algebras and to obtain a large number of Rota-Baxter Leibniz algebras from augmented algebra, bialgebra, and weak Hopf algebra, as well as construct all Rota-Baxter operators of weight 0 and −1 on solvable and nilpotent non-Lie Leibniz algebras of dimension ≤3.

The paper is organized as follows. In the second section, we introduce the concept of Rota-Baxter Leibniz algebras and explore two characterizations of Rota-Baxter Leibniz algebras. One is a generalization of the Atkinson factorization [23, 24]. One is new for a Rota-Baxter Leibniz algebra under the assumption of quasi-idempotency. And we construct a large number of Rota-Baxter Leibniz algebras from Leibniz algebras and associative algebras, respectively, and discover some Rota-Baxter Leibniz algebras from augmented algebra, bialgebra, and weak Hopf algebra. In the third section, we construct all Rota-Baxter operators of weight 0 and −1 on

solvable and nilpotent non-Lie Leibniz algebras of dimension ≤3.

Throughout the paper, all algebras, linear maps, and tensor products are taken over the complex field $\mathscr{C}$ unless otherwise specified.

2. Rota-Baxter Leibniz Algebras

In this section, we mainly give some characterizations of Rota-Baxter Leibniz algebras and construct a large number of Rota-Baxter Leibniz algebras from Leibniz algebras, augmented algebra, and weak Hopf algebra, respectively.

Definition 1. Let A be a vector space. Then, $(A, [-,-])$ is called a (left) *Leibniz algebra defined* as in [5] if there is a bilinear map $[-,-] : A \otimes A \longrightarrow A, a \otimes b \longmapsto [a,b]$ satisfying

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]] \tag{3}$$

for any $a, b, c \in A$.

In the following, our considered Leibniz algebras are left Leibniz algebras unless otherwise specified.

Let $(A, [-,-])$ be a Leibniz algebra. Write $A^1 = A, A^{\kappa+1} = [A, A^{\kappa}]$ and $A^{(1)} = [A, A], A^{(\kappa+1)} = [A^{(\kappa)}, A^{(\kappa)}]$, for any integer $\kappa \geq 1$. In another, we denote a^1 by a and $[a, a^{\kappa}]$ by $a^{\kappa+1}$, for any $a \in A$.

Let $(A, [-,-]_A), (B, [-,-]_B)$ be Leibniz algebras. A linear map $f : A \longrightarrow B$ is called a *Leibniz algebra homomorphism* from A to B, if $f([a,b]_A) = [f(a), f(b)]_B$, for any $a, b \in A$.

Definition 2. Let $(A, [-,-])$ be a Leibniz algebra. If there exists a linear map $P : A \longrightarrow A$ and an element $\lambda \in \mathscr{C}$ satisfying

$$[P(a), P(b)] = P([a, P(b)] + [P(a), b] + \lambda [a,b]) \tag{4}$$

for any $a, b \in A$, then, $(A, [-,-])$ is called a *Rota-Baxter Leibniz algebra of weight* λ, and P is called a *Rota-Baxter operator* on A. In what follows, we simply denote it by $(A, [-,-], P)$.

Let $(A, [-,-], P)$ be a Rota-Baxter Leibniz algebra of weight λ. The subspace M of A is called a *subalgebra*, if M is a Leibniz algebra under the multiplication of A, and P is still a Rota-Baxter operator of weight λ on M.

Example 3. (1) Let A be a 2-dimensional vector space with basis $\{x, y\}$. Define a multiplication on A:

$$\begin{aligned} [x,x] &= 0 = [x,y], \\ [y,x] &= x = [y,y], \end{aligned} \tag{5}$$

and a linear map $P : A \longrightarrow A$ given by

$$\begin{aligned} P(x) &= x, \\ P(y) &= 2x - y. \end{aligned} \tag{6}$$

Then, $(A, [-,-], P)$ is a Rota-Baxter Leibniz algebra of weight -1.

(2) Let $(A, [-,-], P)$ be a Rota-Baxter Leibniz algebra of weight 0. Then, $(A, [-,-], \ell P)$ is also a Rota-Baxter Leibniz algebra of weight 0, for any given element $\ell \in \mathscr{C}$.

(3) Let $(A_i, [-,-], P_i), i \in I$, be a family of Rota-Baxter Leibniz algebras of weight λ. Denote $\bigoplus_{i \in I} A_i$ by A. Now define a linear map $P : A \longrightarrow A$, such that $P((a_i)) = (P_i(a_i))$, for all $(a_i) \in A$. Then $(A, [-,-], P)$ is also a Rota-Baxter Leibniz algebra of weight λ by defining $[(a_i), (b_i)] = ([a_i, b_i])$, for all $(a_i), (b_i) \in A$.

Proof. (1) According to Example 2.1 in [5], we know that $(A, [-,-])$ is a (left) Leibniz algebra, but it is not a (right) Leibniz algebra since $[[y,y],y] \neq [y,[y,y]] + [[y,y],y]$.

It is easy to check that P is a Rota-Baxter operator of weight -1 on A.

(2) For any $a, b \in A$, we have

$$[\ell P(a), \ell P(b)]) = \ell P([\ell P(a), b] + [a, \ell P(b)]). \tag{7}$$

(3) It is straightforward to check that $(A, [-,-], P)$ is a Rota-Baxter Leibniz algebra of weight λ. □

In what follows, we will give some constructions of Rota-Baxter Leibniz algebras.

Proposition 4. *Let A be an algebra and P an algebra map from A to A with $P^2 = P$. Then the following conclusions hold.*

(1) Define a linear map $[-,-]_P : A \otimes A \longrightarrow A$ by

$$[a,b]_P = P(a)b - bP(a). \tag{8}$$

Then $(A, [-,-]_P, P)$ is a Rota-Baxter Leibniz algebra of weight -1.

(2) Define a linear map $[-,-]_P : A \otimes A \longrightarrow A$ by

$$[a,b]_P = P(ab) - bP(a). \tag{9}$$

Then, $(A, [-,-]_P, P)$ is a Rota-Baxter Leibniz algebra of weight -1.

(3) Suppose that A is a commutative algebra. Define a linear map $[-,-]_P : A \otimes A \longrightarrow A$ by

$$[a,b]_P = P(ab) - P(a)b. \tag{10}$$

Then, $(A, [-,-]_P, P)$ is a Rota-Baxter Leibniz algebra of any weight $\ell \in \mathscr{C}$.

Proof. (1) According to Example 2.2 in [5], we know that the conclusion (1) holds.

(2) For any $a, b, c \in A$, we can prove that

$$\begin{aligned} &[a,[b,c]_P]_P = [[a,b]_P, c]_P + [b,[a,c]_P]_P. \\ &P([a, P(b)]_P) + P([P(a), b]_P) - P([a,b]_P) \\ &\quad = [P(a), P(b)]_P. \end{aligned} \tag{11}$$

(3) It is obvious that $(A, [-,-])$ is a Leibniz algebra by (2). □

Remark 5. (1) Let A be an algebra. If A is a augmented algebra as in [25] in the sense that there exists an algebra homomorphism $f : A \longrightarrow \mathscr{C}$, then, by Proposition 4 (2),

$(A, [-,-]_{P_f}, P_f)$ is a Rota-Baxter Leibniz algebra of weight -1, where the operator P_f on A is defined by $P_f(a) = f(a)1_A$ for any $a \in A$, and the operator $[-,-]_{P_f}$ on $A \otimes A$ is given by

$$[a,b]_{P_f} = f(ab)\,1_A - f(a)\,b \tag{12}$$

for any $a, b \in A$.

(2) Let A be a bialgebra or a Hopf algebra as in [26]. Then, the counit map $\varepsilon : A \longrightarrow \mathscr{C}$ is an algebra map. So, by (1), $(A, [-,-]_{P_\varepsilon}, P_\varepsilon)$ is a Rota-Baxter Leibniz algebra of weight -1.

(3) Let A be a weak Hopf algebra with an antipode S given in [27]. Define a linear map $\Pi^L : A \longrightarrow A$ (called the target map) by $\Pi^L(a) = \varepsilon(1_1 a)1_2$, where $\Delta(1_A)$ is denoted by $1_1 \otimes 1_2 \in A \otimes A$.

Then, according to Corollary 2.2 (1) in [28], we know that Π^L is idempotent. Furthermore, if A is commutative, then, by Corollary 2.2 (4) in [28], Π^L is also an algebra map. So, by Proposition 4 (3), $(A, [-,-]_{\Pi^L}, \Pi^L)$ is a Rota-Baxter Leibniz algebra of any weight ℓ, with the product $[a,b]_{\Pi^L} = \Pi^L(ab) - \Pi^L(a)b$.

Proposition 6. *Let $(A, [-,-], P)$ be a Rota-Baxter Leibniz algebra of nonzero weight λ. Then, $(A, [-,-], Q_P)$ is a Rota-Baxter Leibniz algebra of weight $-\ell$, where $Q_P = (\ell/\lambda)P + \ell id$, for any given element $\ell \in \mathscr{C}$.*

Proof. It is straightforward to check that $(A, [-,-], Q_P)$ is a Rota-Baxter Leibniz algebra of weight $-\ell$. □

Example 7. Let A be an algebra and P an algebra map from A to A with $P^2 = P$. Then, according to Proposition 4 (1) and Proposition 6, we know that $(A, [-,-]_P, Q_P = -\ell P + \ell id)$ is a Rota-Baxter Leibniz algebra of weight $-\ell$, for any given element $\ell \in \mathscr{C}$.

Proposition 8. *Let $(A, [-,-], P)$ be a Rota-Baxter Leibniz algebra of weight λ. Define a new binary product $[-,-]_P : A \otimes A \longrightarrow A$ with*

$$[a,b]_P = [a, P(b)] + [P(a), b] + \lambda\,[a,b]. \tag{13}$$

Then we have the following conclusions.

(1) $[P(a), P(b)] = P([a,b]_P)$.

(2) $(A, [-,-]_P, P)$ is a Rota-Baxter Leibniz algebra of weight λ. So P is a Leibniz algebra map from $(A, [-,-]_P)$ to $(A, [-,-])$.

Proof. (1) It is just the Rota-Baxter Leibniz algebra equation.

(2) By the definition of $[-,-]_P$ and the equality of Rota-Baxter Leibniz algebra, we easily prove

$$[a, [b,c]_P]_P = [[a,b]_P, c]_P + [b, [a,c]_P]_P, \tag{14}$$

so $(A, [-,-]_P)$ is a Leibniz algebra. It is easy to see that P is a Rota-Baxter operator of weight λ. □

In the following, we give two differentiated conditions for a Leibniz algebra to be a Rota-Baxter Leibniz algebra.

Theorem 9. *Let $(A, [-,-])$ be a nondegenerate Leibniz algebra and $P : A \longrightarrow A$ be a linear map.*

(1) Suppose that P satisfies $P([a,b]) = [a, P(b)]$, for any $a, b \in A$. Then, $(A, [-,-], P)$ is a Rota-Baxter Leibniz algebra of weight λ, if and only if P is quasi-idempotent of weight λ.

(2) Denote

$$C_A := \{a \in A \mid P([a,b]) = [a, P(b)],\ \forall b \in A\} \tag{15}$$

Then, C_A is a subalgebra of A such that $P([a,b]) = [a, P(b)]$, for all $a \in C_A, b \in A$.

(3) Suppose that $P|_{C_A}$ is a Rota-Baxter operator of weight λ on C_A and C_A is idempotent (i.e., $[C_A, C_A] = C_A$). Then, $P|_{C_A}$ is quasi-idempotent of weight λ. Conversely, if P is quasi-idempotent of weight λ, then $(C_A, [-,-], P|_{C_A})$ is a Rota-Baxter Leibniz algebra of weight λ.

Proof. (1) For any $a, b \in A$, if $(A, [-,-], P)$ is a Rota-Baxter algebra of weight λ, then, we easily prove that

$$\begin{aligned}[P(a), P(b)] = \left[a, P^2(b)\right] + [P(a), P(b)] \\ + \lambda\,[a, P(b)].\end{aligned} \tag{16}$$

So we know that $[a, P^2(b)] + \lambda[a, P(b)] = [a, P^2(b) + \lambda P(b)] = 0$. This implies that $P^2 + \lambda P = 0$.

Conversely, if $P^2 + \lambda P = 0$, then, for any $a, b \in A$, we have

$$P([a, P(b)] + [P(a), b]) + \lambda\,[a,b]) = [P(a), P(b)] \tag{17}$$

as desired.

(2) In order to prove that C_A is a subalgebra of A, we only need to prove that $[a,b] \in C_A$, for all $a, b \in C_A$.

In fact, we have

$$\begin{aligned}[[a,b], P(c)] &= [a, [b, P(c)]] - [b, [a, P(c)]] \\ &= [a, P([b,c])] - [b, P([a,c])] \\ &= P([a, [b,c]]) - P([b, [a,c]]) \\ &= P([[a,b], c]),\end{aligned} \tag{18}$$

so $P([[a,b], c]) = [[a,b], P(c)]$, that is, $[a,b] \in C_A$. Hence C_A is a subalgebra of A.

(3) Suppose that $P|_{C_A}$ is a Rota-Baxter operator of weight λ on C_A. Then, for any $a, b \in C_A$, $P(a) \in C_A$, that is, $[P|_{C_A}(a), P|_{C_A}(b)] = P|_{C_A}([P|_{C_A}(a), b])$, and $[P|_{C_A}(a), P|_{C_A}(b)] - P|_{C_A}([a, P|_{C_A}(b)]) - P|_{C_A}([P|_{C_A}(a), b]) - \lambda P|_{C_A}([a,b]) = 0$. So we get

$$\begin{aligned}0 &= \left[P|_{C_A}(a), P|_{C_A}(b)\right] - P|_{C_A}\left(\left[a, P|_{C_A}(b)\right]\right) \\ &\quad - P|_{C_A}\left(\left[P|_{C_A}(a), b\right]\right) - \lambda P|_{C_A}([a,b]) \\ &= \left[P|_{C_A}(a), P|_{C_A}(b)\right] - P|_{C_A}^2([a,b]) \\ &\quad - P|_{C_A}\left(\left[P|_{C_A}(a), b\right]\right) - \lambda P|_{C_A}([a,b]) \\ &= -P|_{C_A}^2([a,b]) - \lambda P|_{C_A}([a,b]),\end{aligned} \tag{19}$$

that is, $-P|_{C_A}^2([a,b]) - \lambda P|_{C_A}([a,b]) = 0$. Hence $P|_{C_A}^2 = -\lambda P|_{C_A}$ by $[C_A, C_A] = C_A$.

Conversely, if P is quasi-idempotent of weight λ, then, for any $a \in C_A, b \in A$, we have

$$\begin{aligned} 0 &= [P(a), P(b)] - P([a, P(b)]) - P([P(a), b]) \\ &\quad - \lambda P([a, b]) \\ &= [P(a), P(b)] - P^2([a, b]) - P([P(a), b]) \\ &\quad - \lambda P([a, b]) = [P(a), P(b)] - P([P(a), b]), \end{aligned} \tag{20}$$

that is, $[P(a), P(b)] = P[P(a), b]$, so $P(a) \in C_A$. Hence, according to items (1) and (2), we easily see that $P|_{C_A}$ is a Rota-Baxter operator of weight λ on C_A. □

Theorem 10. *Let $(A, [-,-])$ be a Leibniz algebra. If $(A, [-,-], P)$ is a Rota-Baxter Leibniz algebra of nonzero weight λ, then, for any given $a, b \in A$, there is an element $c \in A$, such that*

$$\begin{aligned} \frac{\ell}{\lambda}[P(a), P(b)] &= P(c), \\ [Q(a), Q(b)] &= Q(c), \end{aligned} \tag{21}$$

where $Q = (\ell/\lambda)P + \ell id$ as in Proposition 6.

Conversely, if there exists an element $c \in A$ satisfying the above equalities and the annihilator of $\ell \in \mathscr{C}$ in A has only zero, then, $(A, [-,-], P)$ is a Rota-Baxter Leibniz algebra of nonzero weight λ.

Proof. For any $a, b \in A$, and $\lambda \in \mathscr{C}$, we have

$$[P(a), P(b)] = P([a, P(b)] + [P(a), b] + \lambda[a, b]). \tag{22}$$

Taking $c = (\ell/\lambda)([a, P(b)] + [P(a), b] + \lambda[a, b])$, then, we can obtain

$$\begin{aligned} P(c) &= \frac{\ell}{\lambda} P([a, P(b)] + [P(a), b] + \lambda[a, b]) \\ &= \frac{\ell}{\lambda}[P(a), P(b)], \\ Q(c) &= [Q(a), Q(b)]. \end{aligned} \tag{23}$$

Conversely, if there is an element $c \in A$ such that

$$\begin{aligned} \frac{\ell}{\lambda}[P(a), P(b)] &= P(c), \\ [Q(a), Q(b)] &= Q(c), \end{aligned} \tag{24}$$

for any given $a, b \in A$. Then, when $Q = (\ell/\lambda)P + \ell id$, we have that

$$\begin{aligned} [Q(a), Q(b)] &= \left[\frac{\ell}{\lambda}P(a) + \ell a, \frac{\ell}{\lambda}P(b) + \ell b\right] \\ &= \left[\frac{\ell}{\lambda}P(a), \frac{\ell}{\lambda}P(b) + \ell b\right] \\ &\quad + \left[\ell a, \frac{\ell}{\lambda}P(b) + \ell b\right] \\ &= \left[\frac{\ell}{\lambda}P(a), \frac{\ell}{\lambda}P(b)\right] + \left[\frac{\ell}{\lambda}P(a), \ell b\right] \\ &\quad + \left[\ell a, \frac{\ell}{\lambda}P(b)\right] + [\ell a, \ell b] \\ &= \frac{\ell^2}{\lambda^2}[P(a), P(b)] + \frac{\ell^2}{\lambda}[P(a), b] \\ &\quad + \frac{\ell^2}{\lambda}[a, P(b)] + \ell^2[a, b] = Q(c) \\ &= \frac{\ell}{\lambda}P(c) + \ell c = \frac{\ell^2}{\lambda^2}[P(a), P(b)] + \ell c. \end{aligned} \tag{25}$$

so we have

$$\ell c = \frac{\ell^2}{\lambda}([P(a), b] + [a, P(b)] + \lambda[a, b]). \tag{26}$$

Since ℓ has not trivial annihilator in A, we have

$$c = \frac{\ell}{\lambda}([a, P(b)] + [P(a), b] + \lambda[a, b]). \tag{27}$$

So

$$\begin{aligned} &\frac{\ell}{\lambda}[P(a), P(b)] = P(c) \\ &\quad = P\left(\frac{\ell}{\lambda}([a, P(b)] + [P(a), b] + \lambda[a, b])\right). \end{aligned} \tag{28}$$

This means that $[P(a), P(b)] = P([a, P(b)] + [P(a), b] + \lambda[a, b])$ as desired. □

In the following, we describe some properties of Rota-Baxter Leibniz algebras.

Proposition 11. *Let $(A, [-,-], P)$ be a Rota-Baxter Leibniz algebra of weight λ and P idempotent. Then, for any $a, b \in A$,*

$$\begin{aligned} (1+\lambda)P([a, P(b)]) &= 0, \\ (1+\lambda)P([P(a), b]) &= 0, \\ (1+\lambda)([P(a), P(b)] - \lambda P([a, b])) &= 0. \end{aligned} \tag{29}$$

Proof. This proof is straightforward by Proposition 8. □

Proposition 12. *Let $(A, [-,-], P)$ be a Rota-Baxter Leibniz algebra of weight λ. If P is quasi-idempotent of weight λ, that is, $P^2 = -\lambda P$. Then, for any $x, y \in \operatorname{Im} P$,*

$$[P(x), P(y)] = -\lambda P([x, y]). \tag{30}$$

In particular, if P is idempotent, then, P is a Leibniz algebra homomorphism from $\operatorname{Im} P$ to $\operatorname{Im} P$.

Proof. The proof is left for the readers. □

Proposition 13. *Let $(A, [-,-], P)$ be a Rota-Baxter Leibniz algebra of nonzero weight λ. Then the following conclusions are satisfied.*

(1) $-(\ell^2/\lambda)[a,b]_P = (\ell^2/\lambda^2)[P(a),P(b)] - [Q(a),Q(b)]$.

(2) $-(\ell^2/\lambda)[P(a),P(b)] = P((\ell^2/\lambda^2)[P(a),P(b)] - [Q(a),Q(b)])$.

(3) [29] For any integer $n \geq 2$, and $a_i \in A, i = 1,2,\ldots,n$,

$$\begin{aligned} &-\frac{\ell^n}{\lambda^{n-1}}\left[P(a_1),\left[P(a_2),\left[\cdots\left[P(a_{n-1}),P(a_n)\right]\cdots\right]\right]\right] \\ &= P\left(\frac{\ell^n}{\lambda^n}\left[P(a_1),\right.\right. \\ &\left[P(a_2),\left[\cdots\left[P(a_{n-1}),P(a_n)\right]\cdots\right]\right]-\left[Q(a_1),\right. \\ &\left.\left.\left[Q(a_2),\left[\cdots\left[Q(a_{n-1}),Q(a_n)\right]\cdots\right]\right]\right]\right). \end{aligned} \tag{31}$$

(4) $-(\ell^n/\lambda^{n-1})P(a)^n = P((\ell^n/\lambda^n)P(a)^n - Q(a)^n)$, where $P(a)^n = [P(a),[P(a),[\cdots[P(a),P(a)]\cdots]]]$ and so does $Q(a)^n$.

In particular, taking $\ell = \lambda$, we have

$$-\lambda P(a)^n = P\left(P(a)^n - Q(a)^n\right). \tag{32}$$

Here $Q = (\ell/\lambda)P + \ell id$ is defined in Proposition 6 and $[-,-]_P$ defined in Proposition 8.

Proof. (1) For any $a, b \in A$ and $\ell \in \mathscr{C}$, we easily prove

$$\frac{\ell^2}{\lambda^2}\left[P(a),P(b)\right] - \left[Q(a),Q(b)\right] = -\frac{\ell^2}{\lambda}\left[a,b\right]_P. \tag{33}$$

(2) According to (1) and the equality $P([a,b]_P) = [P(a),P(b)]$ for any $a, b \in A$, we can prove (2).

(3) We can prove this conclusion by using induction on n.

(4) This follows from (3) by taking $a_i = a, i = 1,2,\ldots,n$. □

3. Rota-Baxter Operators on Low-Dimensional Leibniz Algebras

In this section, we mainly focus on the Rota-Baxter operators of weight $\lambda = 0$ and $\lambda = -1$, and give all Rota-Baxter operators on solvable and nilpotent Leibniz algebras of dimension ≤ 3.

Suppose A is a Lebniz algebra with basis $\{a_1, a_2, \ldots, a_n\}$. Then, for any given Rota-Baxter operator P of weight λ on A, it can be presented by a matrix $R = (r_{ij})_{n\times n}$, that is, there are n^2 elements $r_{ij} \in \mathscr{C}$ $(i = 1,2,\ldots n; j = 1,2,\ldots n)$, such that

$$P\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \tag{34}$$

satisfies

$$\left[P(x),P(y)\right] = P\left(\left[x,P(y)\right] + \left[P(x),y\right] + \lambda\left[x,y\right]\right) \tag{35}$$

for any $x, y \in A$.

By [5], we know that any non-Lie Leibniz algebra in dimension ≤ 3 and nilpotent Leibniz algebra are solvable.

In what follows, we choose the low-dimensional nilpotent Leibniz algebras and the solvable ones to construct Rota-Baxter operators. And we denote the set of all the Rota-Baxter operators on A by RBO(A).

Firstly, we recall the conception of nilpotent Leibniz algebras and solvable ones.

Definition 14. A Leibniz algebra A is *solvable* if $A^{(\kappa)} = 0$ for some integers $\kappa \geq 0$.

Definition 15. A Leibniz algebra A is *nilpotent of class κ* if $A^{\kappa+1} = 0$ but $A^{\kappa} \neq 0$ for some integer $\kappa \geq 0$.

Lemma 16. *Let A be a non-Lie Leibniz algebra and $\dim(A) = 2$. Then, by Theorem 6.1 in [5], A is isomorphic to a cyclic Leibniz algebra generated by a single element a with $[a,a^2] = 0$ (hence A is nilpotent), or $[a,a^2] = a^2$ (hence A is solvable).*

In the following, for the proofs of our given results, the readers can see Appendix.

Theorem 17. *The Rota-Baxter operators on 2-dimensional non-Lie Leibniz algebra A are given in Table 1.*

In what follows, Lemmas 18 and 20 follow from Theorems 6.4 and 6.5 in [5], respectively.

Lemma 18. *Let A be a non-Lie nilpotent Leibniz algebra and $\dim(A) = 3$. Then, A is isomorphic to a Leibniz algebra spanned by $\{a,b,c\}$ with the nonzero product given by one of the following:*

(1) $[a,a] = b, [a,b] = c$;

(2) $[a,a] = c$;

(3) $[a,b] = c, [b,a] = c$;

(4) $[a,b] = c, [b,a] = -c, [b,b] = c$;

(5) $[a,b] = c, [b,a] = \alpha c, \alpha \in \mathscr{C}\backslash\{1,-1\}$.

Theorem 19(A). *The Rota-Baxter operators of weight 0 on 3-dimensional non-Lie nilpotent Leibniz algebra A are given in Table 2.*

Theorem 19(B). *The Rota-Baxter operators of weight -1 on 3-dimensional non-Lie nilpotent Leibniz algebra A are given in Table 3.*

Lemma 20. *Let A be a non-Lie nonnilpotent solvable Leibniz algebra and $\dim(A) = 3$. Then, A is isomorphic to a Leibniz algebra spanned by $\{a,b,c\}$ with the nonzero product given by one of the following:*

Table 1

Non-Lie Leibniz algebra A	RBO(A) of weight $\lambda = 0$	RBO(A) of weight $\lambda = -1$
(1) $[a, a^2] = 0$	$\begin{pmatrix} 0 & r_{12} \\ 0 & r_{22} \end{pmatrix}, \begin{pmatrix} 2r_{22} & r_{12} \\ 0 & r_{22} \end{pmatrix}.$	$\begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix} \left(r_{11}^2 = (2r_{11} - 1) r_{22}\right).$
(2) $[a, a^2] = a^2$	$\begin{pmatrix} 0 & r_{12} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} r_{11} & -r_{11} \\ 0 & 0 \end{pmatrix}.$	$\begin{pmatrix} 0 & r_{12} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} r_{11} & -r_{11} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & r_{12} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} r_{11} & 1 - r_{11} \\ 0 & 1 \end{pmatrix}.$

Table 2

Non-Lie nilpotent Leibniz algebra A	RBO(A) of weight $\lambda = 0$
(1) $[a, a] = b; [a, b] = c$	$\begin{pmatrix} 0 & 0 & r_{13} \\ 0 & 0 & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}, \begin{pmatrix} 0 & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2r_{22} & 3r_{23} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & \frac{2}{3} r_{22} \end{pmatrix}.$
(2) $[a, a] = c$	$\begin{pmatrix} 0 & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & r_{32} & r_{33} \end{pmatrix}, \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & \frac{1}{2} r_{11} \end{pmatrix}.$
(3) $[a, b] = c; [b, a] = c$	$\begin{pmatrix} r_{11} & 0 & r_{13} \\ r_{21} & 0 & r_{23} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} r_{11} & \frac{r_{11}^2}{r_{21}} & r_{13} \\ r_{21} & r_{11} & r_{23} \\ 0 & 0 & r_{11} \end{pmatrix} \left(r_{21} \neq 0\right),$ $\begin{pmatrix} r_{11} & 0 & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix} \left((r_{11} + r_{22}) r_{33} = r_{11} r_{22}\right).$
(4) $[a, b] = c; [b, a] = -c; [b, b] = c$	$\begin{pmatrix} r_{11} & 0 & r_{13} \\ r_{21} & 0 & r_{23} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & r_{13} \\ r_{21} & 0 & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}, \begin{pmatrix} 2r_{33} & 0 & r_{13} \\ r_{21} & 2r_{33} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}.$
(5) $[a, b] = c; [b, a] = \alpha c \, (\alpha \in \mathscr{C} \backslash \{1, -1\})$	$\begin{pmatrix} 0 & 0 & r_{13} \\ r_{21} & 0 & r_{23} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} r_{11} & 0 & r_{13} \\ r_{21} & 0 & r_{23} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & 0 \end{pmatrix},$ $\begin{pmatrix} r_{11} & 0 & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix} \left(r_{22} r_{33} + r_{11} r_{33} - r_{11} r_{22} = 0\right).$

(1) $[c, a] = a;$

(2) $[c, a] = \alpha a \, (\alpha \in \mathscr{C} \backslash \{0\}), [c, b] = b, [b, c] = -b;$

(3) $[c, b] = b, [b, c] = -b, [c, c] = a;$

(4) $[c, a] = 2a, [b, b] = a, [c, b] = b, [b, c] = -b, [c, c] = a;$

(5) $[c, a] = a + b, [c, b] = b;$

(6) $[c, a] = b, [c, b] = b, [c, c] = a;$

(7) $[c, b] = b, [c, a] = \alpha a \, (\alpha \in \mathscr{C} \backslash \{0\}).$

Theorem 21(A). *The Rota-Baxter operators of weight* 0 *on 3-dimensional non-Lie and nonnilpotent solvable Leibniz algebra A are given in in Table 4.*

Theorem 21(B). *The Rota-Baxter operators of weight* -1 *on 3-dimensional non-Lie and nonnilpotent solvable Leibniz algebra A are given in Table 5.*

Appendix

In this section, we mainly give the proof of some results in Section 3.

Table 3

Non-Lie nilpotent Leibniz algebra A	RBO(A) of weight $\lambda = -1$
(1) $[a,a] = b; [a,b] = c$	$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & \frac{r_{11}^2}{2r_{11}-1} & \frac{r_{12}(r_{11}-r_{33})}{2r_{11}-1} \\ 0 & 0 & r_{33} \end{pmatrix} \begin{pmatrix} r_{11} \neq \frac{1}{2} \\ (r_{11}^2 + (r_{11}-1)(2r_{11}-1))\, r_{33} = r_{11}^3 \end{pmatrix}.$
(2) $[a,a] = c$	$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & \frac{r_{11}^2}{2r_{11}-1} \end{pmatrix} \left(r_{11} \neq \frac{1}{2}\right).$
(3) $[a,b] = c; [b,a] = c$	$\begin{pmatrix} r_{11} & 0 & r_{13} \\ r_{21} & 0 & r_{23} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} r_{11} & 0 & r_{13} \\ r_{21} & 1 & r_{23} \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & 1 \end{pmatrix},$ $\begin{pmatrix} r_{11} & 0 & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix} \left((r_{11}+r_{22}-1)\, r_{33} = r_{11} r_{22}\right),$ $\begin{pmatrix} r_{11} & \frac{r_{11}^2 - r_{11}}{r_{21}} & r_{13} \\ r_{21} & r_{11} & r_{23} \\ 0 & 0 & r_{11} \end{pmatrix} \left(r_{21} \neq 0\right).$
(4) $[a,b] = c; [b,a] = -c; [b,b] = c$	$\begin{pmatrix} r_{11} & 0 & r_{13} \\ r_{21} & r_{22} & r_{23} \\ 0 & 0 & \frac{r_{22}^2}{2r_{22}-1} \end{pmatrix} \begin{pmatrix} r_{22} \neq \frac{1}{2} \\ (r_{11}+r_{22}-1)\, r_{22}^2 = r_{11} r_{22} (2r_{22}-1) \end{pmatrix}.$
(5) $[a,b] = c; [b,a] = \alpha c\ (\alpha \in \mathscr{C} \backslash \{1,-1\})$	$\begin{pmatrix} 0 & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} r_{11} & 0 & r_{13} \\ r_{21} & 0 & r_{23} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} r_{11} & 0 & r_{13} \\ r_{21} & 1 & r_{23} \\ 0 & 0 & 1 \end{pmatrix},$ $\begin{pmatrix} r_{11} & 0 & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix} \left((r_{11}+r_{22}-1)\, r_{33} = r_{11} r_{22}\right).$

Proof of Theorem 17. We firstly construct Rota-Baxter operators of weight $\lambda = 0$ in the case (1). Assume that

$$P\begin{pmatrix} a \\ a^2 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \begin{pmatrix} a \\ a^2 \end{pmatrix}, \tag{A.1}$$

with $P \in RBO(A)$. Then, we have

$$\begin{aligned} [P(a), P(a)] &= P([a, P(a)] + [P(a), a]), \\ \left[P(a), P\left(a^2\right)\right] &= P\left(\left[a, P\left(a^2\right)\right] + \left[P(a), a^2\right]\right), \\ \left[P\left(a^2\right), P(a)\right] &= P\left(\left[a^2, P(a)\right] + \left[P\left(a^2\right), a\right]\right), \\ \left[P\left(a^2\right), P\left(a^2\right)\right] &= P\left(\left[a^2, P\left(a^2\right)\right] + \left[P\left(a^2\right), a^2\right]\right), \end{aligned} \tag{A.2}$$

that is,

$$\begin{aligned} &\left[r_{11}a + r_{12}a^2, r_{11}a + r_{12}a^2\right] \\ &\quad = P\left(\left[a, r_{11}a + r_{12}a^2\right] + \left[r_{11}a + r_{12}a^2, a\right]\right), \\ &\left[r_{11}a + r_{12}a^2, r_{21}a + r_{22}a^2\right] \\ &\quad = P\left(\left[a, r_{21}a + r_{22}a^2\right] + \left[r_{11}a + r_{12}a^2, a^2\right]\right), \\ &\left[r_{21}a + r_{22}a^2, r_{11}a + r_{12}a^2\right] \\ &\quad = P\left(\left[a^2, r_{11}a + r_{12}a^2\right] + \left[r_{21}a + r_{22}a^2, a\right]\right), \\ &\left[r_{21}a + r_{22}a^2, r_{21}a + r_{22}a^2\right] \\ &\quad = P\left(\left[a^2, r_{21}a + r_{22}a^2\right] + \left[r_{21}a + r_{22}a^2, a^2\right]\right). \end{aligned} \tag{A.3}$$

Table 4

Non-Lie solvable Leibniz algebra A	RBO(A) of weight $\lambda = 0$
(1) $[c,a] = a$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & r_{22} & r_{23} \\ 0 & r_{32} & r_{33} \end{pmatrix}, \begin{pmatrix} 0 & r_{12} & 0 \\ 0 & r_{22} & 0 \\ 0 & r_{32} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}.$
(2) $[c,a] = \alpha a\,(\alpha \in \mathscr{C}\backslash\{0\})$; $[c,b] = b$; $[b,c] = -b$	$\begin{pmatrix} 0 & 0 & 0 \\ r_{21} & 0 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}, \begin{pmatrix} 0 & r_{12} & 0 \\ 0 & 0 & 0 \\ 0 & r_{32} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & r_{22} & r_{23} \\ 0 & r_{32} & -r_{22} \end{pmatrix} \left(r_{22}^2 + r_{23}r_{32} = 0\right).$
(3) $[c,b] = b$; $[b,c] = -b$; $[c,c] = a$	$\begin{pmatrix} r_{11} & 0 & 0 \\ r_{21} & 0 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}, \begin{pmatrix} r_{11} & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 2r_{11} \end{pmatrix}, \begin{pmatrix} r_{11} & r_{12} & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}.$
(4) $[c,a] = 2a$; $[b,b] = a$; $[c,b] = b$; $[b,c] = -b$; $[c,c] = a$	$\begin{pmatrix} 0 & 0 & 0 \\ r_{21} & 0 & 0 \\ r_{31} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ r_{21} & 0 & -2r_{21} \\ 0 & 0 & 0 \end{pmatrix},$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \left(r_{32}^2 + 2r_{33}r_{31} + r_{33}^2 = 0\right),$ $\begin{pmatrix} 0 & 0 & 0 \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \begin{pmatrix} r_{22}^2 + 2r_{23}r_{21} + r_{23}^2 = 0 \\ r_{31} = \dfrac{r_{22}^3 + r_{22}r_{23}^2}{2r_{23}^2}, r_{32} = \dfrac{-r_{22}^2}{r_{23}} \\ r_{22} + r_{33} = 0, r_{23} \neq 0 \end{pmatrix}.$
(5) $[c,a] = a + b$; $[c,b] = b$	$\begin{pmatrix} 0 & r_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r_{33} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ r_{21} & -r_{21} & 0 \\ r_{31} & -r_{31} & 0 \end{pmatrix}, \begin{pmatrix} 0 & r_{12} & 0 \\ 0 & 0 & 0 \\ 0 & r_{32} & 0 \end{pmatrix},$ $\begin{pmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix} \begin{pmatrix} r_{12} = \dfrac{-r_{11}^2 - r_{11}r_{21}}{r_{21}} \\ r_{32} = \dfrac{r_{31}r_{22}}{r_{21}}, r_{21} \neq 0 \\ r_{22} = -r_{11} - r_{21} \end{pmatrix}.$
(6) $[c,a] = b$; $[c,b] = b$; $[c,c] = a$	$\begin{pmatrix} r_{11} & r_{12} & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}, \begin{pmatrix} r_{11} & -r_{11} & 0 \\ r_{21} & -r_{21} & 0 \\ r_{31} & -r_{31} & 0 \end{pmatrix}, \begin{pmatrix} r_{11} & -r_{11} & 0 \\ 0 & 0 & 0 \\ r_{31} & -2r_{11} - r_{31} & 2r_{11} \end{pmatrix}.$
(7) $[c,b] = b$; $[c,a] = \alpha a\,(\alpha \in \mathscr{C}\backslash\{0\})$	$\begin{pmatrix} 0 & 0 & 0 \\ r_{21} & 0 & 0 \\ r_{31} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & r_{12} & 0 \\ 0 & 0 & 0 \\ 0 & r_{32} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r_{33} \end{pmatrix},$ $\begin{pmatrix} r_{11} & r_{12} & 0 \\ r_{21} & -\alpha r_{11} & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix} \begin{pmatrix} r_{31} = \dfrac{r_{11}r_{32}}{r_{12}} \\ r_{21} = -\dfrac{\alpha r_{11}^2}{r_{12}}, r_{12} \neq 0 \end{pmatrix}.$

Table 5

Non-Lie solvable Leibniz algebra A	RBO(A) of weight $\lambda = -1$
(1) $[c, a] = a$	$\begin{pmatrix} 0 & 0 & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & r_{22} & 0 \\ 0 & r_{32} & r_{33} \end{pmatrix}, \begin{pmatrix} 0 & r_{12} & 0 \\ 0 & r_{22} & 0 \\ 0 & r_{32} & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & r_{22} & r_{23} \\ 0 & r_{32} & r_{33} \end{pmatrix},$ $\begin{pmatrix} 1 & r_{12} & 0 \\ 0 & r_{22} & 0 \\ 0 & r_{32} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & r_{22} & r_{23} \\ 0 & r_{32} & r_{33} \end{pmatrix}.$
(2) $[c, a] = \alpha a\,(\alpha \in \mathscr{C}\backslash\{0\})$; $[c, b] = b$; $[b, c] = -b$	$\begin{pmatrix} 0 & 0 & 0 \\ r_{21} & 1 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ r_{21} & 0 & 0 \\ r_{31} & r_{32} & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & r_{32} & r_{33} \end{pmatrix}, \begin{pmatrix} 1 & r_{12} & 0 \\ 0 & 0 & 0 \\ 0 & r_{32} & 0 \end{pmatrix},$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & r_{32} & r_{33} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & r_{32} & r_{33} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 1 \end{pmatrix},$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r_{31} & r_{32} & 1 \end{pmatrix}, \begin{pmatrix} 0 & r_{12} & 0 \\ 0 & 1 & 0 \\ 0 & r_{32} & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & r_{32} & r_{33} \end{pmatrix},$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & r_{22} & r_{23} \\ 0 & \frac{(1-r_{22})\,r_{22}}{r_{23}} & 1-r_{22} \end{pmatrix} \left(r_{23} \neq 0\right),$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & r_{22} & r_{23} \\ 0 & \frac{(1-r_{22})r_{22}}{r_{23}} & 1-r_{22} \end{pmatrix} \left(r_{23} \neq 0\right).$
(3) $[c, b] = b$; $[b, c] = -b$; $[c, c] = a$	$\begin{pmatrix} 1 & 0 & 0 \\ r_{21} & 0 & 0 \\ r_{31} & r_{32} & 1 \end{pmatrix}, \begin{pmatrix} r_{11} & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \left(r_{11}\,(2r_{33}-1) = r_{33}^2\right),$ $\begin{pmatrix} 0 & 0 & 0 \\ r_{21} & 1 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}, \begin{pmatrix} r_{11} & 0 & 0 \\ 0 & 1 & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \left(r_{11}\,(2r_{33}-1) = r_{33}^2\right).$

Table 5: Continued.

Non-Lie solvable Leibniz algebra A	RBO(A) of weight $\lambda = -1$
(4) $[c,a] = 2a$; $[b,b] = a$; $[c,b] = b$; $[b,c] = -b$; $[c,c] = a$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \left(r_{32}^2 + 2r_{33}r_{31} + r_{33}^2 = 0\right),$ $\begin{pmatrix} 0 & 0 & 0 \\ -\dfrac{r_{22}^2 + r_{23}^2}{2r_{23}} & r_{22} & r_{23} \\ -\dfrac{r_{22}^2 r_{33} + r_{23}^2 r_{33}}{2r_{23}^2} & \dfrac{r_{33}r_{22}}{r_{23}} & r_{33} \end{pmatrix} \begin{pmatrix} r_{22} + r_{33} = 1 \\ r_{23} \neq 0 \end{pmatrix}.$
(5) $[c,a] = a + b$; $[c,b] = b$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r_{33} \end{pmatrix}, \begin{pmatrix} 0 & r_{12} & 0 \\ 0 & 1 & 0 \\ 0 & r_{32} & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$ $\begin{pmatrix} 1 & r_{12} & 0 \\ 0 & 0 & 0 \\ 0 & r_{32} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r_{33} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r_{31} & r_{32} & 1 \end{pmatrix},$ $\begin{pmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & r_{33} \end{pmatrix} \begin{pmatrix} r_{12} = \dfrac{r_{33}r_{21} + r_{11}r_{22}}{r_{21}}, r_{21} \neq 0 \\ r_{11}(r_{11} - 1) + (r_{11} + r_{12} + r_{33} - 1) r_{21} = 0 \\ r_{12}(r_{11} - 1) + (r_{11} + r_{12} + r_{33} - 1) r_{22} = r_{33}r_{11} \\ r_{11} + r_{21} + r_{22} - 1 = 0 \end{pmatrix}.$
(6) $[c,a] = b$; $[c,b] = b$; $[c,c] = a$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}, \begin{pmatrix} r_{11} & 1 - r_{11} & 0 \\ r_{11} & 1 - r_{11} & 0 \\ r_{31} & 1 - r_{31} & 0 \end{pmatrix}, \begin{pmatrix} r_{11} & -r_{11} & 0 \\ r_{11} - 1 & 1 - r_{11} & 0 \\ r_{31} & -1 - r_{31} & 1 \end{pmatrix},$ $\begin{pmatrix} r_{11} & 1 - r_{11} & 0 \\ 0 & 1 & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \begin{pmatrix} r_{11} = \dfrac{r_{33}^2}{2r_{33} - 1}, r_{33} \neq \dfrac{1}{2} \\ (2r_{33} - 1)(1 - r_{11}) = (r_{33} - 1)(r_{31} + r_{32}) \end{pmatrix},$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r_{31} & r_{32} & 1 \end{pmatrix}.$

Table 5: Continued.

Non-Lie solvable Leibniz algebra A	RBO(A) of weight $\lambda = -1$
(7) $[c, b] = b$; $[c, a] = \alpha a$ ($\alpha \in \mathscr{C}\backslash\{0\}$)	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r_{33} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ r_{31} & 0 & 0 \end{pmatrix},$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r_{33} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & r_{32} & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & r_{32} & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & r_{32} & 1 \end{pmatrix},$ $\begin{pmatrix} 0 & 0 & 0 \\ r_{21} & 1 & 0 \\ r_{31} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ r_{21} & 0 & 0 \\ 0 & 0 & -\frac{1}{\alpha - 1} \end{pmatrix} (\alpha \neq 1), \begin{pmatrix} 0 & 0 & 0 \\ r_{21} & 1 & 0 \\ 0 & 0 & r_{33} \end{pmatrix} (\alpha = 1),$ $\begin{pmatrix} 1 & 0 & 0 \\ r_{21} & 0 & 0 \\ r_{31} & -\frac{r_{31}}{r_{21}} & 0 \end{pmatrix} \begin{pmatrix} \alpha = 1 \\ r_{21} \neq 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ r_{21} & 0 & 0 \\ r_{31} & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ r_{21} & 1 & 0 \\ 0 & 0 & \frac{\alpha}{\alpha - 1} \end{pmatrix} (\alpha \neq 1),$ $\begin{pmatrix} 0 & r_{12} & 0 \\ 0 & 1 & 0 \\ -\frac{r_{32}}{r_{12}} & r_{32} & 0 \end{pmatrix} \begin{pmatrix} \alpha = 1 \\ r_{12} \neq 0 \end{pmatrix}, \begin{pmatrix} 0 & r_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\alpha}{\alpha - 1} \end{pmatrix} (\alpha \neq 1), \begin{pmatrix} 0 & r_{12} & 0 \\ 0 & 1 & 0 \\ 0 & r_{32} & 1 \end{pmatrix},$ $\begin{pmatrix} 0 & r_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r_{33} \end{pmatrix} (\alpha = 1), \begin{pmatrix} 1 & r_{12} & 0 \\ 0 & 0 & 0 \\ 0 & r_{32} & 0 \end{pmatrix}, \begin{pmatrix} 1 & r_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{\alpha - 1} \end{pmatrix} (\alpha \neq 1),$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r_{33} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r_{33} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r_{31} & r_{32} & 1 \end{pmatrix}, \begin{pmatrix} 1 & r_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r_{33} \end{pmatrix} (\alpha = 1),$ $\begin{pmatrix} 1 & r_{12} & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{31} r_{12} & 1 \end{pmatrix} (\alpha = 1), \begin{pmatrix} 0 & r_{12} & 0 \\ 0 & 1 & 0 \\ 0 & r_{32} & 1 \end{pmatrix} (\alpha = 1),$ $\begin{pmatrix} 0 & 0 & 0 \\ r_{21} & 1 & 0 \\ 0 & 0 & r_{33} \end{pmatrix} (\alpha = 1), \begin{pmatrix} r_{11} & -\frac{r_{11}(r_{11} - 1)}{r_{21}} & 0 \\ r_{21} & 1 - r_{11} & 0 \\ r_{11} & -\frac{r_{11}(r_{11} - 1)}{r_{21}} & 1 \end{pmatrix} \begin{pmatrix} \alpha = 1 \\ r_{21} \neq 0 \end{pmatrix}.$

By computing, we easily get

$$\begin{aligned} r_{11}^2 a^2 &= P\left(2r_{11}a^2\right), \\ r_{11}r_{21}a^2 &= P\left(r_{21}a^2\right), \\ r_{21}^2 a^2 &= P(0), \end{aligned} \tag{A.4}$$

that is, the following equations hold:

$$\begin{aligned} r_{11}^2 a^2 &= 2r_{11}r_{21}a + 2r_{11}r_{22}a^2, \\ r_{11}r_{21}a^2 &= r_{21}^2 a + r_{21}r_{22}a^2, \\ r_{21}^2 a^2 &= 0. \end{aligned} \tag{A.5}$$

Again by computing, we can obtain the following solutions:

$$\begin{aligned} r_{21} &= 0, \\ r_{11} &= 0, \\ r_{21} &= 0, \\ r_{11} &= 2r_{22}. \end{aligned} \tag{A.6}$$

In a similar way, we can show the other cases. □

Proof of Theorem 19(A). In the following, we firstly prove the case (1). Suppose that P is Rota-Baxter operator of weight $\lambda = 0$, with the representation

$$P\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}. \tag{A.7}$$

Applying the above equalities to Rota-Baxter identity, we obtain the following:

$$\begin{aligned} [P(a), P(a)] &= P([a, P(a)] + [P(a), a]), \\ [P(a), P(b)] &= P([a, P(b)] + [P(a), b]), \\ [P(a), P(c)] &= P([a, P(c)] + [P(a), c]), \\ [P(b), P(a)] &= P([b, P(a)] + [P(b), a]), \\ [P(b), P(b)] &= P([b, P(b)] + [P(b), b]), \\ [P(b), P(c)] &= P([b, P(c)] + [P(b), c]), \\ [P(c), P(a)] &= P([c, P(a)] + [P(c), a]), \\ [P(c), P(b)] &= P([c, P(b)] + [P(c), b]), \\ [P(c), P(c)] &= P([c, P(c)] + [P(c), c]). \end{aligned} \tag{A.8}$$

That is, we have

$$\begin{aligned} &[r_{11}a + r_{12}b + r_{13}c, r_{11}a + r_{12}b + r_{13}c] \\ &\quad = P([a, r_{11}a + r_{12}b + r_{13}c] \\ &\quad + [r_{11}a + r_{12}b + r_{13}c, a]), \\ &[r_{11}a + r_{12}b + r_{13}c, r_{21}a + r_{22}b + r_{23}c] \\ &\quad = P([a, r_{21}a + r_{22}b + r_{23}c] \\ &\quad + [r_{11}a + r_{12}b + r_{13}c, b]), \\ &[r_{11}a + r_{12}b + r_{13}c, r_{31}a + r_{32}b + r_{33}c] \\ &\quad = P([a, r_{31}a + r_{32}b + r_{33}c] \\ &\quad + [r_{11}a + r_{12}b + r_{13}c, c]), \\ &[r_{21}a + r_{22}b + r_{23}c, r_{11}a + r_{12}b + r_{13}c] \\ &\quad = P([b, r_{11}a + r_{12}b + r_{13}c] \\ &\quad + [r_{21}a + r_{22}b + r_{23}c, a]), \\ &[r_{21}a + r_{22}b + r_{23}c, r_{21}a + r_{22}b + r_{23}c] \\ &\quad = P([b, r_{21}a + r_{22}b + r_{23}c] \\ &\quad + [r_{21}a + r_{22}b + r_{23}c, b]), \\ &[r_{21}a + r_{22}b + r_{23}c, r_{31}a + r_{32}b + r_{33}c] \\ &\quad = P([b, r_{31}a + r_{32}b + r_{33}c] \\ &\quad + [r_{21}a + r_{22}b + r_{23}c, c]), \\ &[r_{31}a + r_{32}b + r_{33}c, r_{11}a + r_{12}b + r_{13}c] \\ &\quad = P([c, r_{11}a + r_{12}b + r_{13}c] \\ &\quad + [r_{31}a + r_{32}b + r_{33}c, a]), \\ &[r_{31}a + r_{32}b + r_{33}c, r_{21}a + r_{22}b + r_{23}c] \\ &\quad = P([c, r_{21}a + r_{22}b + r_{23}c] \\ &\quad + [r_{31}a + r_{32}b + r_{33}c, b]), \\ &[r_{31}a + r_{32}b + r_{33}c, r_{31}a + r_{32}b + r_{33}c] \\ &\quad = P([c, r_{31}a + r_{32}b + r_{33}c] \\ &\quad + [r_{31}a + r_{32}b + r_{33}c, c]). \end{aligned} \tag{A.9}$$

Since $[a, a] = b$, $[a, b] = c$, we have

$$\begin{aligned} r_{11}^2 b + r_{11}r_{12}c &= (2r_{11}r_{21} + r_{12}r_{31})a \\ &\quad + (2r_{11}r_{22} + r_{12}r_{32})b \\ &\quad + (2r_{11}r_{23} + r_{12}r_{33})c, \\ r_{11}r_{21}b + r_{11}r_{22}c &= \left(r_{21}^2 + r_{22}r_{31} + r_{11}r_{31}\right)a \end{aligned}$$

$$
\begin{aligned}
&+ (r_{21}r_{22} + r_{22}r_{32} + r_{11}r_{32})\, b \\
&+ (r_{21}r_{23} + r_{22}r_{33} + r_{11}r_{33})\, c, \\
r_{11}r_{31}b + r_{11}r_{32}c = {} & (r_{31}r_{21} + r_{32}r_{31})\, a \\
&+ \left(r_{31}r_{22} + r_{32}^2\right) b \\
&+ (r_{31}r_{23} + r_{32}r_{33})\, c, \\
r_{21}r_{11}b + r_{21}r_{12}c = {} & r_{21}^2 a + r_{21}r_{22}b + r_{21}r_{23}c, \\
r_{21}^2 b + r_{21}r_{22}c = {} & r_{21}r_{31}a + r_{21}r_{32}b + r_{21}r_{33}c, \\
r_{21}r_{31}b + r_{21}r_{32}c = {} & 0 \\
r_{31}r_{11}b + r_{31}r_{12}c = {} & r_{31}r_{21}a + r_{31}r_{22}b + r_{31}r_{23}c, \\
r_{31}r_{21}b + r_{31}r_{22}c = {} & r_{31}^2 a + r_{31}r_{32}b + r_{31}r_{33}c, \\
r_{31}^2 b + r_{31}r_{32}c = {} & 0.
\end{aligned}
\tag{A.10}
$$

Transposing and amalgamating, we get that

$$
\begin{aligned}
2r_{11}r_{21} + r_{12}r_{31} &= 0; \\
2r_{11}r_{22} + r_{12}r_{32} - r_{11}^2 &= 0; \\
2r_{11}r_{23} + r_{12}r_{33} - r_{11}r_{12} &= 0, \\
r_{21}^2 + r_{22}r_{31} + r_{11}r_{31} &= 0; \\
r_{21}r_{22} + r_{22}r_{32} + r_{11}r_{32} - r_{11}r_{21} &= 0; \\
r_{21}r_{23} + r_{22}r_{33} + r_{11}r_{33} - r_{11}r_{22} &= 0, \\
r_{31}r_{21} + r_{32}r_{31} &= 0; \\
r_{31}r_{22} + r_{32}^2 - r_{11}r_{31} &= 0; \\
r_{31}r_{23} + r_{32}r_{33} - r_{11}r_{32} &= 0, \\
r_{21}^2 &= 0; \\
r_{21}r_{22} - r_{21}r_{11} &= 0; \\
r_{21}r_{23} - r_{21}r_{12} &= 0, \\
r_{21}r_{31} &= 0; \\
r_{21}r_{32} - r_{21}^2 &= 0; \\
r_{21}r_{33} - r_{21}r_{22} &= 0, \\
r_{21}r_{31} &= 0; \\
r_{21}r_{32} &= 0, \\
r_{31}r_{21} &= 0; \\
r_{31}r_{22} - r_{31}r_{11} &= 0; \\
r_{31}r_{23} - r_{31}r_{12} &= 0, \\
r_{31}^2 &= 0; \\
r_{31}r_{32} - r_{31}r_{21} &= 0; \\
r_{31}r_{33} - r_{31}r_{22} &= 0, \\
r_{31}^2 &= 0; \\
r_{31}r_{32} &= 0.
\end{aligned}
\tag{A.11}
$$

It is easy to see that $r_{21} = r_{31} = 0$ by the above equations. So

$$
\begin{aligned}
2r_{11}r_{22} + r_{12}r_{32} - r_{11}^2 &= 0, \\
2r_{11}r_{23} + r_{12}r_{33} - r_{11}r_{12} &= 0, \\
r_{22}r_{32} + r_{11}r_{32} &= 0, \\
r_{22}r_{33} + r_{11}r_{33} - r_{11}r_{22} &= 0, \\
r_{32}^2 &= 0, \\
r_{32}r_{33} - r_{11}r_{32} &= 0,
\end{aligned}
\tag{A.12}
$$

and $r_{32} = 0$. Hence we get

$$
\begin{aligned}
2r_{11}r_{22} - r_{11}^2 &= 0, \\
2r_{11}r_{23} + r_{12}r_{33} - r_{11}r_{12} &= 0, \\
r_{22}r_{33} + r_{11}r_{33} - r_{11}r_{22} &= 0.
\end{aligned}
\tag{A.13}
$$

In a light of the above first equality: $2r_{11}r_{22} - r_{11}^2 = 0$, we consider two cases: $r_{11} = 0$ or $r_{11} \neq 0$.

Suppose that $r_{11} = 0$. Then, according to the above equalities, it is easy to see that $r_{12}r_{33} = 0$ and $r_{22}r_{33} = 0$. So $r_{12} = r_{22} = 0$ or $r_{33} = 0$.

Suppose that $r_{11} \neq 0$. Then, by the above equalities, we have

$$
\begin{aligned}
r_{11} &= 2r_{22}, \\
2r_{11}r_{23} + r_{12}r_{33} - r_{11}r_{12} &= 0, \\
3r_{22}r_{33} - 2r_{22}^2 &= 0.
\end{aligned}
\tag{A.14}
$$

Since $r_{11} = 2r_{22} \neq 0$, we know that $r_{33} = (2/3)r_{22}$. So $4r_{22}r_{23} + (2/3)r_{12}r_{22} - 2r_{22}r_{12} = 0$. It means that $r_{12} = 3r_{23}$.

Hence we obtain all solutions as follows:

$$\begin{aligned}
&r_{11} = r_{12} = 0,\\
&r_{21} = r_{22} = 0,\\
&r_{31} = r_{32} = 0,\\
&r_{11} = 0,\\
&r_{21} = 0,\\
&r_{31} = r_{32} = r_{33} = 0,\\
&r_{11} = 2r_{22},\\
&r_{12} = 3r_{23},\\
&r_{21} = 0,\\
&r_{31} = r_{32} = 0,\\
&r_{33} = \frac{2}{3}r_{22}.
\end{aligned} \tag{A.15}$$

The other cases can be similarly proved. □

Similar to Theorem 19(A), we can prove Theorems 19(B), 21(A), and 21(B).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (11571173).

References

[1] A. Bloh, "On a generalization of the concept of Lie algebra," *Doklady Akademii Nauk SSSR*, vol. 165, pp. 471–473, 1965.

[2] D. W. Barnes, "Some theorems on Leibniz algebras," *Communications in Algebra*, vol. 39, no. 7, pp. 2463–2472, 2011.

[3] G. Baxter, "An analytic problem whose solution follows from a simple algebraic identity," *Pacific Journal of Mathematics*, vol. 10, pp. 731–742, 1960.

[4] E. M. Canete and A. K. Khudoyberdiyev, "The classification of 4-dimensional Leibniz algebras," *Linear Algebra and its Applications*, vol. 439, no. 1, pp. 273–288, 2013.

[5] I. Demir, K. C. Misra, and E. Stitzinger, "On some structures of Leibniz algebras," in *Recent advances in representation theory, quantum groups, algebraic geometry, and related topics*, vol. 623, pp. 41–54, Contemporary Mathematics, 2014.

[6] B. Dherin and F. Wagemann, "Deformation quantization of Leibniz algebras," *Advances in Mathematics*, vol. 270, pp. 21–48, 2015.

[7] V. Dobrev, *Lie theory and its applications in physics*, vol. 36 of *IX International workshop*, Springer, Tokyo, 2013.

[8] S. Duplij and J. Wess, *Noncommutative Structures in Mathematics and Physics*, vol. 22 of *NATO Science Series II*, Springer Netherlands, Dordrecht, 2001.

[9] M. K. Kinyon, "Leibniz algebras, Lie racks, and digroups," *Journal of Lie Theory*, vol. 17, no. 1, pp. 99–114, 2007.

[10] J. F. Kingman, "Spitzer's identity and its use in probability theory," *Journal Of The London Mathematical Society-Second Series*, vol. 37, pp. 309–316, 1962.

[11] G. E. Andrews, L. Guo, W. Keigher, and K. Ono, "Baxter algebras and Hopf algebras," *Transactions of the American Mathematical Society*, vol. 355, no. 11, pp. 4639–4656, 2003.

[12] E.-F. Kurusch, J. M. Gracia-Bondi'a, and F. Patras, "Rota-Baxter algebras and new combinatorial identities," *Letters in Mathematical Physics*, vol. 81, no. 1, pp. 61–75, 2007.

[13] G.-C. Rota, "Baxter algebras and combinatorial identities," *Bulletin (New Series) of the American Mathematical Society*, vol. 75, pp. 330–334, 1969.

[14] H. An and C. Bai, "From Rota-Baxter algebras to pre-Lie algebras," *Journal of Physics A: Mathematical and Theoretical*, vol. 41, no. 1, Article ID 015201, pp. 145–150, 2008.

[15] R. Bai, L. Guo, J. Li, and Y. Wu, "Rota-Baxter 3-Lie algebras," *Journal of Mathematical Physics*, vol. 54, no. 6, Article ID 063504, pp. 263–277, 2013.

[16] K. Ebrahimi-Fard and L. Guo, "Rota-Baxter algebras and dendriform algebras," *Journal of Pure and Applied Algebra*, vol. 212, no. 2, pp. 320–339, 2008.

[17] T. Zhang, X. Gao, and L. Guo, "Hopf algebras of rooted forests, cocyles, and free Rota-Baxter algebras," *Journal of Mathematical Physics*, vol. 57, no. 10, pp. 492–532, 2016.

[18] L. Guo and W. Keigher, "On free Baxter algebras: completions and the internal construction," *Advances in Mathematics*, vol. 151, no. 1, pp. 101–127, 2000.

[19] L. Guo and W. Keigher, "Baxter algebras and shuffle products," *Advances in Mathematics*, vol. 150, no. 1, pp. 117–149, 2000.

[20] L. Guo and W. Y. Poon, *An introduction to Rota-Baxter algebra*, vol. 4, Higher Education Press, 2012.

[21] L. Qiao, X. Gao, and L. Guo, "Rota-Baxter Modules Toward Derived Functors," *Algebras and Representation Theory*, vol. 178, no. 1, pp. 1–23, 2017.

[22] X. Li, D. Hou, and C. Bai, "Rota-Baxter operators on pre-Lie algebras," *Journal of Nonlinear Mathematical Physics*, vol. 14, no. 2, pp. 269–289, 2007.

[23] F. V. Atkinson, "Some aspects of Baxter's functional equation," *Journal of Mathematical Analysis and Applications*, vol. 7, pp. 1–30, 1963.

[24] H. P. McKean and E. Trubowitz, "Hill's operator and hyperelliptic function theory in the presence of infinitely many branch points," *Communications on Pure and Applied Mathematics*, vol. 29, no. 2, pp. 143–226, 1976.

[25] L. Positselski and A. Vishik, "Koszul duality and Galois cohomology," *Mathematical Research Letters*, vol. 2, no. 6, pp. 771–781, 1995.

[26] S. Montgomery, *Hopf algebras and their actions on rings*, vol. 82 of *CBMS Regional Conference Series in Mathematics*, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1993.

[27] G. Böhm, F. Nill, and K. Szlachányi, "Weak Hopf algebras (I): Integral theory and C^*-structure," *Journal of Algebra*, vol. 221, no. 2, pp. 385–438, 1999.

[28] L.-y. Zhang, "The structure theorem of weak comodule algebras," *Communications in Algebra*, vol. 38, no. 1, pp. 254–260, 2010.

[29] J.-L. Loday and T. Pirashvili, "Universal enveloping algebras of Leibniz algebras and (co)homology," *Mathematische Annalen*, vol. 296, no. 1, pp. 139–158, 1993.