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Research Article

Traveling Wave Solutions of Two Nonlinear Wave Equations by (G'/G)-Expansion Method

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We employ the (G'/G)-expansion method to seek exact traveling wave solutions of two nonlinear wave equations—Padé-II equation and Drinfel'd-Sokolov-Wilson (DSW) equation. As a result, hyperbolic function solution, trigonometric function solution, and rational solution with general parameters are obtained. The interesting thing is that the exact solitary wave solutions and new exact traveling wave solutions can be obtained when the special values of the parameters are taken. Comparing with other methods, the method used in this paper is very direct. The (G'/G)-expansion method presents wide applicability for handling nonlinear wave equations.

1. Introduction

With the development of computer algebra, searching exact solutions for nonlinear partial differential equations (PDEs) has become more and more attractive field for researchers. The reason is that the complicate and tedious algebraic calculation can be completed by computer symbolic system like Maple and Mathematica. As a result, a lot of numerical and exact solutions can be obtained for nonlinear PDEs, especially nonlinear wave equations in mathematical physics. These solutions will play an important role in soliton theory. In order to get exact solutions directly, many powerful methods have been introduced such as inverse scattering method [1], bilinear transformation [2], Bäklund and Darboux transformation [3–5], tanh-sech method [6, 7], extended tanh method [8], Exp-function method [9–11], the sine-cosine method [12–14], the Jacobi elliptic function method [15], F-expansion method [16, 17], auxiliary equation method [18, 19], bifurcation method [20–22], homotopy perturbation method [23], and homogeneous balance method [24, 25]. Recently, Wang et al. [26] introduced a new approach, namely, the (G'/G)-expansion method, for a reliable treatment of the nonlinear wave equations. The useful (G'/G)-expansion method is then widely used by many authors [27–30].

In this paper, our aim is to use the (G'/G)-expansion method to study two nonlinear wave equations, namely, Padé-II equation and Drinfel'd-Sokolov-Wilson (DSW) equation.

The Padé-II equation,

$$u_t + u_x + uu_x - \frac{9}{10}u_{xxx} - \frac{19}{10}u_{xxt} = 0, \tag{1}$$

is a new nonlinear wave equation modeling unidirectional propagation of long wave in dispersive media. It is originally derived by using a padé $(2, 2)$ approximation of the phase velocity that arises in linear water wave theory [31]. To our best knowledge, there are little works on exact solutions of this equation. Here we will use (G'/G)-expansion method to get its exact traveling wave solutions. After that the other nonlinear wave equation will be studied as

$$u_t + pvv_x = 0, \tag{2}$$

$$v_t + qv_{xxx} + ruv_x + su_x v = 0, \tag{3}$$

where $p,\ q,\ r,\ s$ are nonzero parameters. This equation was first proposed by Drinfel'd and Sokolov [32] and Wilson [33], short for DSW. Soliton structure of DSW equation was studied in [34]; the exact solutions and numerical solutions were studied by Adomian decomposition method [35], elliptic equation method [36], and other numerical methods [37, 38]. Here the (G'/G)-expansion method will be used to get new traveling wave solutions of DSW equation.

The rest of the paper is organized as follows. In Section 2, we will shortly present a methodology of the (G'/G)-expansion method. In Section 3, the Padé-II equation and the DSW equation will be studied by the proposed method. And finally, some conclusions will be given in Section 4.

2. Description of the (G'/G)-Expansion Method

The (G'/G)-expansion method is first proposed by Wang [26]. The main steps are as follows:

Suppose that a nonlinear equation is given by

$$P\left(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots\right) = 0, \tag{4}$$

where $u = u(x,t)$ is an unknown function and P is a polynomial in $u = u(x,t)$ and its partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of the (G'/G)-expansion method.

Step 1. The traveling wave variable $u(x,t) = u(\xi), \xi = x - ct$, where c is a constant, permits us to reduce (4) to an ODE for $u = u(\xi)$ in the form

$$P\left(u, -cu', u', c^2u'', -cu'', u'', \ldots\right) = 0. \tag{5}$$

Step 2. Suppose that the solution of (4) can be expressed by a polynomial in (G'/G) as follows:

$$u(\xi) = \alpha_m \left(\frac{G'}{G}\right)^m + \alpha_{m-1}\left(\frac{G'}{G}\right)^{m-1} + \cdots, \tag{6}$$

where $G = G(\xi)$ satisfies the second-order linear differential equation in the form

$$G'' + \lambda G' + \mu G = 0, \tag{7}$$

where $\alpha_m, \alpha_{m-1}, \ldots, \alpha_0$, λ and μ are constants to be determined later; $\alpha_m \neq 0$. The unwritten part in (6) is also a polynomial in (G'/G), but the degree of which is generally equal to or less than $m-1$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (5).

Step 3. Substituting (6) into (5) and using (7), collecting all terms with the same order of (G'/G) together, and then equating each coefficient of the resulting polynomial to zero yield a set of algebraic equations for $\alpha_m, \alpha_{m-1}, \ldots, \alpha_0$, c, λ and μ.

Step 4. Since the general solutions of (7) have been well known for us, then substituting $\alpha_m, \alpha_{m-1}, \ldots, \alpha_0$ and c and the general solutions of (7) into (6), we can get the traveling wave solutions of the nonlinear differential equation (4).

3. Application of (G'/G)-Expansion Method to Padé-II Equation and DSW Equation

In this section, we will use the (G'/G)-expansion method to Padé-II equation and DSW equation to construct exact traveling wave solutions. We first study the Padé-II equation, and then the DSW equation.

3.1. Exact Traveling Wave Solutions of Padé-II Equation. In order to get traveling wave solutions of (1), we need the traveling wave transformation $u(x,t) = u(\xi)$, $\xi = x - ct$ where c is wave speed. Substituting them into (1) and integrating once, we have

$$c_1 + (1-c)\,u + \frac{1}{2}u^2 + \left(\frac{19}{10}c - \frac{9}{10}\right)u'' = 0, \tag{8}$$

where c_1 is integral constant that is to be determined later.

Considering the homogeneous balance between u'' and u^2, we have

$$m + 2 = 2m \Longrightarrow m = 2. \tag{9}$$

We suppose that

$$u(\xi) = \alpha_2\left(\frac{G'}{G}\right)^2 + \alpha_1\left(\frac{G'}{G}\right) + \alpha_0, \quad \alpha_2 \neq 0, \tag{10}$$

where $G = G(\xi)$ satisfies the following equation

$$G'' + \lambda G' + \mu G = 0, \tag{11}$$

and α_0, α_1, α_2, λ and μ are constants to be determined later.

By using (10) and (11), it is derived that

$$\begin{aligned}
u'' &= 6\alpha_2\left(\frac{G'}{G}\right)^4 + \left(2\alpha_1 + 10\lambda\alpha_2\right)\left(\frac{G'}{G}\right)^3 \\
&\quad + \left(8\alpha_2\mu + 3\lambda\alpha_1 + 4\alpha_2\lambda^2\right)\left(\frac{G'}{G}\right)^2 \\
&\quad + \left(6\lambda\mu\alpha_2 + 2\alpha_1\mu + \alpha_1\lambda^2\right)\left(\frac{G'}{G}\right) + 2\alpha_2\mu^2 \\
&\quad + \alpha_1\lambda\mu, \\
u^2 &= \alpha_2^2\left(\frac{G'}{G}\right)^4 + 2\alpha_1\alpha_2\left(\frac{G'}{G}\right)^3 \\
&\quad + \left(2\alpha_0\alpha_2 + \alpha_1^2\right)\left(\frac{G'}{G}\right)^2 + 2\alpha_0\alpha_1\left(\frac{G'}{G}\right) \\
&\quad + \alpha_0^2.
\end{aligned} \tag{12}$$

By substituting (10)–(12) into (8) and collecting all terms with the same power of G'/G together, the left-hand sides of (8) are converted into the polynomials in G'/G. Equating the coefficients of the polynomials to zero yields a set of simultaneous algebraic equations for α_0, α_1, α_2, λ, c, c_1, c_2 and μ as follows:

$$\begin{aligned}
&\frac{19c-9}{10}\times\left(2\alpha_2\mu^2+\alpha_1\lambda\mu\right)+\frac{1}{2}\alpha_0^2+(1-c)\alpha_0\\
&\quad+c_1=0,\\
&\frac{19c-9}{10}\times\left(6\lambda\mu\alpha_2+2\alpha_1\mu+\alpha_1\lambda^2\right)+\alpha_1\alpha_2\\
&\quad+(1-c)\alpha_1=0,\\
&\frac{19c-9}{10}\times\left(8\alpha_2\mu+3\lambda\alpha_1+4\alpha_2\lambda^2\right)+\frac{1}{2}\left(2\alpha_0\alpha_2+\alpha_1^2\right)\\
&\quad+(1-c)\alpha_2=0,\\
&\frac{19c-9}{10}\times\left(2\alpha_1+10\lambda\alpha_2\right)+\alpha_1\alpha_2=0,\\
&\frac{19c-9}{10}\times 6\alpha_2+\frac{1}{2}\alpha_2^2=0.
\end{aligned}\tag{13}$$

Solving the algebraic equations above yields

$$\begin{aligned}
\alpha_2&=-\frac{6}{5}(-9+19c),\\
\alpha_1&=-\frac{6}{5}(-9\lambda+19c\lambda),\\
\alpha_0&=(c-1)+\frac{19c-9}{10}\lambda^2+\frac{72-152c}{10}\mu,\\
&\qquad c\neq\frac{9}{19},
\end{aligned}\tag{14}$$

where λ and μ are constants.

Substituting system (14) into (10), we have the formulae of the solutions of (1) as follows:

$$\begin{aligned}
u(x,t)&=u(\xi)\\
&=-\frac{6}{5}(-9+19c)\left(\frac{G'}{G}\right)^2\\
&\quad-\frac{6}{5}(-9\lambda+19c\lambda)\left(\frac{G'}{G}\right)+(c-1)\\
&\quad+\frac{19c-9}{10}\lambda^2+\frac{72-152c}{10}\mu.
\end{aligned}\tag{15}$$

Solving (11), we deduce after some reduction that

$$\frac{G'}{G}=\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(\frac{A_1\cosh\left((1/2)\sqrt{\lambda^2-4\mu}\xi\right)+A_2\sinh\left((1/2)\sqrt{\lambda^2-4\mu}\xi\right)}{A_1\sinh\left((1/2)\sqrt{\lambda^2-4\mu}\xi\right)+A_2\cosh\left((1/2)\sqrt{\lambda^2-4\mu}\xi\right)}\right)-\frac{\lambda}{2},\tag{16}$$

where A_1 and A_2 are arbitrary constants.

Substituting the solution of (16) into (15), the traveling wave solutions of equation (1) can be obtained as follows.

Case 1. When $\lambda^2-4\mu>0$, then we have the following exact traveling wave solution of (1):

$$\begin{aligned}
u(x,t)&=u(\xi)\\
&=-\frac{3}{10}(19c-9)\left(\lambda^2-4\mu\right)A^2+(c-1)\\
&\quad+\frac{2}{5}\lambda^2(19c-9)+\frac{72-152c}{10}\mu,
\end{aligned}\tag{17}$$

where $A=((A_1\cosh((1/2)\sqrt{\lambda^2-4\mu}\xi)+A_2\sinh((1/2)\sqrt{\lambda^2-4\mu}\xi))/(A_1\sinh((1/2)\sqrt{\lambda^2-4\mu}\xi)+A_2\cosh((1/2)\sqrt{\lambda^2-4\mu}\xi)))$, $\xi=x-ct$.

Case 2. When $\lambda^2-4\mu<0$, then we have the following exact traveling wave solution of (1):

$$\begin{aligned}
u(x,t)&=u(\xi)\\
&=\frac{3}{10}(19c-9)\left(\lambda^2-4\mu\right)B^2+(c-1)\\
&\quad+\frac{2}{5}\lambda^2(19c-9)+\frac{72-152c}{10}\mu,
\end{aligned}\tag{18}$$

where $B=((-A_1\sin((1/2)\sqrt{\lambda^2-4\mu}\xi)+A_2\cos((1/2)\sqrt{\lambda^2-4\mu}\xi))/(A_1\cos((1/2)\sqrt{\lambda^2-4\mu}\xi)+A_2\sin((1/2)\sqrt{\lambda^2-4\mu}\xi)))$, $\xi=x-ct$.

Case 3. When $\lambda^2-4\mu=0$, then we have the following exact rational solution of (1):

$$\begin{aligned}
u(x,t)&=u(\xi)\\
&=-\frac{6}{5}(19c-9)\left(\frac{A_2}{A_1+A_2\xi}\right)^2+(c-1)\\
&\quad+\frac{3}{5}\lambda(19c-9)+\frac{19c-9}{10}\lambda^2+\frac{72-152c}{10}\mu,
\end{aligned}\tag{19}$$

where $\xi=x-ct$, A_1, A_2, λ, μ are constants.

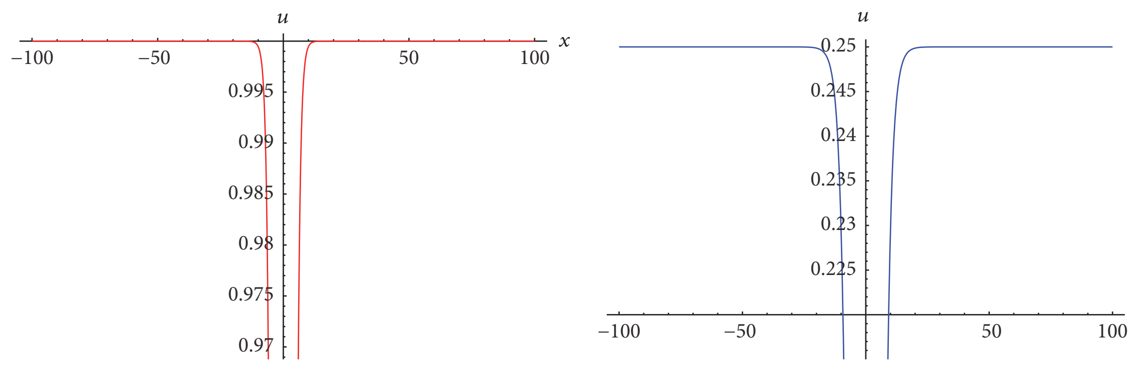

Figure 1: The two-dimensional profiles of the solitary wave solution (21) with $c = 1$, $t = 0$, $\lambda = 1$ ((a) red line) and $c = 1$, $t = 0$, $\lambda = 0.5$ ((b) blue line).

From the above three cases, it is not difficult to construct the solitary wave solutions and rational solutions from the general case traveling wave solutions. For example, let $A_1 = 0$, $\mu = 0$, $A_2 \neq 0$, $\lambda > 0$; then (17) yields the following solitary wave solution:

$$\begin{aligned} u(x,t) &= u(\xi) \\ &= -\frac{3}{10}(19c-9)\lambda^2\tanh^2\left(\frac{\lambda}{2}\xi\right) + (c-1) \\ &\quad + \frac{2}{5}\lambda^2(19c-9). \end{aligned} \tag{20}$$

On the other hand, if we let $A_2 = 0$, $\mu = 0$, $A_1 \neq 0, \lambda > 0$ then (17) yields the following solitary wave solution:

$$\begin{aligned} u(x,t) &= u(\xi) \\ &= -\frac{3}{10}(19c-9)\lambda^2\coth^2\left(\frac{\lambda}{2}\xi\right) + (c-1) \\ &\quad + \frac{2}{5}\lambda^2(19c-9), \end{aligned} \tag{21}$$

where $\xi = x - ct$.

We can see that different traveling wave solutions and rational solutions can be obtained by choosing different parameters. For example, if we choose $A_1 = 0$, $\mu = 0$, $A_2 \neq 0$, $\lambda > 0$, then substituting them into (18), (19), trigonometric function solutions and rational solutions can be obtained in the same manner.

Next we will analyze the nonlinear structure of the solitary wave solutions. We should point out that the solitary wave solution (21) possesses a singular point, while solution (18) has an infinite number of singular points. In the meaning of physics, it shows that there exists blowup phenomenon of the solutions (21) and (18). We show the blowup of solution (21) by the two-dimensional profiles in Figure 1.

3.2. Exact Traveling Wave Solutions of DSW Equation. Now we turn to study the DSW equations (2) and (3). Just as shown above, we have the following traveling wave transformation:

$$\begin{aligned} u(x,t) &= u(\xi), \\ v(x,t) &= v(\xi), \\ \xi &= x - ct. \end{aligned} \tag{22}$$

Substituting (22) into (2) and (3), respectively, we have

$$-cu' + pvv' = 0, \tag{23}$$

$$-cv' + qv''' + ruv' + su'v = 0. \tag{24}$$

Integrating (23) once and substituting it into (24) after integration, we have

$$\frac{rp+2sp}{6c}v^3 + qv'' + \left(\frac{rc_1}{c} - c\right)v + c_2 = 0, \tag{25}$$

where c_1, c_2 are integral constants.

Considering the homogeneous balance between v'' and v^3, we have

$$m + 2 = 3m \Longrightarrow m = 1. \tag{26}$$

We suppose that

$$v(\xi) = \alpha_1\left(\frac{G'}{G}\right) + \alpha_0, \quad \alpha_1 \neq 0, \tag{27}$$

where $G = G(\xi)$ satisfies (11) and α_0, α_1 are constants.

By using (11) and (27), it is derived that

$$v^3(\xi) = \alpha_1^3\left(\frac{G'}{G}\right)^3 + 3\alpha_0\alpha_1^2\left(\frac{G'}{G}\right)^2 + 3\alpha_1\alpha_0^2\left(\frac{G'}{G}\right) + \alpha_0^3,$$
$$v''(\xi) = 2\alpha_1\left(\frac{G'}{G}\right)^3 + 3\alpha_\lambda\alpha_1\left(\frac{G'}{G}\right)^2 + \left(2\alpha_1\mu + \lambda^2 + \alpha_1\right)\left(\frac{G'}{G}\right) + \alpha_1\lambda\mu. \quad (28)$$

By substituting (27) and (28) into (25) and collecting all terms with the same power of G'/G together, the left-hand sides of (25) are converted into the polynomials in G'/G. Equating the coefficients of the polynomials to zero yields a set of simultaneous algebraic equations for α_0, α_1, λ, c, c_1, c_2 and μ as follows:

$$\frac{rp+2sp}{6c}\alpha_0^3 + q\lambda\alpha_1\mu + \left(\frac{rc_1}{c} - c\right)\alpha_0 + c_2 = 0,$$
$$\frac{rp+2sp}{6c}\alpha_0^2\alpha_1 + 2q\alpha_1\mu + q\lambda^2\alpha_1 + \left(\frac{rc_1}{c} - c\right)\alpha_1 = 0,$$
$$\frac{rp+2sp}{6c}\alpha_0\alpha_1^2 + 3q\alpha_1\lambda = 0,$$
$$\frac{rp+2sp}{6c}\alpha_1^3 + 2q\alpha_1 = 0. \quad (29)$$

Solving the algebraic equations above yields

$$\alpha_1 = \pm q\sqrt{\frac{6\left(\lambda^2-4\mu\right)}{p(r+2s)}},$$
$$\alpha_0 = \mp\lambda q\sqrt{\frac{3\left(\lambda^2-4\mu\right)}{2p(r+2s)}},$$
$$c = \frac{1}{2}\left(-q\lambda^2 + 4q\mu\right),$$
$$c_1 = 0,$$
$$c_2 = 0, \quad (30)$$

or

$$\alpha_1 = \frac{2\alpha_0}{\lambda},$$
$$c = \frac{-rp\alpha_0^2 - 2sp\alpha_0^2}{3q\lambda^2},$$
$$c_1 = \frac{c\left(2c - q\lambda^2 - 4q\mu\right)}{2r},$$
$$c_2 = 0, \quad (31)$$

where λ, α_0 are nonzero constants.

Substituting system (30) or (31) into (27), we have the formulae of the solutions of (25) as follows:

$$v(x,t) = v(\xi) = \pm q\sqrt{\frac{6\left(\lambda^2-4\mu\right)}{p(r+2s)}}\left(\frac{G'}{G}\right) \mp \lambda q\sqrt{\frac{3\left(\lambda^2-4\mu\right)}{2p(r+2s)}},$$
$$\xi = x - \frac{1}{2}\left(-q\lambda^2 + 4q\mu\right)t, \quad (32)$$

or

$$v(x,t) = v(\xi) = \frac{2\alpha_0}{\lambda}\left(\frac{G'}{G}\right) + \alpha_0,$$
$$\xi = x - \frac{-rp\alpha_0^2 - 2sp\alpha_0^2}{3q\lambda^2}t, \quad (33)$$

where α_0, λ are nonzero constants.

Substituting the solution of (16) into (32) or (33), we can get the exact solutions of $v(x,t)$. Then substituting $v(x,t)$ into (23), we will get the solutions of $u(x,t)$. From the expression of (32) and (33), we conclude that the only difference of the exact solution $v(x,t)$ between these two cases are their coefficients. Without loss of generality, we calculate the case (33). Now substituting solution (16) into (33), the traveling wave solutions of DSW equation can be obtained as follows.

Case 1. When $\lambda^2 - 4\mu > 0$, then we have the following exact traveling wave solution of DSW equation:

$$v(x,t) = v(\xi) = \frac{\alpha_0}{\lambda}\sqrt{\lambda^2-4\mu}\left(\frac{A_1\cosh\left((1/2)\sqrt{\lambda^2-4\mu}\xi\right) + A_2\sinh\left((1/2)\sqrt{\lambda^2-4\mu}\xi\right)}{A_1\sinh\left((1/2)\sqrt{\lambda^2-4\mu}\xi\right) + A_2\cosh\left((1/2)\sqrt{\lambda^2-4\mu}\xi\right)}\right),$$
$$u(x,t) = u(\xi) = \frac{p}{2c}v^2(\xi) + \frac{c_1}{c}, \quad (34)$$

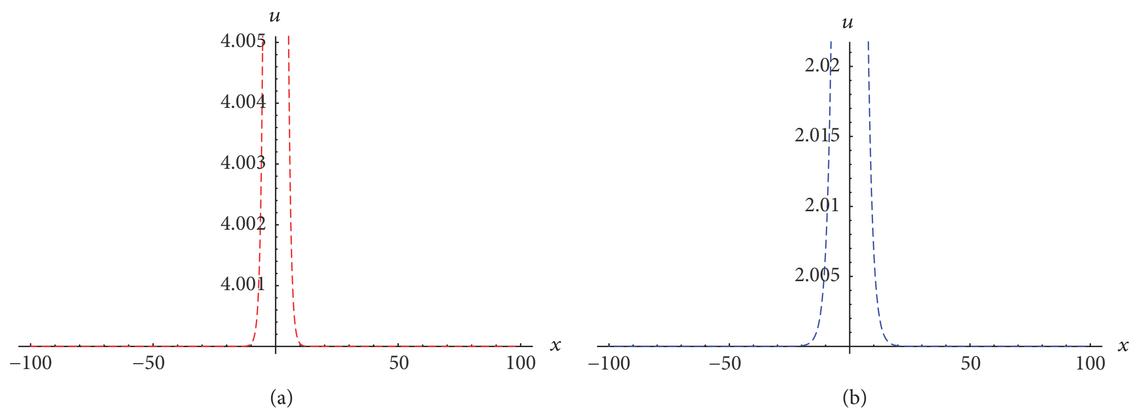

FIGURE 2: The two-dimensional profiles of the solitary wave solution (39) with $c = 1$, $t = 0$, $\lambda = 1$, $r = 0.5$, $p = q = 1$, $\alpha = 0.5$ ((a) red dashed line) and $c = 1$, $t = 0$, $\lambda = 0.5$, $r = 0.5$, $p = q = 1$, $\alpha = 0.5$ ((b) blue dashed line).

where $c = (-rp\alpha_0^2 - 2sp\alpha_0^2)/3q\lambda^2$, $c_1 = c(2c - q\lambda^2 - 4q\mu)/2r$, $\xi = x - ((-rp\alpha_0^2 - 2sp\alpha_0^2)/3q\lambda^2)t$, $\lambda, \mu, \alpha_0, A_1$ and A_2 are constants.

Case 2. When $\lambda^2 - 4\mu < 0$, then we have the following exact traveling wave solution of DSW equation:

$$v(x,t) = v(\xi) = \frac{\alpha_0}{\lambda}\sqrt{\lambda^2 - 4\mu}\left(\frac{-A_1 \sin\left((1/2)\sqrt{\lambda^2 - 4\mu}\xi\right) + A_2 \cos\left((1/2)\sqrt{\lambda^2 - 4\mu}\xi\right)}{A_1 \cos\left((1/2)\sqrt{\lambda^2 - 4\mu}\xi\right) + A_2 \sin\left((1/2)\sqrt{\lambda^2 - 4\mu}\xi\right)}\right),$$
$$u(x,t) = u(\xi) = \frac{p}{2c}v^2(\xi) + \frac{c_1}{c}, \tag{35}$$

where $c = (-rp\alpha_0^2 - 2sp\alpha_0^2)/3q\lambda^2$, $c_1 = c(2c - q\lambda^2 - 4q\mu)/2r$, $\xi = x - ((-rp\alpha_0^2 - 2sp\alpha_0^2)/3q\lambda^2)t$, λ, μ, α_0, A_1 and A_2 are constants.

Case 3. When $\lambda^2 - 4\mu = 0$, then we have the following exact traveling wave solution of DSW equation:

$$v(x,t) = v(\xi) = \frac{2\alpha_0 A_2}{\lambda(A_1 + A_2\xi)},$$
$$u(x,t) = u(\xi) = \frac{p}{2c}v^2(\xi) + \frac{c_1}{c}, \tag{36}$$

where $c = (-rp\alpha_0^2 - 2sp\alpha_0^2)/3q\lambda^2$, $c_1 = c(2c - q\lambda^2 - 4q\mu)/2r$, $\xi = x - ((-rp\alpha_0^2 - 2sp\alpha_0^2)/3q\lambda^2)t$, λ, μ, α_0, A_1 and A_2 are constants.

From the above three general cases, it is not difficult to get the exact solitary wave solutions of DSW equation. For example, let $A_1 = 0$, $\mu = 0$, $A_2 \neq 0$, $\lambda > 0$; then (34) yield the following solitary wave solution:

$$v(x,t) = v(\xi) = \alpha_0 \tanh\left(\frac{\lambda}{2}\xi\right),$$
$$u(x,t) = u(\xi) = \frac{p\alpha_0^2}{2r}\tanh^2\left(\frac{\lambda}{2}\xi\right) + \frac{2c - q\lambda^2}{2r}. \tag{37}$$

On the other hand, if we let $A_2 = 0$, $\mu = 0$, $A_1 \neq 0$, $\lambda > 0$; then (17) yields the following solitary wave solution:

$$v(x,t) = v(\xi) = \alpha_0 \coth\left(\frac{\lambda}{2}\xi\right), \tag{38}$$

$$u(x,t) = u(\xi) = \frac{p\alpha_0^2}{2r}\coth^2\left(\frac{\lambda}{2}\xi\right) + \frac{2c - q\lambda^2}{2r}, \tag{39}$$

where $c = (-rp\alpha_0^2 - 2sp\alpha_0^2)/3q\lambda^2$, $c_1 = c(2c - q\lambda^2)/2r$, $\xi = x - ((-rp\alpha_0^2 - 2sp\alpha_0^2)/3q\lambda^2)t$, λ and α_0 are nonzero constants.

We can see that a number of traveling wave solutions and rational solutions can be obtained by choosing different parameters. For example, if we choose $A_1 = 0$, $\mu = 0$, $A_2 \neq 0$, $\lambda > 0$ then substituting them into (34)–(36), trigonometric function solutions and rational solutions can be obtained. To our best knowledge, the rational solutions appear the first time; they are new exact solutions of DSW equation. These new exact solutions will enrich the previous results.

We should also point out that the solitary wave solutions (38) and (39) possess a singular point, while solutions (35) have an infinite number of singular points. In the meaning of physics, it shows there exists blowup of these solutions. We also show the two-dimensional profiles of solutions (39) in Figure 2.

4. Conclusion and Discussion

In this paper, we investigate two nonlinear water wave equations which have important applications in several areas of physics and engineering by using the (G'/G)-expansion method. As a result, several pairs of exact traveling wave solutions are given directly. The most important thing is that we can get new explicit solitary wave solutions when choosing different parameters. These new exact solitary wave solutions not only enrich the previous results but also are helpful to further study these two nonlinear wave equations. Especially, for the Padé-II equation, it is a new nonlinear wave equation modeling unidirectional propagation of long wave in dispersive media. It is worth further studying. At the same time, we analyze the nonlinear structure of the solitary wave solutions. We show the blowup of the solutions (21) and (39) by plotting their two-dimensional profiles. There should be much more interesting structures of these two nonlinear wave equations. We will further study them in the near future. However, according to our study, we can conclude that the method used in this paper is a powerful mathematical tool for solving nonlinear wave equations in mathematical physics and engineering fields. It is direct and concise. Much tedious algebraic calculations can be finished by computer programmes like Mathematica, Maple, and so on. Many well known nonlinear wave equations can be handled by this method.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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