Research Article

Blow-Up and Global Existence Analysis for the Viscoelastic Wave Equation with a Frictional and a Kelvin-Voigt Damping

Fosheng Wang¹ and Chengqiang Wang²

¹*School of Mathematics and Physics, Mianyang Teachers' College, Mianyang 621000, China*

²*School of Mathematics, Chengdu Normal University, Chengdu 611130, China*

Correspondence should be addressed to Chengqiang Wang; cqwung@foxmail.com

Received 25 February 2018; Accepted 28 March 2018; Published 6 May 2018

Academic Editor: Ming Mei

Copyright © 2018 Fosheng Wang and Chengqiang Wang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We are concerned in this paper with the initial boundary value problem for a quasilinear viscoelastic wave equation which is subject to a nonlinear action, to a nonlinear frictional damping, and to a Kelvin-Voigt damping, simultaneously. By utilizing a carefully chosen Lyapunov functional, we establish first by the celebrated convexity argument a finite time blow-up criterion for the initial boundary value problem in question; we prove second by an a priori estimate argument that some solutions to the problem exists globally if the nonlinearity is “weaker,” in a certain sense, than the frictional damping, and if the viscoelastic damping is sufficiently strong.

1. Introduction

In this paper, we are concerned with the initial boundary value problem (IBVP) for a quasilinear viscoelastic wave equation whose energy is inhibited by a nonlinear frictional damping and by a Kelvin-Voigt damping. More precisely, we consider IBVP

$$\begin{aligned} & |\partial_t u|^\rho \partial_t^2 u - \Delta u + \int_0^t g(t-\tau) \Delta u(\cdot, \tau) d\tau - \Delta \partial_t u \\ & + |\partial_t u|^{m-2} \partial_t u = |u|^{p-2} u \quad \text{in } \Omega \times (0, +\infty), \\ & u = 0 \quad \text{on } \Gamma \times (0, +\infty), \\ & u(\cdot, 0) = u^0, \\ & \partial_t u(\cdot, 0) = u^1 \end{aligned} \quad (1)$$

in Ω ,

where $u = u(x, t)$ is the unknown, Ω , having the boundary $\Gamma \in \mathbb{R}^2$, is a nonempty bounded open subset of $\mathbb{R}^N$ with N a positive integer, $\rho > 0$, $m > 2$, $p > 2$ are constants, and $g : [0, +\infty) \rightarrow [0, +\infty)$, the so-called relaxation function or

viscoelastic kernel, is a decreasing function. The restrictions on ρ , p , m and g would be added when necessary. Eq. (1)₁ was used to describe the vibration of beams or plates, whose wave propagation speed depends on the velocity of vibration, which is subject to nonlinear frictional damping force and viscoelastic damping force; see [1–3]. In particular, the term $|\partial_t u|^\rho \partial_t^2 u$ with $\rho > 0$, by Boltzmann's superposition principle, indicates that the wave speed depends on the velocity of the wave; $\int_0^t g(t-\tau) \Delta u(\cdot, \tau) d\tau$, the so-called viscoelastic damping, is incorporated in IBVP (1) to characterize the hereditary properties.

In recent years, the wave equation has been frequently chosen as the model to study the competition between different damping terms and the source term $|u|^{p-2}u$. In [4], Messaoudi considered the general decay problem for IBVP

$$\begin{aligned} & \partial_t^2 u - \Delta u + \int_0^t g(t-\tau) \Delta u(\cdot, \tau) d\tau = 0 \\ & \quad \text{in } \Omega \times (0, +\infty), \\ & u = 0 \\ & \quad \text{on } \Gamma \times (0, +\infty), \end{aligned}$$

$$\begin{aligned}
u(\cdot, 0) &= u^0, \\
\partial_t u(\cdot, 0) &= u^1 \\
&\text{in } \Omega,
\end{aligned} \tag{2}$$

and he proved, under certain additional restrictions on the relaxation function g , that the energy of IBVP (2) decreases to zero. The results and methods in [4] were later extended by Messaoudi and Al-Khulaifi [5] to IBVP

$$\begin{aligned}
|\partial_t u|^\rho \partial_t^2 u - \Delta u - \Delta \partial_t^2 u + \int_0^t g(t-\tau) \Delta u(\cdot, \tau) d\tau &= 0 \\
&\text{in } \Omega \times (0, +\infty), \\
u &= 0 \\
&\text{on } \Gamma \times (0, +\infty), \\
u(\cdot, 0) &= u^0, \\
\partial_t u(\cdot, 0) &= u^1 \\
&\text{in } \Omega;
\end{aligned} \tag{3}$$

more precisely, they established some general decay results for IBVP (3). It was shown in both [4, 5] that the energy decay rate is closely related to the peculiarity of the relaxation function g . Different from (2)₁, the wave speed of (3)₁ depends on the velocity of the wave. The study of [5] was highly inspired by [6], in which Cavalcanti, Domingos Cavalcanti, and Ferreira studied the IBVP

$$\begin{aligned}
|\partial_t u|^\rho \partial_t^2 u - \Delta u - \Delta \partial_t^2 u + \int_0^t g(t-\tau) \Delta u(\cdot, \tau) d\tau \\
- \gamma \Delta \partial_t u &= 0 \quad \text{in } \Omega \times (0, +\infty), \\
u &= 0 \quad \text{on } \Gamma \times (0, +\infty), \\
u(\cdot, 0) &= u^0, \\
\partial_t u(\cdot, 0) &= u^1 \\
&\text{in } \Omega,
\end{aligned} \tag{4}$$

and they proved that the solutions of IBVP (4) exist globally if $\gamma \geq 0$ and their energy decreases to zero if $\gamma > 0$. Amongst the afore-mentioned references, there are vast literatures concerning energy decay for viscoelastic wave equations; see [7–14] and the references cited therein. The key ingredient in proving energy decay estimates is an appropriate modified energy functional associated with the initial boundary value problem under consideration. The usual idea to obtain decay for the afore-chosen energy functional is to exploit various perturbed energy functionals for the above energy functional. This idea is important in our later development.

In the afore-mentioned references, the source term $|u|^{p-2}u$, in one form or another, is “beaten” by the damping terms $\int_0^t g(t-\tau) \Delta u(\cdot, \tau) d\tau$, $-\Delta \partial_t u$ and/or $|\partial_t u|^{m-2} \partial_t u$. In some cases, these damping terms were also proved to be

“beaten” by the above pure power source term; see [15–25]. As with the procedure to obtain energy decay, the main step in proving finite time blow-up results is to choose suitable Lyapunov functional $L(t)$ which is closely related to the so-called modified energy functional frequently used in obtaining energy decay, and to employ an convexity argument to obtain $L'(t) \geq C_0(L(t))^{1/(1-\alpha_0)}$ with $C_0 > 0$ and $\alpha_0 \in (0, 1)$. This idea would be useful in proving our finite time blow-up result.

More recently, Song [26] established a global nonexistence of solutions with positive initial data for IBVP

$$\begin{aligned}
|\partial_t u|^\rho \partial_t^2 u - \Delta u + \int_0^t g(t-\tau) \Delta u(\cdot, \tau) d\tau \\
+ |\partial_t u|^{m-2} \partial_t u &= |u|^{p-2} u \quad \text{in } \Omega \times (0, +\infty), \\
u &= 0 \quad \text{on } \Gamma \times (0, +\infty), \\
u(\cdot, 0) &= u^0, \\
\partial_t u(\cdot, 0) &= u^1 \\
&\text{in } \Omega.
\end{aligned} \tag{5}$$

Inspired by [26], Hao and Wei [27] studied IBVP

$$\begin{aligned}
|\partial_t u|^\rho \partial_t^2 u - \Delta u + \int_0^t g(t-\tau) \Delta u(\cdot, \tau) d\tau - \Delta \partial_t u \\
= |u|^{p-2} u \quad \text{in } \Omega \times (0, +\infty), \\
u &= 0 \quad \text{on } \Gamma \times (0, +\infty), \\
u(\cdot, 0) &= u^0, \\
\partial_t u(\cdot, 0) &= u^1 \\
&\text{in } \Omega,
\end{aligned} \tag{6}$$

and they proved a global existence and a blow-up criterion for solutions to IBVP (6). Both Refs. [26, 27] assume the same restrictions on ρ , g , m and p . The results obtained by [26, 27] mean that the damping terms $|\partial_t u|^{m-2} \partial_t u$ and $-\Delta \partial_t u$ are beaten, in the sense that some solutions blow up in finite time, independently by the source term $|u|^{p-2}u$ in the “arena” $|\partial_t u|^\rho \partial_t^2 u - \Delta u + \int_0^t g(t-\tau) \Delta u(\cdot, \tau) d\tau = 0$. We are tempted to compare the damping term $|\partial_t u|^{m-2} \partial_t u - \Delta \partial_t u$ and the source term $|u|^{p-2}u$. The later one was shown in a certain condition to be “stronger” than the former one in the arena of viscoelastic plate equation $\partial_t^2 u + \Delta^2 u - \int_0^t g(t-\tau) \Delta^2 u(\cdot, \tau) d\tau - \Delta \partial_t^2 u = 0$. More, precisely, Li et al. [28] provided a finite time blow-up result for IBVP

$$\begin{aligned}
\partial_t^2 u + \Delta^2 u - \int_0^t g(t-\tau) \Delta^2 u(\cdot, \tau) d\tau - \Delta \partial_t u - \Delta \partial_t^2 u \\
- |\partial_t u|^{m-1} \partial_t u &= |u|^{p-1} u \quad \text{in } \Omega \times (0, +\infty), \\
u &= 0, \\
\partial_\nu u &= 0 \\
&\text{on } \Gamma \times (0, +\infty),
\end{aligned}$$

$$\begin{aligned}
u(\cdot, 0) &= u^0, \\
\partial_t u(\cdot, 0) &= u^1 \\
&\text{in } \Omega.
\end{aligned} \tag{7}$$

Inspired by [26–28], we seek to prove that some solutions with positive initial data to IBVP (1) blow up in finite time under certain conditions (see Theorem 11); all solutions to IBVP (1) exist globally under other conditions (see Theorem 12). Different from [26], we provide a global existence result under some conditions, in addition to a finite time blow-up result. Compared with the finite time blow-up results in [26, 27], our finite time blow-up result shows that the source beats the combination of the frictional damping and the Kelvin-Voigt damping.

Notations 1. Throughout the paper, C is a generic positive constant and it can take a different value at every new occurrence. $L^p(\Omega)$ (resp. $H_0^1(\Omega)$) is the classical Lebesgue (resp., Sobolev) space.

The rest of our paper is planned as follows. In Section 2, we collect the main results and some preliminaries necessary in proving the results. In Section 3, we prove in a certain detail all the main results. In Section 4, we present an open question which confuses us recently.

2. Preliminaries and the Main Results

In this section, we collect the main assumptions on the relaxation function g and the parameters in IBVP (1), some results which can simplify our later development and the main results of this paper. Due to their pervasion in the textbooks on PDEs or in the references cited below, all the proofs of the following lemmas are omitted here. Let us recall first Young's inequality with a $\delta > 0$.

Lemma 2. For every pair $(a, b) \in [0, +\infty)^2$, and every $\theta \in (0, 1)$, we have

$$a^{1-\theta} b^\theta \leq \delta a + \theta \left(\frac{1-\theta}{\delta} \right)^{1/\theta-1} b, \quad \forall \delta \in (0, +\infty). \tag{8}$$

We list here all the assumptions which are necessary for our claimed results.

Assumption 3. The relaxation function $g : [0, +\infty) \rightarrow [0, +\infty)$ is absolutely continuous and decreasing and satisfies

$$\ell = 1 - \int_0^{+\infty} g(\tau) d\tau > 0. \tag{9}$$

Assumption 4. $m > 2$; $p > 2$ and $\rho > 0$ if $N = 1, 2$; $2 < p \leq 2(n-1)/(n-2)$ and $0 < \rho \leq 2/(n-2)$ if $N \geq 3$.

Assumption 5. $p > \max(m, \rho + 2)$ and $\int_0^{+\infty} g(\tau) d\tau < (p/2 - 1)/(p/2 - 1 + 1/2p)$.

Assumption 6. $2 < p \leq \rho + 2$ and $\rho + 2 \leq 2(n-1)/(n-2)$.

Assumption 7. $E(0) < [1 - (1/(p-2)p)((1-\ell)/\ell)](1/2 - 1/p)(B_1)^{-2p/(p-2)}$ and $\|\nabla u^0\|_{L^2(\Omega; \mathbb{R}^N)} \geq (B_1)^{-p/(p-2)}$ where $E(t)$ is given by (14):

$$B_1 = \frac{B}{\sqrt{\ell}}, \tag{10}$$

$$\begin{aligned}
B &= \inf \{ C \in (0, +\infty) \mid \|\varphi\|_{L^p(\Omega)} \\
&\leq C \|\nabla \varphi\|_{L^2(\Omega; \mathbb{R}^N)} \text{ whenever } \varphi \in H_0^1(\Omega) \}.
\end{aligned} \tag{11}$$

Remark 8. By the well-known Poincaré's embedding, B is a number in $(0, +\infty)$.

With the above assumptions prepared, we are now in a position to state the main results of this paper. What comes first is the local existence theory.

Theorem 9. Let Assumptions 3 and 4 be fulfilled. For every pair $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists a (unique) $T_{\max} \in (0, +\infty]$ such that IBVP (1) admits a unique solution,

$$u \in \mathcal{C}([0, T_{\max}); H_0^1(\Omega)) \cap \mathcal{C}^1([0, T_{\max}); L^2(\Omega)), \tag{12}$$

and that $[0, T_{\max})$ is the maximal time interval of existence of u .

Remark 10. $T_{\max}$ is called the lifespan of u . In the situation $T_{\max} = +\infty$, the solution u is said to exist globally; in the situation $T_{\max} < +\infty$, the solution u is said to blow up in finite time.

Our second main result is the following finite time blow-up result.

Theorem 11. Let Assumptions 3, 4, and 5 be fulfilled. Then every solution u to IBVP (1) blows up in finite time whenever it satisfies Assumption 7: If (u^0, u^1) satisfies Assumption 7, then we have $T_{\max} < +\infty$. And in this situation, we have furthermore

$$\begin{aligned}
&\lim_{t \rightarrow T_{\max}} \left(\|\partial_t u(\cdot, t)\|_{L^{\rho+2}(\Omega)}^{\rho+2} + \|\nabla u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 \right. \\
&\quad \left. + \|u(\cdot, t)\|_{L^p(\Omega)}^p \right) = +\infty.
\end{aligned} \tag{13}$$

The third main result is the following global existence result.

Theorem 12. Let Assumptions 3, 4, and 6 be fulfilled. Then for every pair $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, IBVP (1) admits a unique global solution $u \in \mathcal{C}([0, +\infty); H_0^1(\Omega)) \cap \mathcal{C}^1([0, +\infty); L^2(\Omega))$.

In the proofs of the above theorems, the following modified energy functionals are very important:

$$\begin{aligned}
E(t) &= \frac{1}{\rho+2} \|\partial_t u(\cdot, t)\|_{L^{\rho+2}(\Omega)}^{\rho+2} \\
&\quad + \frac{1}{2} \left(1 - \int_0^t g(\tau) d\tau\right) \|\nabla u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 \\
&\quad + \frac{1}{2} (g \diamond \nabla u)(t) - \frac{1}{p} \|u(\cdot, t)\|_{L^p(\Omega)}^p, \\
&\quad \forall t \in [0, +\infty);
\end{aligned} \tag{14}$$

here and hereafter, we write for every $\psi \in L_{\text{loc}}^2([0, +\infty); L^2(\Omega; \mathbb{R}^n))$

$$\begin{aligned}
(g \diamond \psi)(t) &= \int_0^t g(t-\tau) \|\psi(\cdot, t) - \psi(\cdot, \tau)\|_{L^2(\Omega; \mathbb{R}^n)}^2 d\tau, \\
&\quad \forall t \in [0, +\infty).
\end{aligned} \tag{15}$$

The following lemma would be used several times later.

Lemma 13. *Let $E(t)$ be defined by (14). Then along the solution u to IBVP (1), we have*

$$\begin{aligned}
E'(t) &= -\|\partial_t u(\cdot, t)\|_{L^m(\Omega)}^m - \|\nabla \partial_t u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 \\
&\quad + \frac{1}{2} (g' \diamond \nabla u)(t) \\
&\quad - \frac{1}{2} g(t) \|\nabla u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2, \quad \forall t \in [0, +\infty).
\end{aligned} \tag{16}$$

Remark 14. It should be noted that $E(t)$ is decreasing whenever g fulfills Assumption 3.

3. Proofs of the Main Results

In this section, we present the proofs of our main results.

Proof of Theorem 9. Since one can prove the local existence by mimicking the steps in [6] and by using the classical Faedo-Galerkin method, the details are omitted here. By combining the local existence result and a standard continuation method, we can show that there exists a maximal time $T_{\max}$ of existence of the solution u for every pair $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$. $\square$

Before proving Theorem 11, we introduce the following lemma.

Lemma 15 (see [26]). *Let Assumptions 3, 4, and 5 be fulfilled. There exists a $\beta \in ((B_1)^{-p/(p-2)}, +\infty)$ such that*

$$\begin{aligned}
\left(1 - \int_0^t g(\tau) d\tau\right) \|\nabla u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 &\geq \beta^2, \\
\|u(\cdot, t)\|_{L^p(\Omega)} &\geq B_1 \beta, \\
&\quad \forall t \in [0, +\infty)
\end{aligned} \tag{17}$$

whenever u is a solution to IBVP (1) verifying $E(0) < E_1$ and $\|\nabla u^0\|_{L^2(\Omega; \mathbb{R}^n)} \geq (B_1)^{-p/(p-2)}$ where E_1 is given by

$$E_1 = \left(\frac{1}{2} - \frac{1}{p}\right) (B_1)^{-2p/(p-2)}. \tag{18}$$

Proof of Theorem 11. Let $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$. We write u for the solution to IBVP (1) with its initial datum (u^0, u^1) and write T^* for the maximal time of existence of u . Our goal is to show by a contradiction argument that $T^* < +\infty$.

Assume $T^* = +\infty$. Let us introduce the following functional for u :

$$\begin{aligned}
L(t) &= (E_2 - E(t))^{1-\alpha} + \frac{\varepsilon}{2} \|\nabla u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 \\
&\quad + \frac{\varepsilon}{\rho+1} \int_{\Omega} u(x, t) |\partial_t u(x, t)|^{\rho} \partial_t u(x, t) dx, \\
&\quad \forall t \in [0, +\infty),
\end{aligned} \tag{19}$$

where $E(t)$ is given by (14) and $E_2 \in (E(0), E_1)$, $\varepsilon > 0$ and $\alpha \in (0, 1/m - 1/p)$ are yet to be determined later. Differentiate both hand sides of (19), and conduct some routine but tedious calculations, to obtain

$$\begin{aligned}
L'(t) &= -(1-\alpha)(E_2 - E(t))^{-\alpha} E'(t) + \varepsilon \int_{\Omega} \nabla u(x, t) \\
&\quad \cdot \nabla \partial_t u(x, t) dx + \frac{\varepsilon}{\rho+1} \|\partial_t u(\cdot, t)\|_{L^{\rho+2}(\Omega)}^{\rho+2} \\
&\quad + \varepsilon \langle u(\cdot, t), |\partial_t u(\cdot, t)|^{\rho} \partial_t^2 u(\cdot, t) \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} = (1 \\
&\quad - \alpha)(E_2 - E(t))^{-\alpha} \left[\|\partial_t u(\cdot, t)\|_{L^m(\Omega)}^m \right. \\
&\quad + \|\nabla \partial_t u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 - \frac{1}{2} (g' \diamond \nabla u)(t) \\
&\quad + \frac{1}{2} g(t) \|\nabla u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 \left. \right] + \varepsilon \int_{\Omega} \nabla u(x, t) \\
&\quad \cdot \nabla \partial_t u(x, t) dx + \frac{\varepsilon}{\rho+1} \|\partial_t u(\cdot, t)\|_{L^{\rho+2}(\Omega)}^{\rho+2} \\
&\quad - \varepsilon \|\nabla u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 + \varepsilon \int_0^t g(t-\tau) \int_{\Omega} \nabla u(x, t) \\
&\quad \cdot \nabla u(x, \tau) dx d\tau \\
&\quad - \varepsilon \int_{\Omega} u(x, t) |\partial_t u(x, t)|^{m-2} \partial_t u(x, t) dx \\
&\quad - \varepsilon \int_{\Omega} \nabla u(x, t) \cdot \nabla \partial_t u(x, t) dx + \varepsilon \|u(\cdot, t)\|_{L^p(\Omega)}^p \\
&\quad \geq (1-\alpha)(E_2 - E(t))^{-\alpha} \|\partial_t u(\cdot, t)\|_{L^m(\Omega)}^m \\
&\quad + \frac{\varepsilon}{\rho+1} \|\partial_t u(\cdot, t)\|_{L^{\rho+2}(\Omega)}^{\rho+2} - \varepsilon \|\nabla u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2
\end{aligned}$$

$$\begin{aligned}
& + \varepsilon \int_0^t g(t-\tau) \int_{\Omega} \nabla u(x, t) \cdot \nabla u(x, \tau) dx d\tau \\
& - \varepsilon \int_{\Omega} u(x, t) |\partial_t u(x, t)|^{m-2} \partial_t u(x, t) dx \\
& + \varepsilon \|u(\cdot, t)\|_{L^p(\Omega)}^p, \quad \forall t \in [0, +\infty).
\end{aligned} \tag{20}$$

We next treat $\int_{\Omega} u(x, t) |\partial_t u(x, t)|^{m-2} \partial_t u(x, t) dx$ and $\int_0^t g(t-\tau) \int_{\Omega} \nabla u(x, t) \cdot \nabla u(x, \tau) dx d\tau$ term-by-term. By utilizing the algebraic identity

$$\begin{aligned}
ab &= a^2 - a(a-b) = a^2 - 2 \cdot \left[a \sqrt{\frac{1}{2p(1-\delta_1)}} \right] \\
&\cdot \left[(a-b) \sqrt{\frac{p(1-\delta_1)}{2}} \right] = a^2 \\
&+ \left\{ \left[a \sqrt{\frac{1}{2p(1-\delta_1)}} - (a-b) \sqrt{\frac{p(1-\delta_1)}{2}} \right]^2 \right. \\
&\left. - \frac{a^2}{2p(1-\delta_1)} - \frac{p(1-\delta_1)(a-b)^2}{2} \right\} \\
&= \left[a \sqrt{\frac{1}{2p(1-\delta_1)}} - (a-b) \sqrt{\frac{p(1-\delta_1)}{2}} \right]^2 + \left[1 \right. \\
&\left. - \frac{1}{2p(1-\delta_1)} \right] a^2 - \frac{p(1-\delta_1)}{2} (a-b)^2,
\end{aligned} \tag{21}$$

for every $a, b \in \mathbb{R}$, we conclude that

$$\begin{aligned}
& \int_0^t g(t-\tau) \int_{\Omega} \nabla u(x, t) \cdot \nabla u(x, \tau) dx d\tau = \int_0^t g(t-\tau) \int_{\Omega} \left[\left| \sqrt{\frac{1}{2p(1-\delta_1)}} \nabla u(x, t) \right. \right. \\
& \left. \left. - \sqrt{\frac{p(1-\delta_1)}{2}} [\nabla u(x, t) - \nabla u(x, \tau)] \right|^2 + \left[1 \right. \right. \\
& \left. \left. - \frac{1}{2p(1-\delta_1)} \right] |\nabla u(x, t)|^2 - \frac{p(1-\delta_1)}{2} |\nabla u(x, t) \right. \\
& \left. - \nabla u(x, \tau)|^2 \right] dx d\tau \geq \left[1 \right. \\
& \left. - \frac{1}{2p(1-\delta_1)} \right] \int_0^t g(\tau) d\tau \|\nabla u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 \\
& - \frac{p(1-\delta_1)}{2} (g \diamond \nabla u)(t), \quad \forall t \in [0, +\infty),
\end{aligned} \tag{22}$$

where $\delta_1 \in (0, 1)$ is yet to be selected later. By the definition (14) of $E(t)$, we have

$$\begin{aligned}
& - \frac{p}{2} (g \diamond \nabla u)(t) \\
& = -pE_2 + p(E_2 - E(t)) + \frac{p}{\rho+2} \|\partial_t u(\cdot, t)\|_{L^{\rho+2}(\Omega)}^{\rho+2} \\
& + \frac{p}{2} \left(1 - \int_0^t g(\tau) d\tau \right) \|\nabla u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 \\
& - \|u(\cdot, t)\|_{L^p(\Omega)}^p, \quad \forall t \in [0, +\infty).
\end{aligned} \tag{23}$$

This, together with (22), implies

$$\begin{aligned}
& \int_0^t g(t-\tau) \int_{\Omega} \nabla u(x, t) \cdot \nabla u(x, \tau) dx d\tau \geq \left[1 - \frac{1}{2p(1-\delta_1)} \right] \int_0^t g(\tau) d\tau \|\nabla u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 - p(1-\delta_1) E_2 \\
& + p(1-\delta_1) (E_2 - E(t)) + \frac{p(1-\delta_1)}{\rho+2} \|\partial_t u(\cdot, t)\|_{L^{\rho+2}(\Omega)}^{\rho+2} + \frac{p(1-\delta_1)}{2} \left(1 - \int_0^t g(\tau) d\tau \right) \|\nabla u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 \\
& - (1-\delta_1) \|u(\cdot, t)\|_{L^p(\Omega)}^p \\
& = \frac{[p(1-\delta_1)/2 - 1] \left(1 - \int_0^t g(\tau) d\tau \right) - (1/2p(1-\delta_1)) \int_0^t g(\tau) d\tau}{1 - \int_0^t g(\tau) d\tau} \left(1 - \int_0^t g(\tau) d\tau \right) \|\nabla u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 \\
& - p(1-\delta_1) E_2 + p(1-\delta_1) (E_2 - E(t)) + \frac{p(1-\delta_1)}{\rho+2} \|\partial_t u(\cdot, t)\|_{L^{\rho+2}(\Omega)}^{\rho+2} - (1-\delta_1) \|u(\cdot, t)\|_{L^p(\Omega)}^p + \|\nabla u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2, \\
& \quad \forall t \in [0, +\infty).
\end{aligned} \tag{24}$$

Set

$$\delta_1^* = \frac{p-1-\sqrt{1/\ell}}{p}. \quad (25)$$

Thanks to

$$1-\ell = \int_0^{+\infty} g(\tau) d\tau < \frac{p/2-1}{p/2-1+1/2p}, \quad (26)$$

we have, for every $\delta_1 \in (0, \delta_1^*)$,

$$\begin{aligned} & \frac{[p(1-\delta_1)/2-1] \left(1 - \int_0^t g(\tau) d\tau\right) - (1/2p(1-\delta_1)) \int_0^t g(\tau) d\tau}{1 - \int_0^t g(\tau) d\tau} \left(1 - \int_0^t g(\tau) d\tau\right) \|\nabla u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 \\ & \geq \frac{\ell [p(1-\delta_1)/2-1] - (1-\ell)/2p(1-\delta_1)}{\ell} \beta^2 = \frac{\ell [p(1-\delta_1)/2-1] - (1-\ell)/2p(1-\delta_1)}{\ell} (B_1)^{-2p/(p-2)} \\ & - p(1-\delta_1) E_2 + p(1-\delta_1) E_2, \quad \forall t \in [0, +\infty), \end{aligned} \quad (27)$$

where “ $\geq$ ” follows from Lemma 15. We fix $\delta_1^{**} \in (0, \delta_1^*)$ and $E_2^* \in (E(0), E_1)$ such that

$$-p(1-\delta_1^{**}) E_2^* > 0, \quad (28)$$

$$\frac{\ell [p(1-\delta_1^{**})/2-1] - (1-\ell)/2p(1-\delta_1^{**})}{\ell} (B_1)^{-2p/(p-2)}$$

and therefore

$$\frac{[p(1-\delta_1)/2-1] \left(1 - \int_0^t g(\tau) d\tau\right) - (1/2p(1-\delta_1)) \int_0^t g(\tau) d\tau}{1 - \int_0^t g(\tau) d\tau} \left(1 - \int_0^t g(\tau) d\tau\right) \|\nabla u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 > p(1-\delta_1) E_2. \quad (29)$$

The existence of δ_1^{**} and E_2^* can be justified by the following observation

$$\begin{aligned} & \lim_{(\delta_1, E_2) \in (0, \delta_1^*) \times (E(0), E_2), (\delta_1, E_2) \rightarrow (0, E(0))} \left\{ \frac{\ell [p(1-\delta_1)/2-1] - (1-\ell)/2p(1-\delta_1)}{\ell} (B_1)^{-2p/(p-2)} - p(1-\delta_1) E_2 \right\} \\ & = p \left\{ \left[1 - \frac{1}{(p-2)p} \frac{1-\ell}{\ell} \right] \left(\frac{1}{2} - \frac{1}{p} \right) (B_1)^{-2p/(p-2)} - E(0) \right\} > 0. \end{aligned} \quad (30)$$

By Lemma 2, for every $\delta_2 > 0$,

$$\left| \int_{\Omega} u(x, t) |\partial_t u(x, t)|^{m-2} \partial_t u(x, t) dx \right| \leq \|u(\cdot, t)\|_{L^m(\Omega)} \quad (31)$$

$$\begin{aligned} & \cdot \left\| |\partial_t u(x, t)|^{m-2} \partial_t u(x, t) \right\|_{L^{m/(m-1)}(\Omega)} \\ & = \|u(\cdot, t)\|_{L^m(\Omega)} \left\| \partial_t u(\cdot, t) \right\|_{L^m(\Omega)}^{m-1} \leq C \|u(\cdot, t)\|_{L^p(\Omega)} \\ & \cdot \left\| \partial_t u(\cdot, t) \right\|_{L^m(\Omega)}^{m-1} = C \left(\|u(\cdot, t)\|_{L^p(\Omega)}^p \right)^{1/p-1/m} \\ & \cdot \|u(\cdot, t)\|_{L^p(\Omega)}^{p/m} \left\| \partial_t u(\cdot, t) \right\|_{L^m(\Omega)}^{m-1} \\ & \leq C \delta_2 (E_2 - E(0))^{1/p-1/m} \|u(\cdot, t)\|_{L^p(\Omega)}^p \end{aligned}$$

$$+ CC_{\delta_2} (E_2 - E(t))^{1/p-1/m} \left\| \partial_t u(\cdot, t) \right\|_{L^m(\Omega)}^m,$$

$$\forall t \in [0, +\infty).$$

By Remark 14 and by the assumption that $p > m$, we have

$$\begin{aligned} & (E_2 - E(t))^{1/p-1/m} \\ & = (E_2 - E(t))^{-\alpha} (E_2 - E(t))^{\alpha-(1/m-1/p)} \\ & \leq (E_2 - E(t))^{-\alpha} (E_2 - E(t))^{\alpha-(1/m-1/p)} \\ & \leq (E_2 - E(t))^{-\alpha} (E_2 - E(0))^{\alpha-(1/m-1/p)}, \end{aligned} \quad (32)$$

$$\forall t \in [0, +\infty).$$

This, together with (31), implies

$$\begin{aligned} & \left| \int_{\Omega} u(x, t) |\partial_t u(x, t)|^{m-2} \partial_t u(x, t) dx \right| \\ & \leq C \delta_2 (E_2 - E(0))^{1/p-1/m} \|u(\cdot, t)\|_{L^p(\Omega)}^p \\ & + CC_{\delta_2} (E_2 - E(0))^{\alpha-(1/m-1/p)} (E_2 - E(t))^{-\alpha} \\ & \cdot \|\partial_t u(\cdot, t)\|_{L^m(\Omega)}^m, \quad \forall t \in [0, +\infty). \end{aligned} \quad (33)$$

Combine (20), (24), (29), and (33), to obtain for $\delta_1 = \delta_1^{**}$, $E_2 = E_2^*$ and every $\delta_2 > 0$ and every $\varepsilon > 0$,

$$\begin{aligned} L'(t) & \geq \left[(1-\alpha) - \varepsilon CC_{\delta_2} (E_2 - E(0))^{\alpha-(1/m-1/p)} \right] \\ & \cdot (E_2 - E(t))^{-\alpha} \|\partial_t u(\cdot, t)\|_{L^m(\Omega)}^m \\ & + \varepsilon \left[\frac{1}{\rho+1} + \frac{p(1-\delta_1)}{\rho+2} \right] \|\partial_t u(\cdot, t)\|_{L^{\rho+2}(\Omega)}^{\rho+2} \\ & + p\varepsilon(1-\delta_1)(E_2 - E(t)) \\ & + \varepsilon \left[\delta_1 - C\delta_2 (E_2 - E(0))^{1/p-1/m} \right] \\ & \cdot \|u(\cdot, t)\|_{L^p(\Omega)}^p, \quad \forall t \in [0, +\infty). \end{aligned} \quad (34)$$

Let $\delta_1 = \delta_1^{**}$, $E_2 = E_2^*$ and choose sufficiently small $\delta_2 > 0$ and $\varepsilon > 0$ such that

$$\begin{aligned} L'(t) & \geq C \left[\|\partial_t u(\cdot, t)\|_{L^{\rho+2}(\Omega)}^{\rho+2} + (E_2 - E(t)) \right. \\ & \left. + \|u(\cdot, t)\|_{L^p(\Omega)}^p \right], \quad \forall t \in [0, +\infty), \end{aligned} \quad (35)$$

where $C > 0$ depends on δ_1^{**} , E_2^* , δ_2 , ε , $E(0)$, p , ρ and α . Since the function $[0, +\infty) \ni \zeta \mapsto \zeta^{1/(1-\alpha)} \in [0, +\infty)$ is convex, by Jensen's inequality, we have

$$\begin{aligned} (L(t))^{1/(1-\alpha)} & \leq \left[(E_2 - E(t))^{1-\alpha} + \frac{\varepsilon}{2} \|\nabla u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 \right. \\ & \left. + \frac{\varepsilon}{\rho+1} \left| \int_{\Omega} u(x, t) |\partial_t u(x, t)|^p \right. \right. \\ & \left. \cdot \partial_t u(x, t) dx \right|^{1/(1-\alpha)} \leq 3^{\alpha/(1-\alpha)} \left[(E_2 - E(t)) \right. \\ & \left. + \left(\frac{\varepsilon}{2} \right)^{1/(1-\alpha)} \left(\|\nabla u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 \right)^{1/(1-\alpha)} \right. \\ & \left. + \left(\frac{\varepsilon}{\rho+1} \right)^{1/(1-\alpha)} \left| \int_{\Omega} u(x, t) |\partial_t u(x, t)|^p \right. \right. \\ & \left. \cdot \partial_t u(x, t) dx \right|^{1/(1-\alpha)} \leq C_{\alpha} \left[(E_2 - E(t)) \right. \\ & \left. + \left(\|\nabla u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 \right)^{1/(1-\alpha)} + \left| \int_{\Omega} u(x, t) \right. \right. \end{aligned}$$

$$\begin{aligned} & \left. \cdot |\partial_t u(x, t)|^p \partial_t u(x, t) dx \right|^{1/(1-\alpha)} \Big], \\ & \quad \forall t \in [0, +\infty). \end{aligned} \quad (36)$$

Since $T^* = +\infty$, $\|\partial_t u(\cdot, t)\|_{L^{\rho+2}(\Omega)}^{\rho+2} + \|\nabla u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 + \|u(\cdot, t)\|_{L^p(\Omega)}^p$ is bounded by some constant C . Since $E_2 - E(t)$ is increasing, we obtain

$$\begin{aligned} & \left(\|\nabla u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 \right)^{1/(1-\alpha)} \leq C^{1/(1-\alpha)} \\ & \leq \frac{C^{1/(1-\alpha)}}{E_2 - E(0)} (E_2 - E(t)), \quad \forall t \in [0, +\infty). \end{aligned} \quad (37)$$

On the other hand, we have the following sequence of inequalities:

$$\begin{aligned} & \left| \int_{\Omega} u(x, t) |\partial_t u(x, t)|^p \partial_t u(x, t) dx \right|^{1/(1-\alpha)} \\ & \leq \left(\|u(\cdot, t)\|_{L^{\rho+2}(\Omega)} \right. \\ & \cdot \left. \left\| |\partial_t u(x, t)|^p \partial_t u(x, t) \right\|_{L^{(\rho+2)/(\rho+1)}(\Omega)} \right)^{1/(1-\alpha)} \\ & = \|u(\cdot, t)\|_{L^{\rho+2}(\Omega)}^{1/(1-\alpha)} \|\partial_t u(\cdot, t)\|_{L^{\rho+2}(\Omega)}^{(\rho+1)/(1-\alpha)} \\ & \leq C \|u(\cdot, t)\|_{L^p(\Omega)}^{1/(1-\alpha)} \|\partial_t u(\cdot, t)\|_{L^{\rho+2}(\Omega)}^{(\rho+1)/(1-\alpha)} \\ & \leq C \left(\|u(\cdot, t)\|_{L^p(\Omega)}^{1/(1-\alpha) \cdot ((1-\alpha)(\rho+2)/(1-\alpha\rho-2\alpha))} \right. \\ & \left. + \|\partial_t u(\cdot, t)\|_{L^{\rho+2}(\Omega)}^{(\rho+1)/(1-\alpha) \cdot ((1-\alpha)(\rho+2)/(\rho+1))} \right) \\ & = C \left(\|u(\cdot, t)\|_{L^p(\Omega)}^{(\rho+2)/(1-\alpha\rho-2\alpha)} + \|\partial_t u(\cdot, t)\|_{L^{\rho+2}(\Omega)}^{\rho+2} \right), \\ & \quad \forall t \in [0, +\infty), \end{aligned} \quad (38)$$

where $C > 0$ depends on the measure $\text{meas } \Omega$ of Ω ; “ $\leq$ ” in the first line follows from Hölder's inequality; “ $\leq$ ” in the third line follows from the continuous embedding $L^{\rho+2}(\Omega) \hookrightarrow L^p(\Omega)$; “ $\leq$ ” follows from Hölder's inequality; “ $\leq$ ” in the fourth line follows from Young's inequality (see Lemma 2) and from

$$\frac{1-\alpha\rho-2\alpha}{(1-\alpha)(\rho+2)} + \frac{\rho+1}{(1-\alpha)(\rho+2)} = 1. \quad (39)$$

We fix α such that

$$0 < \alpha < \min \left(\frac{1}{\rho+2} - \frac{1}{p}, \frac{1}{m} - \frac{1}{p} \right), \quad (40)$$

and hence

$$\begin{aligned} & \frac{\rho+2}{1-\alpha\rho-2\alpha} = \left(\frac{\rho+2}{1-\alpha\rho-2\alpha} - p \right) + p \\ & = \frac{p(\rho+2)}{1-\alpha\rho-2\alpha} \left[\alpha - \left(\frac{1}{\rho+2} - \frac{1}{p} \right) \right] + p \\ & < p. \end{aligned} \quad (41)$$

By Remark 14, this implies

$$\begin{aligned}
& \|u(\cdot, t)\|_{L^p(\Omega)}^{(\rho+2)/(1-\alpha\rho-2\alpha)} = \|u(\cdot, t)\|_{L^p(\Omega)}^p \\
& \cdot \|u(\cdot, t)\|_{L^p(\Omega)}^{(\rho+2)/(1-\alpha\rho-2\alpha)-p} = \|u(\cdot, t)\|_{L^p(\Omega)}^p \\
& \cdot \left(\|u(\cdot, t)\|_{L^p(\Omega)}^p \right)^{((\rho+2)/(1-\alpha\rho-2\alpha))[\alpha-(1/(\rho+2)-1/p)]} \\
& \leq C \|u(\cdot, t)\|_{L^p(\Omega)}^p \\
& \cdot (E_2 - E(t))^{((\rho+2)/(1-\alpha\rho-2\alpha))[\alpha-(1/(\rho+2)-1/p)]} \\
& \leq C \|u(\cdot, t)\|_{L^p(\Omega)}^p \\
& \cdot (E_2 - E(0))^{((\rho+2)/(1-\alpha\rho-2\alpha))[\alpha-(1/(\rho+2)-1/p)]}, \\
& \quad \forall t \in [0, +\infty),
\end{aligned} \tag{42}$$

where “ $\leq$ ” in the third line follows from

$$\begin{aligned}
E_2 - E(t) &= E_2 - \frac{1}{\rho+2} \|\partial_t u(\cdot, t)\|_{L^{\rho+2}(\Omega)}^{\rho+2} \\
&\quad - \frac{1}{2} \left(1 - \int_0^t g(\tau) d\tau \right) \|\nabla u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 \\
&\quad - \frac{1}{2} (g \diamond \nabla u)(t) + \frac{1}{p} \|u(\cdot, t)\|_{L^p(\Omega)}^p \\
&\leq E_1 \\
&\quad - \frac{1}{2} \left(1 - \int_0^t g(\tau) d\tau \right) \|\nabla u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 \\
&\quad + \frac{1}{p} \|u(\cdot, t)\|_{L^p(\Omega)}^p \\
&\leq E_1 - \frac{1}{2} \beta^2 + \frac{1}{p} \|u(\cdot, t)\|_{L^p(\Omega)}^p \\
&\leq \left(\frac{1}{2} - \frac{1}{p} \right) (B_1)^{-2p/(p-2)} - \frac{1}{2} (B_1)^{-2p/(p-2)} \\
&\quad + \frac{1}{p} \|u(\cdot, t)\|_{L^p(\Omega)}^p \\
&= -\frac{1}{p} (B_1)^{-2p/(p-2)} + \frac{1}{p} \|u(\cdot, t)\|_{L^p(\Omega)}^p \\
&\leq \frac{1}{p} \|u(\cdot, t)\|_{L^p(\Omega)}^p, \quad \forall t \in [0, +\infty).
\end{aligned} \tag{43}$$

Combine (38) and (42), to yield

$$\begin{aligned}
& \left| \int_{\Omega} u(x, t) |\partial_t u(x, t)|^p \partial_t u(x, t) dx \right|^{1/(1-\alpha)} \\
& \leq C \left\{ \|\partial_t u(\cdot, t)\|_{L^{\rho+2}(\Omega)}^{\rho+2} + \|u(\cdot, t)\|_{L^p(\Omega)}^p \right. \\
& \quad \cdot (E_2 - E(0))^{((\rho+2)/(1-\alpha\rho-2\alpha))[\alpha-(1/(\rho+2)-1/p)]} \Big\}, \\
& \quad \forall t \in [0, +\infty).
\end{aligned} \tag{44}$$

This, together with (37), implies directly the following differential inequality:

$$L'(t) \geq C(L(t))^{1/(1-\alpha)}, \quad \forall t \in [0, +\infty), \tag{45}$$

which implies in turn

$$L(t) \geq \left[(L(0))^{\alpha/(\alpha-1)} - \frac{C\alpha t}{1-\alpha} \right]^{-(1-\alpha)/\alpha}, \tag{46}$$

$$\forall t \in [0, +\infty),$$

where $C > 0$ depends on p, ρ and α . The proof is complete. $\square$

Proof of Theorem 12. Let $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, and let $u \in \mathcal{C}([0, T_{\max}); H_0^1(\Omega)) \cap \mathcal{C}^1([0, T_{\max}); L^2(\Omega))$, where $T_{\max}$ is the maximal time of existence, be the solution to IBVP (1) associated with the initial datum $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$. Our objective is to prove $T_{\max} = +\infty$. We achieve this goal by contradiction. Assume to the contrary $T_{\max} < +\infty$, and we associate the following functional to u

$$\begin{aligned}
F(t) &= \frac{1}{\rho+2} \|\partial_t u(\cdot, t)\|_{L^{\rho+2}(\Omega)}^{\rho+2} \\
&\quad + \frac{1}{2} \left(1 - \int_0^t g(\tau) d\tau \right) \|\nabla u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 \\
&\quad + \frac{1}{2} (g \diamond \nabla u)(t) + \frac{1}{p} \|u(\cdot, t)\|_{L^p(\Omega)}^p \\
&= E(t) + \frac{2}{p} \|u(\cdot, t)\|_{L^p(\Omega)}^p, \quad \forall t \in [0, T_{\max}),
\end{aligned} \tag{47}$$

where $E(t)$ is given as in (14). By using (16), we can deduce from Assumption 3 the following inequality

$$\begin{aligned}
F'(t) &= E'(t) - 2 \int_{\Omega} |u(x, t)|^{p-2} u(x, t) \partial_t u(x, t) dx \\
&= -\|\partial_t u(\cdot, t)\|_{L^m(\Omega)}^m - \|\nabla \partial_t u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 \\
&\quad + \frac{1}{2} (g' \diamond \nabla u)(t) - \frac{1}{2} g(t) \|\nabla u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 \\
&\quad + 2 \int_{\Omega} |u(x, t)|^{p-2} u(x, t) \partial_t u(x, t) dx \\
&\leq 2 \int_{\Omega} |u(x, t)|^{p-2} u(x, t) \partial_t u(x, t) dx \\
&\quad - \|\nabla \partial_t u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2, \quad \forall t \in [0, T_{\max}).
\end{aligned} \tag{48}$$

By Young's inequality (see Lemma 2), we have

$$\begin{aligned}
& \int_{\Omega} |u(x, t)|^{p-2} u(x, t) \partial_t u(x, t) dx \\
& \leq \left| \int_{\Omega} |u(x, t)|^{p-2} u(x, t) \partial_t u(x, t) dx \right| \\
& \leq \int_{\Omega} |u(x, t)|^{p-1} |\partial_t u(x, t)| dx \\
& \leq \int_{\Omega} (\delta_3 |\partial_t u(x, t)|^p + C_{\delta_3} |u(x, t)|^p) dx
\end{aligned}$$

$$\begin{aligned}
&= \delta_3 \|\partial_t u(\cdot, t)\|_{L^p(\Omega)}^p + C_{\delta_3} \|u(\cdot, t)\|_{L^p(\Omega)}^p \\
&= \frac{\delta_3 (\rho + 2 - p)}{\rho} \|\partial_t u(\cdot, t)\|_{L^p(\Omega)}^2 \\
&\quad + \frac{\delta_3 (p - 2)}{\rho} \|\partial_t u(\cdot, t)\|_{L^p(\Omega)}^{\rho+2} + C_{\delta_3} \|u(\cdot, t)\|_{L^p(\Omega)}^p \\
&\leq \delta_3 B^2 \|\nabla \partial_t u(\cdot, t)\|_{L^p(\Omega; \mathbb{R}^N)}^2 + C_{\delta_3} \|\partial_t u(\cdot, t)\|_{L^{\rho+2}(\Omega)}^{\rho+2} \\
&\quad + C_{\delta_3} \|u(\cdot, t)\|_{L^p(\Omega)}^p, \quad \forall t \in [0, T_{\max}), \tag{49}
\end{aligned}$$

where δ_3 is a positive number which would be specified explicitly later and B is the best Poincaré's embedding constant (see (11) for the definition). Plug (49) into (48) and write the result compactly, to yield

$$\begin{aligned}
F'(t) &\leq - (1 - 2\delta_3 B^2) \|\nabla \partial_t u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 \\
&\quad + 2C_{\delta_3} \|\partial_t u(\cdot, t)\|_{L^{\rho+2}(\Omega)}^{\rho+2} \\
&\quad + 2C_{\delta_3} \|u(\cdot, t)\|_{L^p(\Omega)}^p, \quad \forall t \in [0, T_{\max}). \tag{50}
\end{aligned}$$

By choosing $\delta_3 = 1/2B^2$, we obtain the following differential inequality

$$F'(t) \leq CF(t), \quad \forall t \in [0, T_{\max}), \tag{51}$$

which implies immediately

$$F(t) \leq e^{Ct} F(0) \leq e^{CT_{\max}} F(0), \quad \forall t \in [0, T_{\max}), \tag{52}$$

where $C > 0$ is a constant depending on B , p and ρ . By the definition (47), we have

$$\begin{aligned}
&\|\partial_t u(\cdot, t)\|_{L^{\rho+2}(\Omega)}^{\rho+2} + \|\nabla u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)}^2 \\
&\leq \max\left(\rho + 2, \frac{2}{\ell}\right) F(t) \\
&\leq \max\left(\rho + 2, \frac{2}{\ell}\right) e^{CT_{\max}} F(0), \quad \forall t \in [0, T_{\max}). \tag{53}
\end{aligned}$$

Now choose $t_0 \in (0, T_{\max})$ such that

$$T_{\max} - t_0 < T^{\max}, \tag{54}$$

where $T^{\max}$ is the maximal time of existence of solutions whose initial data's norms are bounded by $\max(\rho + 2, 2/\ell)e^{CT_{\max}}F(0)$. By Theorem 9, IBVP (1) supplemented by the initial condition

$$(\tilde{u}(\cdot, 0), \partial_t \tilde{u}(\cdot, 0)) = (u(\cdot, t_0), \partial_t u(\cdot, t_0)) \tag{55}$$

admits a unique solution

$$\tilde{u} \in \mathcal{C}([0, T_{\max}); H_0^1(\Omega)) \cap \mathcal{C}^1([0, T_{\max}); L^2(\Omega)) \tag{56}$$

with $T^{\max}$ the maximal time of existence of the solution. Write

$$\hat{u}(x, t) = \begin{cases} u(x, t) & \text{if } (x, t) \in \Omega \times [0, t_0], \\ \tilde{u}(x, t) & \text{if } (x, t) \in \Omega \times [t_0, t_0 + T^{\max}]. \end{cases} \tag{57}$$

By the construction, $\hat{u}$ is the unique solution to IBVP (1) on $[0, t_0 + T^{\max})$ associated with the initial datum $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ which is given at the very beginning of the proof. By the definition of $T_{\max}$, $T_{\max} \geq t_0 + T^{\max}$. This contradicts (54).

Therefore, $T_{\max} = +\infty$. The proof is complete. $\square$

4. Conclusion

In this paper, we studied a class of viscoelastic wave equations which incorporate a pure power source term, a nonlinear frictional damping term, and a strong linear damping term of Kelvin-Voigt type. We proved first a finite time blow-up result by a convexity argument and obtained second a global existence result by a priori estimates for the problem in question. The modified energy functional (14) played an important role in finishing the proofs of the finite time blow-up result and the global existence result.

There remains a problem concerning IBVP (1), that is, the problem of finding or estimating the lifespan for every solution u to IBVP (1). When Assumptions 3, 4, and 6 are fulfilled, all the solutions exist globally, and hence their lifespans are $+\infty$. When Assumptions 3, 4, and 5 are fulfilled, the problem of finding lifespans becomes very complicated; in this case, for those initial data satisfying the hypothesis of Theorem 12, we can deduce from (46) the upper bound estimate for the lifespans:

$$T^u = \frac{1 - \alpha}{C\alpha} (L(0))^{\alpha/(\alpha-1)}, \tag{58}$$

where $C > 0$ depends on p , ρ , α and the initial data (u^0, u^1) . In view of this, we are tempted to understand better the following questions:

- (i) How to obtain an estimate better than (58) upon lifespans?
- (ii) Let Assumptions 3, 4, and 5 be fulfilled. How to obtain an estimate upon lifespans for solutions whose initial data do not satisfy the hypothesis of Theorem 12?

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

Fosheng is supported by the Initial Foundation of Mianyang Teachers' College (Grant no. QD2016A003). Chengqiang is supported by NSFC (nos. 11701050 and 11571244), by JG Program (no. 2017JG13) of Chengdu Normal University, and by SCJYT Program (no. 18ZB0098) of Sichuan Province, China.

References

- [1] M. Fabrizio and A. Morro, *Mathematical problems in linear viscoelasticity*. SIAM Studies in Applied Mathematics, Society for

- Industrial and Applied Mathematics (SIAM), Philadelphia, PA, USA, 1992.
- [2] R. M. Christensen, *Theory of Viscoelasticity: An Introduction*, Academic Press, New York, NY, USA, 1977.
 - [3] L. Boltzmann, "Zur Theorie der Elastischen Nachwirkung," *Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften. Mathematisch-Naturwissenschaftliche Classe*, vol. 70, no. 2, pp. 1–174, 1874.
 - [4] S. A. Messaoudi, "General decay of solutions of a viscoelastic equation," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 2, pp. 1457–1467, 2008.
 - [5] S. A. Messaoudi and W. Al-Khulaifi, "General and optimal decay for a quasilinear viscoelastic equation," *Applied Mathematics Letters*, vol. 66, pp. 16–22, 2017.
 - [6] M. M. Cavalcanti, V. N. Domingos Cavalcanti, and J. Ferreira, "Existence and uniform decay for a non-linear viscoelastic equation with strong damping," *Mathematical Methods in the Applied Sciences*, vol. 24, no. 14, pp. 1043–1053, 2001.
 - [7] J. E. Muñoz Rivera, E. C. Lapa, and R. Barreto, "Decay rates for viscoelastic plates with memory," *Journal of Elasticity*, vol. 44, no. 1, pp. 61–87, 1996.
 - [8] M. M. Cavalcanti, V. N. Domingos Cavalcanti, and J. A. Soriano, "Global existence and asymptotic stability for viscoelastic problems," *Differential and Integral Equations*, vol. 15, no. 6, pp. 731–748, 2002.
 - [9] M. M. Cavalcanti, V. N. D. Cavalcanti, and J. A. Soriano, "Exponential decay for the solution of semilinear viscoelastic wave equation with localized damping," *Electronic Journal of Differential Equations*, vol. 44, pp. 1–14, 2002.
 - [10] S. Berrimi and S. A. Messaoudi, "Existence and decay of solutions of a viscoelastic equation with a nonlinear source," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 10, pp. 2314–2331, 2006.
 - [11] S. A. Messaoudi and N.-E. Tatar, "Exponential and polynomial decay for a quasilinear viscoelastic equation," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 4, pp. 785–793, 2008.
 - [12] M. M. Cavalcanti, V. N. Domingos Cavalcanti, and P. Martinez, "General decay rate estimates for viscoelastic dissipative systems," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 68, no. 1, pp. 177–193, 2008.
 - [13] Y. Wang and Y. Wang, "Exponential energy decay of solutions of viscoelastic wave equations," *Journal of Mathematical Analysis and Applications*, vol. 347, no. 1, pp. 18–25, 2008.
 - [14] S. A. Messaoudi and M. I. Mustafa, "A general stability result for a quasilinear wave equation with memory," *Nonlinear Analysis: Real World Applications*, vol. 14, no. 4, pp. 1854–1864, 2013.
 - [15] H. A. Levine, "Some additional remarks on the nonexistence of global solutions to nonlinear wave equations," *SIAM Journal on Mathematical Analysis*, vol. 5, no. 1, pp. 138–146, 1974.
 - [16] H. A. Levine and J. Serrin, "Global nonexistence theorems for quasilinear evolution equations with dissipation," *Archive for Rational Mechanics and Analysis*, vol. 137, no. 4, pp. 341–361, 1997.
 - [17] G. Todorova, "Stable and unstable sets for the Cauchy problem for a nonlinear wave equation with nonlinear damping and source terms," *Journal of Mathematical Analysis and Applications*, vol. 239, no. 2, pp. 213–226, 1999.
 - [18] E. Vitillaro, "Global nonexistence theorems for a class of evolution equations with dissipation," *Archive for Rational Mechanics and Analysis*, vol. 149, no. 2, pp. 151–182, 1999.
 - [19] G. Todorova, "Cauchy problem for a nonlinear wave equation with nonlinear damping and source terms," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 41, no. 7-8, pp. 891–905, 2000.
 - [20] S. A. Messaoudi, "Blow up and global existence in a nonlinear viscoelastic wave equation," *Mathematische Nachrichten*, vol. 260, pp. 58–66, 2003.
 - [21] S. A. Messaoudi, "Blow-up of positive-initial-energy solutions of a nonlinear viscoelastic hyperbolic equation," *Journal of Mathematical Analysis and Applications*, vol. 320, no. 2, pp. 902–915, 2006.
 - [22] H. Song and C. Zhong, "Blow-up of solutions of a nonlinear viscoelastic wave equation," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 5, pp. 3877–3883, 2010.
 - [23] H. T. Song and D. S. Xue, "Blow up of solutions for a viscoelastic wave equation with strong damping," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 109, pp. 245–251, 2014.
 - [24] M. M. Cavalcanti and H. P. Oquendo, "Frictional versus viscoelastic damping in a semilinear wave equation," *SIAM Journal on Control and Optimization*, vol. 42, no. 4, pp. 1310–1324, 2003.
 - [25] W. Liu, "General decay and blow-up of solution for a quasilinear viscoelastic problem with nonlinear source," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 6, pp. 1890–1904, 2010.
 - [26] H. Song, "Global nonexistence of positive initial energy solutions for a viscoelastic wave equation," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 125, pp. 260–269, 2015.
 - [27] J. Hao and H. Wei, "Blow-up and global existence for solution of quasilinear viscoelastic wave equation with strong damping and source term," *Boundary Value Problems*, vol. 65, pp. 1–12, 2017.
 - [28] G. Li, Y. Sun, and W. Liu, "On asymptotic behavior and blow-up of solutions for a nonlinear viscoelastic petrovsky equation with positive initial energy," *Journal of Function Spaces and Applications*, vol. 2013, Article ID 905867, pp. 1–7, 2013.

Submit your manuscripts at
www.hindawi.com