Hindawi
Advances in Mathematical Physics
Volume 2018, Article ID 9282905, 11 pages
https://doi.org/10.1155/2018/9282905

Research Article

Constructions of L_∞ Algebras and Their Field Theory Realizations

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Received 2 July 2018; Revised 3 October 2018; Accepted 14 October 2018; Published 1 November 2018

Academic Editor: Andrei D. Mironov

We construct L_∞ algebras for general "initial data" given by a vector space equipped with an antisymmetric bracket not necessarily satisfying the Jacobi identity. We prove that any such bracket can be extended to a 2-term L_∞ algebra on a graded vector space of twice the dimension, with the 3-bracket being related to the Jacobiator. While these L_∞ algebras always exist, they generally do not realize a nontrivial symmetry in a field theory. In order to define L_∞ algebras with genuine field theory realizations, we prove the significantly more general theorem that if the Jacobiator takes values in the image of any linear map that defines an ideal there is a 3-term L_∞ algebra with a generally nontrivial 4-bracket. We discuss special cases such as the commutator algebra of octonions, its contraction to the "R-flux algebra," and the Courant algebroid.

1. Introduction

Lie groups are ubiquitous in mathematics and theoretical physics as the structures formalizing the notion of continuous symmetries. Their infinitesimal objects are Lie algebras: vector spaces equipped with an antisymmetric bracket satisfying the Jacobi identity. In various contexts it is advantageous (if not strictly required) to generalize the notion of a Lie algebra so that the brackets do not satisfy the Jacobi identity. Rather, in addition to the "2-bracket," general "n-brackets" ℓ_n are introduced on a graded vector space for $n = 1, 2, 3, \ldots$, satisfying generalized Jacobi identities involving all brackets. Such structures, referred to as L_∞ or strongly homotopy Lie algebras, first appeared in the physics literature in closed string field theory [1] and in the mathematics literature in topology [2–4]. A closely related cousin of L_∞ algebras is A_∞ algebras, which generalize associative algebras to structures without associativity [5, 6].

Our goal in this paper is to prove general theorems about the existence of L_∞ structures for given "initial data" such as an antisymmetric bracket and to discuss their possible field theory realizations. First, as a warm-up, we answer the following natural question: Given a vector space V with an antisymmetric bracket $[\cdot,\cdot]$, under which conditions can this algebra be extended to an L_∞ algebra with $\ell_2(v,w) = [v,w]$? We will show that this is always possible. More specifically, we will prove the following theorem: The graded vector space $X = X_1 + X_0$, where $X_0 = V$ is the space of degree zero and $X_1 = V^*$ is isomorphic to V and of degree one, carries a 2-term L_∞ structure, meaning that the highest nontrivial product is ℓ_3, which encodes the "Jacobiator" (i.e., the anomaly due to the failure of the original bracket to satisfy the Jacobi identity). We have been informed that this theorem is known to some experts, and one instance of it has been stated in [7], but we have not been able to find a proof in the literature. (See also [8, 9] for examples of finite-dimensional L_∞ algebras.)

At first sight the above theorem may shed doubt on the usefulness of L_∞ algebras, since it states that *any* generally non-Lie algebra can be extended to an L_∞ algebra. It should be emphasized, however, that for a generic bracket the resulting structure is quite degenerate in that the 2-term L_∞ algebra may not be extendable further in a nontrivial way, say by including a vector space X_{-1}. Such extensions are particularly important for applications in theoretical physics as here X_{-1} encodes the "space of physical fields", X_0 the space of "gauge parameters," and X_1 the space of "trivial parameters" whose action on fields vanishes [10]. Thus, if X_1 is isomorphic to X_0 there is no nontrivial action of X_0 on the physical fields and hence no genuine field theory realization of the L_∞ algebra. In order to obtain nontrivial field theory realizations we will next prove a much more general theorem that covers the case of the Jacobiator being of a special form. Specifically, we will prove that if the Jacobiator takes values in the image of a linear operator that defines an ideal of the original algebra then there exists a 3-term L_∞ algebra whose highest bracket in general is a nontrivial ℓ_4. A special case is the Courant bracket investigated by Roytenberg and Weinstein [11], for which the 4-bracket trivializes, but which is extendable and realized in string theory, in the form of double field theory [10, 12, 13].

We will illustrate these results with examples. Our investigation arose in fact out of the question whether the nonassociative octonions (more precisely, the 7-dimensional commutator algebra of imaginary octonions) can be viewed as part of an L_∞ algebra. Our first theorem implies that the answer is affirmative, with the total graded space being 14-dimensional, which we will see is minimal. However, given the theorem, the existence of this L_∞ structure does not express a nontrivial fact about the octonions. Moreover, this L_∞ structure is not extendable, which implies with the results of [10] that the octonions, at least when realized as a 2-term L_∞ algebra, cannot realize a nontrivial gauge symmetry in field theory.

As recently discovered in [14] and further investigated in [15, 16], the octonions are related to the phase space of nongeometric backgrounds in M-theory (nongeometric R-flux or non-geometric Kaluza-Klein monopoles in M-theory). Furthermore, a contraction of the octonions leads to the string theory "R-flux algebra" of [7, 17–20] and also to the "magnetic monopole algebra" of [20–26]. The Jacobiator of the R-flux algebra only takes values in a one-dimensional subspace, and therefore these contracted nonassociative algebras may in fact be extendable. Here it is sufficient to take X_1 to be one-dimensional, leading to an 8-dimensional L_∞ algebra. (A 14-dimensional and hence nonminimal L_∞ realization of the R-flux algebra has already been given in [7].)

The remainder of this paper is organized as follows. In Section 2 we briefly review the axioms of L_∞ algebras. In Section 3 we prove the theorem that for arbitrary 2-bracket as initial data there is an L_∞ structure on the "doubled" vector space. This theorem will then be significantly generalized in Section 4. In Section 5 we discuss examples, such as the octonions, the "R-flux algebra," and the Courant algebroid. In the appendix we prove an analogous result for A_∞ algebras.

2. Axioms of L_∞ Algebras

We begin by stating the axioms of an L_∞ algebra. It is defined on a graded vector space

$$X = \bigoplus_{n\in\mathbb{Z}} X_n, \tag{1}$$

and we refer to elements in X_n as having degree n. We also refer to algebras with $X_n = 0$ for all n with $|n| \geq k$ as a k-term L_∞ algebra. There are a potentially infinite number of generalized multilinear products or brackets ℓ_k having k inputs and intrinsic degree $k-2$, meaning that they take values in a vector space whose degree is given by

$$\deg\left(\ell_k\left(x_1,\ldots,x_k\right)\right) = k-2+\sum_{i=1}^{k}\deg\left(x_i\right). \tag{2}$$

For instance, ℓ_1 has intrinsic degree -1, implying that it acts on the graded vector space according to

$$\cdots \longrightarrow X_1 \xrightarrow{\ell_1} X_0 \xrightarrow{\ell_1} X_{-1} \longrightarrow \cdots \tag{3}$$

Moreover, the brackets are *graded (anti-)commutative* in that, e.g., ℓ_2 satisfies

$$\ell_2\left(x_1,x_2\right) = (-1)^{1+x_1x_2}\,\ell_2\left(x_2,x_1\right), \tag{4}$$

and similarly for all other brackets.

The brackets have to satisfy a (potentially infinite) number of generalized Jacobi identities. In order to state these identities we have to define the Koszul sign $\epsilon(\sigma;x)$ for any σ in the permutation group of k objects and a choice $x = (x_1,\ldots,x_k)$ of k such objects. It can be defined implicitly by considering a graded commutative algebra with

$$x_i\wedge x_j = (-1)^{x_ix_j}\,x_j\wedge x_i, \quad \forall i,j, \tag{5}$$

where in exponents x_i denotes the degree of the corresponding element. The Koszul sign is then inferred from

$$x_1\wedge\ldots\wedge x_k = \epsilon\left(\sigma;x\right)x_{\sigma(1)}\wedge\ldots\wedge x_{\sigma(k)}. \tag{6}$$

The L_∞ relations are given by

$$\begin{aligned}\sum_{i+j=n+1}(-1)^{i(j-1)}\sum_{\sigma}(-1)^{\sigma}\,\epsilon\left(\sigma;x\right)\\ \cdot\,\ell_j\left(\ell_i\left(x_{\sigma(1)},\ldots,x_{\sigma(i)}\right),x_{\sigma(i+1)},\ldots x_{\sigma(n)}\right) = 0,\end{aligned} \tag{7}$$

for each $n = 1,2,3,\ldots$, which indicates the total number of inputs. Here $(-1)^\sigma$ gives a plus sign if the permutation is even and a minus sign if the permutation is odd. Moreover, the inner sum runs, for a given $i,j\geq 1$, over all permutations σ of n objects whose arguments are partially ordered ("unshuffles"), satisfying

$$\begin{aligned}\sigma\left(1\right)\leq\cdots\leq\sigma\left(i\right),\\ \sigma\left(i+1\right)\leq\cdots\leq\sigma\left(n\right).\end{aligned} \tag{8}$$

We will now state these relations explicitly for the values of n relevant for our subsequent analysis. For $n = 1$ the identity reduces to

$$\ell_1(\ell_1(x)) = 0, \tag{9}$$

stating that ℓ_1 is nilpotent, so that (3) is a chain complex. For $n = 2$ the identity reads

$$\begin{aligned}\ell_1(\ell_2(x_1, x_2)) = {} & \ell_2(\ell_1(x_1), x_2) \\ & + (-1)^{x_1} \ell_2(x_1, \ell_1(x_2)),\end{aligned} \tag{10}$$

meaning that ℓ_1 acts like a derivation on the product ℓ_2. For $n = 3$ one obtains

$$\begin{aligned}0 = {} & \ell_1(\ell_3(x_1, x_2, x_3)) + \ell_3(\ell_1(x_1), x_2, x_3) \\ & + (-1)^{x_1} \ell_3(x_1, \ell_1(x_2), x_3) \\ & + (-1)^{x_1+x_2} \ell_3(x_1, x_2, \ell_1(x_3)) \\ & + \ell_2(\ell_2(x_1, x_2), x_3) \\ & + (-1)^{(x_1+x_2)x_3} \ell_2(\ell_2(x_3, x_1), x_2) \\ & + (-1)^{(x_2+x_3)x_1} \ell_2(\ell_2(x_2, x_3), x_1).\end{aligned} \tag{11}$$

We recognize the last three lines as the usual Jacobiator. Thus, this relation encodes the failure of the 2-bracket to satisfy the Jacobi identity in terms of a 1- and 3-bracket and the failure of ℓ_1 to act as a derivation on ℓ_3. Finally, the $n = 4$ relations read

$$\begin{aligned}& \mathcal{O}(x_1, \ldots, x_4) \\ & \quad \equiv -\ell_2(\ell_3(x_1, x_2, x_3), x_4) \\ & \qquad + (-1)^{x_3x_4} \ell_2(\ell_3(x_1, x_2, x_4), x_3) \\ & \qquad + (-1)^{(1+x_1)x_2} \ell_2(x_2, \ell_3(x_1, x_3, x_4)) \\ & \qquad - (-1)^{x_1} \ell_2(x_1, \ell_3(x_2, x_3, x_4)) \\ & \qquad + \ell_3(\ell_2(x_1, x_2), x_3, x_4) \\ & \qquad + (-1)^{1+x_2x_3} \ell_3(\ell_2(x_1, x_3), x_2, x_4) \\ & \qquad + (-1)^{x_4(x_2+x_3)} \ell_3(\ell_2(x_1, x_4), x_2, x_3) \\ & \qquad - \ell_3(x_1, \ell_2(x_2, x_3), x_4) \\ & \qquad + (-1)^{x_3x_4} \ell_3(x_1, \ell_2(x_2, x_4), x_3) \\ & \qquad + \ell_3(x_1, x_2, \ell_2(x_3, x_4)) \\ & \quad = -\ell_1(\ell_4(x_1, x_2, x_3, x_4)) + \ell_4(\ell_1(x_1), x_2, x_3, x_4) \\ & \qquad + (-1)^{x_1} \ell_4(x_1, \ell_1(x_2), x_3, x_4) \\ & \qquad + (-1)^{x_1+x_2} \ell_4(x_1, x_2, \ell_1(x_3), x_4) \\ & \qquad + (-1)^{x_1+x_2+x_3} \ell_4(x_1, x_2, x_3, \ell_1(x_4)),\end{aligned} \tag{12}$$

where we named the l.h.s. $\mathcal{O}(x_1, \ldots, x_4)$ for later convenience. For a 2-term L_∞ algebra there are no 4-brackets and hence the above right-hand side is zero. The $n = 4$ relation then poses a nontrivial constraint on ℓ_2 and ℓ_3, while all higher L_∞ relations will be automatically satisfied.

3. A Warm-Up Theorem

We now prove the first theorem stated in the introduction. Consider an algebra $(V, [\cdot,\cdot])$ with bilinear antisymmetric 2-bracket, i.e.,

$$[v, w] = -[w, v] \quad \forall v, w \in V, \tag{13}$$

but we do *not* assume that the bracket satisfies any further constraints. In particular, the Jacobi identity is generally not satisfied, so that the *Jacobiator*

$$\mathrm{Jac}(u, v, w) \equiv [[u, v], w] + [[v, w], u] + [[w, u], v], \tag{14}$$

in general is nonzero. We then have the following.

Theorem 1. *The graded vector space*

$$X = X_1 + X_0, \tag{15}$$

where $X_0 = V$ and $X_1 = V^*$ with V^* a vector space isomorphic to V, carries a 2-term L_∞ structure whose nontrivial brackets are given by

$$\ell_1(v^*) = v, \tag{16}$$

$$\ell_2(v, w) = [v, w], \tag{17}$$

$$\ell_2(v^*, w) = [v, w]^*, \tag{18}$$

$$\ell_3(u, v, w) = -\mathrm{Jac}(u, v, w)^*. \tag{19}$$

Comment. We denote the elements of V^* by v^*, w^*, etc., and the isomorphism by

$$\begin{aligned}{}^* : V &\longrightarrow V^*, \\ v &\longmapsto v^*,\end{aligned} \tag{20}$$

and similarly for its inverse. For instance, if V carries a nondegenerate metric we can take V^* to be the dual vector space of V and the isomorphism to be the canonical isomorphism. (More simply, we can think of V^* as a second copy of V and of the isomorphism as the identity, but at least for notational reasons it is important to view V and V^* as different objects.)

Proof. The proof proceeds straightforwardly by fixing the products so that the $n = 1, 2, 3$ relations are partially satisfied and then verifying that in fact all L_∞ relations are satisfied. First, ℓ_1 maps $X_1 = V^*$ to $X_0 = V$, and we take it to be given by the (inverse) isomorphism (20),

$$\begin{aligned}\forall v^* \in X_1,\ v \in X_0 : \quad & \ell_1(v^*) = v, \\ & \ell_1(v) = 0.\end{aligned} \tag{21}$$

The second relation in here is necessary because there is no space X_{-1} in (15). The $n = 1$ relations $\ell_1^2 = 0$ then hold trivially.

Next, we fix the ℓ_2 product by requiring $\ell_2(v, w) = [v, w]$ on $X_0 = V$ and imposing the $n = 2$ relation (10). For arguments of total degree 0 this relation is trivial because of the second relation in (21). For arguments of total degree 1 we have

$$\begin{aligned}\ell_1(\ell_2(v^*, w)) &= \ell_2(\ell_1(v^*), w) - \ell_2(v^*, \ell_1(w)) \\ &= \ell_2(v, w) = [v, w],\end{aligned} \tag{22}$$

where we used (21). Using (21) on the l.h.s. we infer

$$\ell_2(v^*, w) = [v, w]^* \Longleftrightarrow \ell_2(w, v^*) = [w, v]^*. \tag{23}$$

Since there is no space X_2 we have $\ell_2(v^*, w^*) = 0$. This is consistent with the $n = 2$ relation (10) for arguments of total degree 2:

$$\begin{aligned}0 &= \ell_1(\ell_2(v^*, w^*)) = \ell_2(v, w^*) - \ell_2(v^*, w) \\ &= [v, w]^* - [v, w]^* = 0,\end{aligned} \tag{24}$$

where we used (23). Thus, all $n = 2$ relations are satisfied.

Let us now consider the $n = 3$ relations (11). For arguments of total degree 0 (i.e., all taking values in X_0), it reads

$$\begin{aligned}0 &= \ell_1(\ell_3(u, v, w)) + \ell_2(\ell_2(u, v), w) \\ &\quad + \ell_2(\ell_2(v, w), u) + \ell_2(\ell_2(w, u), v) \\ &= \ell_1(\ell_3(u, v, w)) + \mathrm{Jac}(u, v, w),\end{aligned} \tag{25}$$

from which we infer

$$\ell_3(u, v, w) = -\mathrm{Jac}(u, v, w)^* \in X_1. \tag{26}$$

Due to the antisymmetry of the bracket $[\cdot, \cdot]$, the Jacobiator is completely antisymmetric in all arguments, and (26) is consistent with the required graded commutativity of ℓ_3. Since there is no space X_2, ℓ_3 is trivial for any arguments in X_1. We have thus determined all nontrivial n-brackets.

So far we have verified the $n = 1, 2$ relations and the $n = 3$ relation for arguments of total degree 0. We now verify the remaining L_∞ relations. The $n = 3$ relation for arguments of total degree 1 reads

$$\begin{aligned}0 &= \ell_3(\ell_1(u^*), v, w) + \ell_2(\ell_2(u^*, v), w) \\ &\quad + \ell_2(\ell_2(w, u^*), v) + \ell_2(\ell_2(v, w), u^*) \\ &= -\mathrm{Jac}(u, v, w)^* + \mathrm{Jac}(u, v, w)^*,\end{aligned} \tag{27}$$

and is thus satisfied. The $n = 3$ relations for arguments of total degree larger than 1 are trivially satisfied, completing the proof of all $n = 3$ relations.

Finally, we have to verify the $n = 4$ relations. Since there is no nontrivial ℓ_4 these require that the left-hand side of (12) vanishes identically for ℓ_2 and ℓ_3 defined above. This follows by a direct computation that we display in detail. First, for arguments $v_1, v_2, v_3, v_4 \in X_0$ of total degree 0 one may verify that (12) is completely antisymmetric in the four arguments. Writing $\sum_{\mathrm{anti}}$ for the totally antisymmetrized sum (carrying $4! = 24$ terms and pre-factor $1/4!$) we then compute for the left-hand side of (12)

$$\begin{aligned}\mathcal{O}(v_1, \ldots, v_4) &= \sum_{\mathrm{anti}} (-4\ell_2(\ell_3(v_1, v_2, v_3), v_4) \\ &\quad + 6\ell_3(\ell_2(v_1, v_2), v_3, v_4)) \\ &= \sum_{\mathrm{anti}} \big(4[\mathrm{Jac}(v_1, v_2, v_3), v_4]^* \\ &\quad - 6\,\mathrm{Jac}([v_1, v_2], v_3, v_4)^*\big) \\ &= \sum_{\mathrm{anti}} \big(12[[[v_1, v_2], v_3], v_4]^* \\ &\quad - 6\big(2[[[v_1, v_2], v_3], v_4]^* + [[v_3, v_4], [v_1, v_2]]^*\big)\big) \\ &= 0.\end{aligned} \tag{28}$$

Here we used repeatedly the total antisymmetry in the four arguments, in particular in the last step that under the sum $[[v_3, v_4], [v_1, v_2]]^*$ then vanishes. The $n = 4$ relations for arguments of total degree 1 or higher are trivially satisfied because they would have to take values in spaces of degree 2 or higher, which do not exist. The L_∞ relations for $n > 4$ are trivially satisfied for the same reason. This completes the proof. □

4. Main Theorem

The above theorem states that an arbitrary bracket can be extended to an L_∞ algebra. For generic brackets, this L_∞ structure is, however, quite degenerate in that it may not be extendable further, say by adding a further space X_{-1}. Indeed, if the violation of the Jacobi identity is "maximal" and the Jacobiator takes values in all of V, the space X_1 has to be as large as V, and the image of the map $\ell_1 : X_1 \longrightarrow X_0$ equals $X_0 = V$. Consequently, one cannot introduce a further space X_{-1} together with a nontrivial $\ell_1 : X_0 \longrightarrow X_{-1}$ satisfying $\ell_1^2 = 0$. Since in physical applications X_{-1} serves as the space of fields, such brackets do not lead to L_∞ algebras encoding a nontrivial gauge symmetry.

More interesting situations arise when the Jacobiator takes values in a proper subspace $U \subset V$, for then it is sufficient to set $X_1 = U$ and to take $\ell_1 = \iota$ to be the "inclusion" defined for any $u \in U$ by $\iota(u) = u$, viewing u as an element of V. Indeed, it is easy to verify, provided the subspace forms an ideal (i.e., $\forall u \in U, v \in V : [u, v] \in U$), that the above proof goes through as before. In this case, further extensions of the L_∞ algebra may exist. In the following we will prove a yet more general theorem that is applicable to situations where the Jacobiator takes values in the image of a linear map that itself may have a nontrivial kernel. Then there is an extension to a 3-term L_∞ algebra that generally requires a nontrivial 4-bracket.

Theorem 2. *Let $(V, [\cdot,\cdot])$ be an algebra with bilinear antisymmetric 2-bracket as in Section 3, and let $\mathscr{D} : U \longrightarrow V$ be a linear map satisfying the closure conditions*

$$[\mathrm{Im}(\mathscr{D}), V] \subset \mathrm{Im}(\mathscr{D}), \tag{29}$$

together with the Jacobiator relation

$$\forall v_1, v_2, v_3 \in V: \quad \mathrm{Jac}(v_1, v_2, v_3) \in \mathrm{Im}(\mathscr{D}), \tag{30}$$

where $\mathrm{Im}(\mathscr{D})$ and $\ker(\mathscr{D})$ denote image and kernel of $\mathscr{D}$, respectively. Then there exists a 3-term L_∞ structure with $\ell_2(v, w) = [v, w]$ on the graded vector space with

$$X_2 \xrightarrow{\ell_1=\iota} X_1 \xrightarrow{\ell_1=\mathscr{D}} X_0, \tag{31}$$

where $X_0 = V$, $X_1 = U$, $X_2 = \ker(\mathscr{D})$ and ι denotes the inclusion of $\ker(\mathscr{D})$ into U. The highest nontrivial bracket in general is given by the 4-bracket (and the complete list of nontrivial brackets is given in eq. (54) below).

Notation and Comments. We denote the elements of V by $v, w, \ldots$, the elements of U by $\alpha, \beta, \ldots$ and the elements of $\ker(\mathscr{D})$ by $c, c', \ldots$ The condition (30) implies that there is a multilinear and totally antisymmetric map $f : V^{\otimes 3} \longrightarrow U$ so that

$$\forall v_1, v_2, v_3 \in V: \quad \mathrm{Jac}(v_1, v_2, v_3) = \mathscr{D} f(v_1, v_2, v_3). \tag{32}$$

The condition (29) states that the bracket of an arbitrary $v \in V$ with $\mathscr{D}\alpha$, $\alpha \in U$, lies in the image of $\mathscr{D}$, i.e., we can write

$$\begin{aligned} \forall v \in V, \alpha \in U: \quad & [\mathscr{D}\alpha, v] = \mathscr{D}(v(\alpha)), \\ & v(\alpha) \in U. \end{aligned} \tag{33}$$

We can think of the operation on the r.h.s. as defining for each $v \in V$ a map on U, $\alpha \longmapsto v(\alpha) \in U$. This map is defined by (33) only up contributions in the kernel, as is the function f in (32), but the following construction goes through for any choice of functions satisfying (33), (32). (The algebras resulting for different choices of these functions are almost certainly equivalent under suitably defined L_∞ isomorphisms; see, e.g., [27], but we leave a detailed analysis for future work.)

Proof. As for Theorem 1, the proof proceeds by determining the n-brackets from the L_∞ relations as far as possible and then proving that in fact all relations are satisfied. The $n = 1$ relations $\ell_1^2 = 0$ for ℓ_1 defined in (31) are satisfied by definition since $\mathscr{D}(\iota(c)) = 0$ for all $c \in \ker(\mathscr{D})$. In the following we systematically go through all relations for $n = 1, \ldots, 5$.

$n = 2$ relations: The $n = 2$ relations are satisfied for arguments of total degree zero, since ℓ_1 acts trivially on X_0. For arguments $\alpha \in X_1$, $v \in X_0$ of total degree 1 we need

$$\ell_1(\ell_2(\alpha, v)) = \ell_2(\ell_1(\alpha), v) = [\mathscr{D}\alpha, v] = \mathscr{D}(v(\alpha)), \tag{34}$$

where we used (33). As the l.h.s. equals $\mathscr{D}(\ell_2(\alpha, v))$, this relation is satisfied if we set

$$\ell_2(\alpha, v) = v(\alpha) \in X_1. \tag{35}$$

For arguments $\alpha, \beta \in X_1$ of total weight 2 we compute

$$\begin{aligned} \ell_1(\ell_2(\alpha, \beta)) &= \ell_2(\ell_1(\alpha), \beta) - \ell_2(\alpha, \ell_1(\beta)) \\ &= -\ell_2(\beta, \mathscr{D}\alpha) - \ell_2(\alpha, \mathscr{D}\beta) \\ &= -(\mathscr{D}\alpha)(\beta) - (\mathscr{D}\beta)(\alpha), \end{aligned} \tag{36}$$

using (35) in the last step. As ℓ_1 on the l.h.s. acts by inclusion, we can satisfy this relation by setting

$$\ell_2(\alpha, \beta) = -(\mathscr{D}\alpha)(\beta) - (\mathscr{D}\beta)(\alpha) \in \ker(\mathscr{D}), \tag{37}$$

but it remains to prove that the r.h.s. indeed takes values in the kernel. This follows by setting $v = \mathscr{D}\beta$ in (33):

$$\begin{aligned} & [\mathscr{D}\alpha, \mathscr{D}\beta] = \mathscr{D}((\mathscr{D}\beta)(\alpha)) \Longrightarrow \\ & \mathscr{D}((\mathscr{D}\alpha)(\beta) + (\mathscr{D}\beta)(\alpha)) = 0, \end{aligned} \tag{38}$$

using the fact that the bracket is antisymmetric. Note that (37) is properly symmetric in its two arguments, in agreement with the graded commutativity (4). Another choice of arguments of total degree 2 is $v \in X_0$, $c \in X_2$, for which we require

$$\begin{aligned} \ell_1(\ell_2(v, c)) &= \ell_2(\ell_1(v), c) + \ell_2(v, \ell_1(c)) \\ &= \ell_2(v, \iota(c)) = -v(\iota(c)), \end{aligned} \tag{39}$$

where we used (35) in the last step, recalling $\iota(c) \in X_1$. Thus, using $\ell_1 = \iota$ on the l.h.s. together with the graded symmetry we have

$$\iota(\ell_2(c, v)) = v(\iota(c)). \tag{40}$$

We can also write this as (Here we employ the map on X_2 induced by $v(\alpha)$ via $v(c) := v(\iota(c))$, which lies in $\ker(\mathscr{D})$ as a consequence of $\mathscr{D}c = 0$ and (33))

$$\forall c \in X_2, \ v \in X_0: \quad \ell_2(c, v) = v(c) \in X_2. \tag{41}$$

We next consider arguments $c \in X_2$, $\alpha \in X_1$ of total degree 3, for which ℓ_2 must vanish as there is no vector space X_3. This leads to a constraint from the $n = 2$ relation:

$$\begin{aligned} 0 = \ell_1(\ell_2(c, \alpha)) &= \ell_2(\iota(c), \alpha) + \ell_2(c, \mathscr{D}\alpha) \\ &= -(\mathscr{D}\alpha)(c) + \ell_2(c, \mathscr{D}\alpha), \end{aligned} \tag{42}$$

where we used (37) and $\mathscr{D}c = 0$. This relation is satisfied for (41). Finally, the $n = 2$ relations are trivially satisfied for arguments of total degree 4 or higher, completing the proof of all $n = 2$ relations.

$n = 3$ relations: We now consider the $n = 3$ relations for arguments $v_1, v_2, v_3 \in X_0$ of total degree zero:

$$0 = \ell_1(\ell_3(v_1, v_2, v_3)) + \mathrm{Jac}(v_1, v_2, v_3). \tag{43}$$

Recalling (32) and that $\ell_1 = \mathscr{D}$ when acting on X_1, we infer that this relation is satisfied for

$$\ell_3(v_1, v_2, v_3) = -f(v_1, v_2, v_3) \in X_1. \tag{44}$$

Next, for arguments $\alpha \in X_1, v_1, v_2 \in X_0$ of total weight 1 the $n = 3$ relation reads

$$\begin{aligned} 0 &= \ell_1(\ell_3(\alpha, v_1, v_2)) + \ell_3(\ell_1(\alpha), v_1, v_2) \\ &\quad + \ell_2(\ell_2(\alpha, v_1), v_2) + \ell_2(\ell_2(v_2, \alpha), v_1) \\ &\quad + \ell_2(\ell_2(v_1, v_2), \alpha) \\ &= \iota(\ell_3(\alpha, v_1, v_2)) - f(\mathscr{D}\alpha, v_1, v_2) + v_2(v_1(\alpha)) \\ &\quad - v_1(v_2(\alpha)) - [v_1, v_2](\alpha), \end{aligned} \tag{45}$$

where we used repeatedly (35). Moreover, we used (44) and that $\ell_3(\alpha, v_1, v_2) \in X_2$ on which ℓ_1 acts as the inclusion. We will next prove that the function

$$\begin{aligned} g(\alpha, v_1, v_2) \equiv f(\mathscr{D}\alpha, v_1, v_2) + [v_1, v_2](\alpha) \\ + v_1(v_2(\alpha)) - v_2(v_1(\alpha)), \end{aligned} \tag{46}$$

takes values in the subspace $\ker(\mathscr{D})$. We have to prove that the r.h.s. is annihilated by $\mathscr{D}$. To this end we compute for the first term with (32)

$$\begin{aligned} \mathscr{D} f(\mathscr{D}\alpha, v_1, v_2) &= \operatorname{Jac}(\mathscr{D}\alpha, v_1, v_2) \\ &= [[\mathscr{D}\alpha, v_1], v_2] + [[v_2, \mathscr{D}\alpha], v_1] + [[v_1, v_2], \mathscr{D}\alpha] \\ &= [\mathscr{D}(v_1(\alpha)), v_2] - [\mathscr{D}(v_2(\alpha)), v_1] \\ &\quad - \mathscr{D}([v_1, v_2](\alpha)) \\ &= \mathscr{D}(v_2(v_1(\alpha)) - v_1(v_2(\alpha)) - [v_1, v_2](\alpha)), \end{aligned} \tag{47}$$

where we repeatedly used (33). This show that the r.h.s. of (46) is annihilated by $\mathscr{D}$, proving that g takes values in $X_2 = \ker(\mathscr{D})$. We can thus satisfy (45) by setting

$$\ell_3(\alpha, v_1, v_2) = g(\alpha, v_1, v_2) \in X_2. \tag{48}$$

We next recall that there can be no nontrivial ℓ_3 for arguments $\alpha_1, \alpha_2 \in X_1, v \in X_0$ of total degree 2. Thus, the $n = 3$ relation for these arguments has to be satisfied for the products already defined. We then compute from (11), noting that it is symmetric in α_1, α_2 and writing $\sum_{\text{sym}}$ for the symmetrized sum,

$$\begin{aligned} 0 &= \sum_{\text{sym}} (2\ell_3(\ell_1(\alpha_1), \alpha_2, v) + \ell_2(\ell_2(\alpha_1, \alpha_2), v) \\ &\quad + 2\ell_2(\ell_2(v, \alpha_1), \alpha_2)) = \sum_{\text{sym}} (-2\ell_3(\alpha_1, \mathscr{D}\alpha_2, v) \\ &\quad - 2\ell_2((\mathscr{D}\alpha_1)(\alpha_2), v) - 2\ell_2(v(\alpha_1), \alpha_2)) \\ &= \sum_{\text{sym}} (-2f(\mathscr{D}\alpha_1, \mathscr{D}\alpha_2, v) - 2[\mathscr{D}\alpha_2, v](\alpha_1) \\ &\quad - 2(\mathscr{D}\alpha_2)(v(\alpha_1)) + 2v((\mathscr{D}\alpha_2)(\alpha_1)) \\ &\quad - 2v((\mathscr{D}\alpha_1)(\alpha_2)) + 2(\mathscr{D}(v(\alpha_1)))(\alpha_2) \\ &\quad + 2(\mathscr{D}\alpha_2)(v(\alpha_1))), \end{aligned} \tag{49}$$

where we used (35), (37), and, in the third equality, (48). It is now easy to see that under the symmetrized sum all terms cancel, using in particular that f is totally antisymmetric. Thus, this $n = 3$ relation is satisfied. Since the $n = 3$ relations for total degree 3 or higher are trivially satisfied, we have completed the proof of all $n = 3$ relations.

$n = 4$ relations: We consider the $n = 4$ relations (12) for arguments of total degree 0. Precisely as in (28) we compute

$$\begin{aligned} \mathscr{O}(v_1, \ldots, v_4) &= \sum_{\text{anti}} (-4\ell_2(\ell_3(v_1, v_2, v_3), v_4) \\ &\quad + 6\ell_3(\ell_2(v_1, v_2), v_3, v_4)) \\ &= \sum_{\text{anti}} (-4v_1(f(v_2, v_3, v_4)) - 6f([v_1, v_2], v_3, v_4)). \end{aligned} \tag{50}$$

In contrast to (28) this is not zero in general, but we can now have a nontrivial ℓ_4 taking values in X_2. We next prove that the function defined by

$$\begin{aligned} &h(v_1, \ldots, v_4) \\ &\quad \equiv \sum_{\text{anti}} (4v_1(f(v_2, v_3, v_4)) + 6f([v_1, v_2], v_3, v_4)), \end{aligned} \tag{51}$$

takes values in $\ker(\mathscr{D})$. To this end we have to show that it is annihilated by $\mathscr{D}$:

$$\begin{aligned} \mathscr{D}(h(v_1, \ldots, v_4)) &= \sum_{\text{anti}} (4\mathscr{D}(v_1(f(v_2, v_3, v_4))) \\ &\quad + 6\mathscr{D} f([v_1, v_2], v_3, v_4)) \\ &= \sum_{\text{anti}} (4[\operatorname{Jac}(v_2, v_3, v_4), v_1] \\ &\quad + 6\operatorname{Jac}([v_1, v_2], v_3, v_4)) = 0. \end{aligned} \tag{52}$$

Thus, the $n = 4$ relation can be satisfied by setting

$$\ell_4(v_1, \ldots, v_4) = h(v_1, \ldots, v_4) \in X_2. \tag{53}$$

We have now determined all nontrivial brackets, which we summarize here:

$$\begin{aligned} c \in X_2 : &\quad \ell_1(c) = \iota(c) = c \in X_1, \\ \alpha \in X_1 : &\quad \ell_1(\alpha) = \mathscr{D}\alpha \in X_0, \\ v, w \in X_0 : &\quad \ell_2(v, w) = [v, w] \in X_0, \\ \alpha \in X_1,\ v \in X_0 : &\quad \ell_2(\alpha, v) = v(\alpha) \in X_1, \\ c \in X_2,\ v \in X_0 : &\quad \ell_2(c, v) = v(c) \in X_2, \\ \alpha, \beta \in X_1 : &\quad \ell_2(\alpha, \beta) \\ &\quad = -(\mathscr{D}\alpha)(\beta) - (\mathscr{D}\beta)(\alpha) \\ &\quad \in X_2, \end{aligned}$$

$$\begin{aligned} v_1, v_2, v_3 \in X_0 : \quad & \ell_3(v_1, v_2, v_3) \\ & = -f(v_1, v_2, v_3) \in X_1, \\ \alpha \in X_1,\ v_1, v_2 \in X_0 : \quad & \ell_3(\alpha, v_1, v_2) = g(\alpha, v_1, v_2) \\ & \in X_2, \\ v_1, \ldots, v_4 \in X_0 : \quad & \ell_4(v_1, \ldots, v_4) \\ & = h(v_1, \ldots, v_4) \in X_2, \end{aligned} \tag{54}$$

with the functions g, h defined in (46) and (51), respectively. All further L_∞ relations have to be satisfied identically. Let us next consider the $n = 4$ relations (12) for arguments $v_1, v_2, v_3 \in X_0, \alpha \in X_1$ of total degree 1. It is easy to see that (12) is then totally antisymmetric in v_1, v_2, v_3, and writing $\sum_{\text{anti}}$ for the antisymmetric sum over these three arguments we compute

$$\begin{aligned} \mathcal{O}(v_1, v_2, v_3, \alpha) &= \sum_{\text{anti}} (-\ell_2(\ell_3(v_1, v_2, v_3), \alpha) \\ &\quad + 3\ell_2(\ell_3(\alpha, v_1, v_2), v_3) + 3\ell_3(\ell_2(v_1, v_2), v_3, \alpha) \\ &\quad + 3\ell_3(v_1, v_2, \ell_2(v_3, \alpha))) \\ &= \sum_{\text{anti}} (\ell_2(f(v_1, v_2, v_3), \alpha) + 3v_3(\ell_3(\alpha, v_1, v_2)) \\ &\quad + 3\ell_3([v_1, v_2], v_3, \alpha) - 3\ell_3(v_1, v_2, v_3(\alpha))) \\ &= \sum_{\text{anti}} (-(\mathcal{D}f(v_1, v_2, v_3))(\alpha) - (\mathcal{D}\alpha) \\ &\quad \cdot (f(v_1, v_2, v_3)) + 3v_3(f(\mathcal{D}\alpha, v_1, v_2) \\ &\quad + [v_1, v_2](\alpha) + 2v_1(v_2(\alpha))) \\ &\quad + 3(f(\mathcal{D}\alpha, [v_1, v_2], v_3) + [[v_1, v_2], v_3](\alpha) \\ &\quad + [v_1, v_2](v_3(\alpha)) - v_3([v_1, v_2](\alpha))) \\ &\quad - 3(f(\mathcal{D}(v_3(\alpha)), v_1, v_2) + [v_1, v_2](v_3(\alpha)) \\ &\quad + 2v_1(v_2(v_3(\alpha))))) = \sum_{\text{anti}} (-(\mathcal{D}\alpha)(f(v_1, v_2, v_3)) \\ &\quad + 3v_3(f(\mathcal{D}\alpha, v_1, v_2)) + 3f(\mathcal{D}\alpha, [v_1, v_2], v_3) \\ &\quad - 3f([\mathcal{D}\alpha, v_3], v_1, v_2)), \end{aligned} \tag{55}$$

where we used the products already defined, in particular (48), and the relation (32) for the Jacobiator. We observe that various terms cancelled under the totally antisymmetric sum. In order to satisfy the $n = 4$ relation (12), the remaining terms need to be equal to $\ell_4(v_1, v_2, v_3, \mathcal{D}\alpha)$. To see this note that writing (53) with an antisymmetrized sum over only the first three arguments one obtains

$$\begin{aligned} \ell_4(v_1, \ldots, v_4) &= \sum_{\text{anti}[v_1, v_2, v_3]} (3v_1(f(v_2, v_3, v_4)) \\ &\quad - v_4(f(v_1, v_2, v_3)) + 3f([v_1, v_2], v_3, v_4) \\ &\quad - 3f([v_4, v_1], v_2, v_3)). \end{aligned} \tag{56}$$

Specializing this to $\ell_4(v_1, v_2, v_3, \mathcal{D}\alpha)$ we infer that it equals (55), completing the proof of this $n = 4$ relation. It is easy to see that for arguments of total degree 2 or higher the $n = 4$ relations are trivially satisfied. Thus, we have verified all $n = 4$ relations.

$n = 5$ relations: We have not displayed the L_∞ relations in Section 2 for $n \geq 5$ explicitly as these get increasingly laborious. However, it is easy to see that here the only nontrivial $n = 5$ relation has arguments $v_1, \ldots, v_5 \in X_0$, which are of even degree so that the Koszul sign becomes $\epsilon(\sigma; v) = 1$. Moreover, ℓ_5 is trivial, and it is then easy to verify that (7) reduces to

$$\begin{aligned} \sum_{\text{anti}} (&10\ell_4(\ell_2(v_1, v_2), v_3, v_4, v_5) \\ &+ 5\ell_2(\ell_4(v_1, v_2, v_3, v_4), v_5) \\ &+ 10\ell_3(\ell_3(v_1, v_2, v_3), v_4, v_5)) = 0, \end{aligned} \tag{57}$$

where the sum antisymmetrizes over all five arguments. Upon inserting the products in (54), it is a straightforward direct calculation, largely analogous to (55), to verify that this relation is identically satisfied. As these are the only nontrivial L_∞ relations for $n = 5$ or higher, this completes the proof. □

Specializations. As a special case of Theorem 2 let us assume that the Jacobiator takes values in a subspace $U \subset V$, which forms an ideal of the bracket. In this case we can take $\mathcal{D} = \iota$ to be the inclusion map $U \longrightarrow V$. Since its kernel is trivial, we have $X_2 = \{0\}$, and the algebra can be reduced to a 2-term L_∞ algebra. Indeed, the action of $v \in V$ on U that is implicit in (33) then reduces to

$$u \longmapsto v(u) \equiv -[v, u] \in U. \tag{58}$$

Using this and $\text{Jac}(v_1, v_2, v_3) = f(v_1, v_2, v_3)$, it is straightforward to verify that all products in (54) that take values in X_2 trivialize. In particular, ℓ_4 trivializes. Theorem 1 is contained as a special case, for which $U = V$.

5. Examples

We will now discuss a few examples, which get increasingly less trivial, with the goal to illustrate the scope of the above theorems.

The octonions: The seven imaginary octonions e_a, $a = 1, \ldots, 7$ satisfy the algebra

$$e_a e_b = -\delta_{ab} 1 + \eta_{abc} e_c, \tag{59}$$

and thus the commutation relations

$$[e_a, e_b] = 2\eta_{abc} e_c, \tag{60}$$

where the structure constants are defined as follows. Splitting the index as $a = (i, \bar{i}, 7)$, where $i, \bar{i} = 1, 2, 3$, η_{abc} is the totally antisymmetric tensor defined by

$$\begin{aligned} \eta_{ijk} &= \epsilon_{ijk}, \\ \eta_{i\bar{j}\bar{k}} &= -\epsilon_{ijk}, \\ \eta_{7i\bar{j}} &= \delta_{ij}, \end{aligned} \tag{61}$$

with the three-dimensional Levi-Civita symbol satisfying $\epsilon_{123} = 1$. (This coincides with the conventions of [14].) η_{abc} satisfy the following relations

$$\begin{aligned} \eta_{abe}\eta_{cde} &= 2\delta_{c[a}\delta_{b]d} - \Theta_{abcd}, \\ \Theta_{abcd} &\equiv \frac{1}{3!}\epsilon_{abcdefg}\eta_{efg}. \end{aligned} \tag{62}$$

Using these it is straightforward to compute the Jacobiator:

$$\mathrm{Jac}\left(e_a, e_b, e_c\right) = -12\Theta_{abcd}e_d. \tag{63}$$

It is easy to verify with this expression that each generator appears on the right-hand side, see [14]. Thus, the Jacobiator does not take values in a proper subspace, and therefore the L_∞ extension requires a doubling to a 14-dimensional space (with basis $\{e_a, e_a^*\}$) as in Theorem 1, with the nontrivial brackets being given in addition to (60) by

$$\begin{aligned} \ell_1\left(e_a^*\right) &= e_a, \\ \ell_2\left(e_a^*, e_b\right) &= 2\eta_{abc}e_c^*, \\ \ell_3\left(e_a, e_b, e_c\right) &= 12\Theta_{abcd}e_d^*. \end{aligned} \tag{64}$$

There is no further nontrivial extension; in particular, this algebra cannot describe a nontrivial gauge symmetry in a field theory.

The R-flux algebra: This algebra, introduced in [17–19], is a contraction of the algebra of imaginary octonions in the following sense [14]: (As shown in [16], the algebra of octonions can be also contracted in an analogous way to the magnetic monopole algebra, which is isomorphic to the R-flux algebra upon exchange of position and momentum variables.) We decompose $e_a = (e_i, f_i, e_7)$, with $i = 1, 2, 3$, and introduce a scaling parameter λ to define

$$\begin{aligned} p_i &\equiv -\frac{1}{2}\lambda i e_i, \\ x^i &\equiv \frac{1}{2}\sqrt{\lambda} i f_i, \\ I &\equiv \frac{1}{2} i \lambda^{3/2} e_7. \end{aligned} \tag{65}$$

Expressing the algebra (60) now in terms of x, p, I and sending $\lambda \longrightarrow 0$ one obtains the R-flux algebra

$$\begin{aligned} \left[x^i, p_j\right] &= i\delta^i{}_j I, \\ \left[x^i, x^j\right] &= i\epsilon^{ijk}p_k, \\ \left[p_i, p_j\right] &= 0, \end{aligned} \tag{66}$$

where I is a central element that commutes with everything. It is easy to see that the only nonvanishing Jacobiator is

$$\mathrm{Jac}\left(x^i, x^j, x^k\right) = 3\epsilon^{ijk}I. \tag{67}$$

Thus, the Jacobiator takes values in the one-dimensional subspace spanned by I. According to the specialization discussed after the proof of Theorem 2, we can then define an L_∞ structure on $X_1 + X_0$, where $X_0 = \{x^i, p_i, I\}$ and $X_1 = \{I^*\}$. In addition to the 2-brackets defined by (66) we have the nontrivial products

$$\begin{aligned} \ell_1\left(I^*\right) &= I, \\ \ell_3\left(x^i, x^j, x^k\right) &= -3\epsilon^{ijk}I^*. \end{aligned} \tag{68}$$

The Courant algebroid: The Courant bracket of generalized geometry or the "C-bracket" of double field theory has a nonvanishing Jacobiator. Denoting the arguments of this bracket, i.e., the elements of X_0, by ξ_1, ξ_2, etc., it is given by

$$\begin{aligned} \mathrm{Jac}\left(\xi_1, \xi_2, \xi_3\right) &= \mathscr{D} f\left(\xi_1, \xi_2, \xi_3\right), \\ f\left(\xi_1, \xi_2, \xi_3\right) &\equiv \frac{1}{2}\sum_{\mathrm{anti}} \left\langle \left[\xi_1, \xi_2\right], \xi_3 \right\rangle, \end{aligned} \tag{69}$$

where $\langle , \rangle$ denotes the $O(d, d)$ invariant metric and $\mathscr{D}$ is the exterior derivative in generalized geometry or the doubled partial derivative in double field theory. The bracket satisfies for a function χ

$$\left[\mathscr{D}\chi, \xi\right] = -\frac{1}{2}\mathscr{D}\left\langle \mathscr{D}\chi, \xi \right\rangle, \tag{70}$$

so that for our current notation we read off with (33)

$$\xi\left(\chi\right) = -\frac{1}{2}\left\langle \mathscr{D}\chi, \xi \right\rangle. \tag{71}$$

It was established by Roytenberg and Weinstein that the Courant algebroid defines a 2-term L_∞ algebra with the highest bracket being ℓ_3, which is defined by f, and X_1 being the space of functions [11]. The space X_2 of constants (the kernel of the differential operator $\mathscr{D}$) is not needed as all brackets in (54) taking values in X_2 vanish. For instance, ℓ_2 for two functions $\chi_1, \chi_2 \in X_1$ becomes

$$\begin{aligned} \ell_2\left(\chi_1, \chi_2\right) &= -\left(\mathscr{D}\chi_1\right)\left(\chi_2\right) - \left(\mathscr{D}\chi_2\right)\left(\chi_1\right) \\ &= \left\langle \mathscr{D}\chi_1, \mathscr{D}\chi_2 \right\rangle = 0. \end{aligned} \tag{72}$$

In double field theory language this is zero because of the "strong constraint," and it is also one of the axioms of a Courant algebroid (see definition 3.2, axiom 4 in [11]). The vanishing of all other products taking values in X_2 can be verified similarly using the relations given, for instance, in [10]. Thus, the existence of an L_∞ structure on the Courant algebroid is a corollary of the more general Theorem 2.

6. Conclusions

We established general theorems about the existence of L_∞ algebras for a given bracket and discussed possible field theory realizations. This includes well-known examples such as the Courant algebroid as special cases. Most importantly, it then remains to construct explicit examples of algebras that obey the conditions of Theorem 2 and that really do use the full structure possible, particularly a nontrivial 4-bracket. This may require identifying a structure that relaxes some of the axioms of a Courant algebroid.

Moreover, it is clear that there will be further generalizations of this theorem. For instance, the construction of Theorem 2 could be extended by taking the map $\ell_1 : X_2 \longrightarrow X_1$ not to be the inclusion map but rather a nontrivial operator that again could have a nontrivial kernel, which in turn would necessitate a new space X_3 and higher brackets beyond a 4-bracket. These may be useful for generalizations of double and exceptional field theory [28, 29]. Indeed, it is to be expected that the gauge structure of exceptional field theory requires L_∞ algebras with arbitrarily high brackets [30], as is also the case in closed string field theory [1]. Moreover, in order to obtain interesting L_∞ algebras with nontrivial field theory realizations, for special cases it is instrumental to take an appropriate bracket as starting point. For instance, for the $E_{8(8)}$ theory in [31] the naive bracket does not yield a Jacobiator living in the image of an appropriate operator (or, equivalently, the naive bracket does not transform covariantly under its own "adjoint" action [32]), but rather the vector space has to be suitably enlarged from the beginning, leading to a so-called Leibniz-Loday structure [33].

Appendix

A_∞ and Nonassociative Algebras

In analogy to the doubling of vector spaces introduced for the L_∞ realization of Theorem 1 we will show that every nonassociative algebra has a realization as an A_∞ algebra. An A_∞ algebra is a graded vector space V together with a collection $\{m_k \mid k \in \mathbb{N}\}$ of multilinear maps $m_k : \bigotimes^k V \longrightarrow V$ of internal degree $k-2$ satisfying the following fundamental identity [4]

$$\sum_{\lambda=0}^{n-1}\sum_{j=1}^{n-\lambda}(-1)^{j+\lambda+j\lambda+nj+j(|a_1|+\cdots+|a_\lambda|)}\, m_{n-j+1}\big(a_1,\ldots,a_\lambda, m_j(a_{\lambda+1},\ldots,a_{\lambda+k}),a_{\lambda+k+1},\ldots,a_n\big)=0, \tag{A.1}$$

for every $n \in \mathbb{N}$. The first four equations read explicitly

(i) $n = 1$, deg $= -2$:

$$0 = m_1(m_1(a_1)). \tag{A.2}$$

(ii) $n = 2$, deg $= -1$:

$$\begin{aligned} 0 &= -m_1(m_2(a_1,a_2)) + m_2(m_1(a_1),a_2) \\ &\quad + (-1)^{|a_1|} m_2(a_1, m_1(a_2)). \end{aligned} \tag{A.3}$$

(iii) $n = 3$, deg $= 0$:

$$\begin{aligned} 0 &= m_1(m_3(a_1,a_2,a_3)) + m_3(m_1(a_1),a_2,a_3) \\ &\quad + (-1)^{|a_1|} m_3(a_1, m_1(a_2), a_3) \\ &\quad + (-1)^{|a_1|+|a_2|} m_3(a_1,a_2,m_1(a_3)) \\ &\quad + m_2(m_2(a_1,a_2),a_3) - m_2(a_1, m_2(a_2,a_3)). \end{aligned} \tag{A.4}$$

(iv) $n = 4$, deg $= 1$:

$$\begin{aligned} 0 &= -m_1(m_4(a_1,a_2,a_3,a_4)) \\ &\quad + m_4(m_1(a_1),a_2,a_3,a_4) \\ &\quad + (-1)^{|a_1|} m_4(a_1, m_1(a_2), a_3, a_4) \\ &\quad + (-1)^{|a_1|+|a_2|} m_4(a_1,a_2,m_1(a_3),a_4) \\ &\quad + (-1)^{|a_1|+|a_2|+|a_3|} m_4(a_1,a_2,a_3,m_1(a_4)) \\ &\quad - m_3(m_2(a_1,a_2),a_3,a_4) \\ &\quad + m_3(a_1, m_2(a_2,a_3),a_4) \\ &\quad - m_3(a_1,a_2,m_2(a_3,a_4)) \\ &\quad + m_2(m_3(a_1,a_2,a_3),a_4) \\ &\quad + (-1)^{|a_1|} m_2(a_1, m_3(a_2,a_3,a_4)). \end{aligned} \tag{A.5}$$

Let $(V, \star)$ be a nonassociative algebra and V^* a vector space isomorphic to V with the isomorphism denoted by $V \ni a \longmapsto a^* \in V^*$. The graded vector space of the A_∞ algebra is then defined as

$$\begin{aligned} X_1 &= V^*, \\ X_0 &= V. \end{aligned} \tag{A.6}$$

In addition we define the following products

$$\begin{aligned} m_1(a^*) &= a, \\ m_2(a_1,a_2) &= a_1 \star a_2. \end{aligned} \tag{A.7}$$

Using this construction, the $n = 1$ A_∞ equation is trivially satisfied. For the second equation we compute

$$\begin{aligned} 0 &= -m_1(m_2(a_1^*,a_2)) + m_2(m_1(a_1^*),a_2) \\ &\quad + (-1)^{|a_1^*|} m_2(a_1^*, m_1(a_2)) \end{aligned} \tag{A.8}$$

$$= -m_1(m_2(a_1^*,a_2)) + a_1 \star a_2, \tag{A.9}$$

from which we conclude

$$m_2(a_1^*,a_2) = (a_1 \star a_2)^*. \tag{A.10}$$

For two arguments of degree 1 we compute

$$\begin{aligned} 0 &= -m_1(m_2(a_1^*,a_2^*)) + m_2(m_1(a_1^*),a_2^*) \\ &\quad + (-1)^{|a_1^*|} m_2(a_1^*, m_1(a_2^*)) \end{aligned} \tag{A.11}$$

$$= m_2(a_1,a_2^*) - (a_1 \star a_2)^*, \tag{A.12}$$

from which we conclude

$$m_2(a_1, a_2^*) = (a_1 \star a_2)^* . \tag{A.13}$$

Note that the m-products have no a priori symmetry properties, so the m_2-product has to be specified for each order of entries individually.

The $n = 3$ equations read

$$\begin{aligned} 0 &= m_1(m_3(a_1, a_2, a_3)) + m_2(m_2(a_1, a_2), a_3) \\ &\quad - m_2(a_1, m_2(a_2, a_3)) \end{aligned} \tag{A.14}$$

$$\begin{aligned} &= m_1(m_3(a_1, a_2, a_3)) + (a_1 \star a_2) \star a_3 - a_1 \\ &\quad \star (a_2 \star a_3), \end{aligned} \tag{A.15}$$

from which we infer that the 3-product is defined by the associator:

$$m_3(a_1, a_2, a_3) = -\mathrm{Ass}(a_1, a_2, a_3)^* . \tag{A.16}$$

Moreover, for total degree 1 we compute

$$\begin{aligned} 0 &= m_3(m_1(a_1^*), a_2, a_3) + m_2(m_2(a_1^*, a_2), a_3) \\ &\quad - m_2(a_1^*, m_2(a_2, a_3)) \end{aligned} \tag{A.17}$$

$$\begin{aligned} &= -\mathrm{Ass}(a_1, a_2, a_3)^* + ((a_1 \star a_2) \star a_3)^* \\ &\quad - (a_1 \star (a_2 \star a_3))^* , \end{aligned} \tag{A.18}$$

which is therefore satisfied.

We claim that the $n = 4$ equations are satisfied for $m_4 \equiv 0$, which we verify by a direct computation:

$$\begin{aligned} 0 &= -m_3(m_2(a_1, a_2), a_3, a_4) \\ &\quad + m_3(a_1, m_2(a_2, a_3), a_4) \\ &\quad - m_3(a_1, a_2, m_2(a_3, a_4)) \\ &\quad + m_2(m_3(a_1, a_2, a_3), a_4) \\ &\quad + m_2(a_1, m_3(a_2, a_3, a_4) \\ &= \mathrm{Ass}((a_1 \star a_2), a_3, a_4)^* - \mathrm{Ass}(a_1, a_2 \\ &\quad \star a_3, a_4)^* + \mathrm{Ass}(a_1, a_2, a_3 \star a_4)^* \\ &\quad - (\mathrm{Ass}(a_1, a_2, a_3) \star a_4)^* - (a_1 \\ &\quad \star \mathrm{Ass}(a_2, a_3, a_4))^* = [((a_1 \star a_2) \star a_3) \star a_4 \\ &\quad - (a_1 \star a_2) \star (a_3 \star a_4) - (a_1 \star (a_2 \star a_3)) \star a_4 \\ &\quad + a_1 \star ((a_2 \star a_3) \star a_4) + (a_1 \star a_2) \star (a_3 \star a_4) \\ &\quad - a_1 \star (a_2 \star (a_3 \star a_4)) - ((a_1 \star a_2) \star a_3) \star a_4 \\ &\quad + (a_1 \star (a_2 \star a_3)) \star a_4 - a_1 \star ((a_2 \star a_3) \star a_4) \\ &\quad + a_1 \star (a_2 \star (a_3 \star a_4))]^* . \end{aligned} \tag{A.19}$$

The terms exactly cancel. This completes the proof that any nonassociative algebra can be embedded into an A_∞ algebra.

Data Availability

This work is theoretical and does not use any data.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

We would like to thank Ralph Blumenhagen, Michael Fuchs, Ezra Getzler, Tom Lada, Martin Rocek, Christian Saemann, Jim Stasheff, Richard Szabo, and Barton Zwiebach for useful discussions and comments. Olaf Hohm is supported by a DFG Heisenberg Fellowship of the German Science Foundation (DFG). Vladislav Kupriyanov is supported by the Capes-Humboldt Fellowship No. 0079/16-2. The work of Dieter Lüst is supported by the ERC Advanced Grant No. 320045 "Strings and Gravity."

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