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Research Article

The Application of Minimal Length in Klein-Gordon Equation with Hulthen Potential Using Asymptotic Iteration Method

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The application of minimal length formalism in Klein-Gordon equation with Hulthen potential was studied in the case of scalar potential that was equal to vector potential. The approximate solution was used to solve the Klein-Gordon equation within the minimal length formalism. The relativistic energy and wave functions of Klein-Gordon equation were obtained by using the Asymptotic Iteration Method. By using the Matlab software, the relativistic energies were calculated numerically. The unnormalized wave functions were expressed in hypergeometric terms. The results showed the relativistic energy increased by the increase of the minimal length parameter. The unnormalized wave function amplitude increased for the larger minimal length parameter.

1. Introduction

The relativistic effect gives the correction in the nonrelativistic quantum mechanics by applying the strong potential field in the particles dynamic. The particles dynamic in relativistic effect can be described by using Klein-Gordon equation, particularly for a zero spin particle. The Klein-Gordon equation is formed by two potentials coupling which are the four-vector potential ($V(r)$) and scalar potential ($S(r)$). From these two potentials, the Klein-Gordon equation has two framework conditions, which are as follows: if the scalar potential was equal to vector potential ($S(r) = V(r)$) and if the scalar potential was equal to minus vector potential ($S(r) = -V(r)$). This condition appears in nuclear and high energy physics problem [1–3]. Some of researchers have investigated both these conditions with the certainty vector potential. The main case of that research was how to reduce the Klein-Gordon equation into the Schrodinger-like equation, so we can solve it by using the certainty suitable methods. The methods which are usually used are such as Supersymmetric Quantum Mechanics (SUSY) [4], Nikiforov-Uvarov [1, 5], and Asymptotic Iteration Method [6, 7]. Various potentials are used to explain the dynamic particle, such as the harmonic potential [8], Makarov potential [2], Hulthen potential [1, 5], Kratzer potential [6], and Trigonometric Poschl-Teller potential [7].

The particles dynamic in quantum mechanics corresponds to the position and momentum of particles. And, as we have known, the study of commutation relations between position and momentum operators is explained using Heisenberg uncertainty principle [9], which is given by

$$\left[\widehat{X}, \widehat{P}\right] \geq i\hbar \tag{1}$$

where $\widehat{X}$ is position operator, $\widehat{P}$ is momentum operator, i is imaginer number, and $\hbar = h/2\pi$ with h being Planck constant. The presence of a quantum gravity on quantum mechanics has the consequence of the existence of a minimal observable distance on the order of the Planck length. Therefore the Heisenberg uncertainty principle gets additional correction due to the presence of a quantum gravity, which is well known as Generalized Uncertainty Principle (GUP) [10, 11], given by

$$\left[\widehat{X}, \widehat{P}\right] \geq i\hbar\left(1 + \alpha_{ML} P^2\right) \tag{2}$$

where α_{ML} is minimal length parameter that has value $0 \leq \alpha_{ML} \leq 1$ and P is magnitude of the momentum [11]. When the energy is much smaller than the Planck mass, α_{ML} goes to zero and we recover Heisenberg uncertainty principle [12].

In 2009, Jana and Roy have solved Klein-Gordon equation in the presence of minimal length for scalar potential was equal to vector potential using Algebraic Approach [13]. The Klein-Gordon equation in the presence of a minimal length is solved by using Feynman for scalar potential was equal to vector potential [14]. In addition, the hypergeometric method is used to solve Klein-Gordon equation using hyperbolic cotangent potential [15] and Asymptotic Iteration Method in trigonometric cotangent potential [16, 17].

In this paper, we solved the minimal length formalism in radial part of Klein-Gordon equation for the condition $S(r) = V(r)$ with Hulthen potential by using approximate solution. The approximate solution is used by Chabab et al. to solve the Bohr Mottelson Hamiltonian in the presence of a minimal length formalism by introducing the new wave function [11]. The minimal length formalism in the Klein-Gordon equation is reduced into second-order differential equation. The relativistic energy equation and wave functions of Klein-Gordon equation are obtained by using Asymptotic Iteration Method.

The study is organized as follows. In Section 2, the approximate form of Klein-Gordon equation within minimal length formalism is presented. The Hulthen potential is introduced in Section 3. We describe Asymptotic Iteration Method in Section 4. The result and discussion are presented in Section 5. At last, conclusion is given in Section 6.

2. Approximate Form of Klein-Gordon Equation within Minimal Length Formalism

Generalized Uncertainty Principle is called the minimal length that is deformed from the commutation relations between position and momentum operators in quantum mechanics. In (2), the commutation relations can be rewritten as follows [9, 12]:

$$\widehat{X}_i = \widehat{x}_i \tag{3}$$

$$\widehat{P}_i = \left(1 + \alpha_{ML}\widehat{p}^2\right)\widehat{p}_i \tag{4}$$

where $\widehat{P}_i$ and $\widehat{p}_i$ are momentum operators at high and low energy, respectively. The magnitude of $\widehat{p}_i$ is expressed by p. The occurrence of minimal length is in string theory, black hole, quantum gravity, and noncommutative geometry, which yield new correction to Heisenberg uncertainty principle and imply a finite minimal uncertainty in position measurements, e.g., at the Planck scale [14].

The general Klein-Gordon equation with scalar potential $S(r)$ and vector potential $V(r)$ is given as

$$\begin{aligned}(E - V(r))^2 \psi(r,\theta,\varphi) \\ = \left[P^2c^2 + \left(M_oc^2 + S(r)\right)^2\right]\psi(r,\theta,\varphi)\end{aligned} \tag{5}$$

where E is relativistic energy and M_o is rest mass. By setting $S(r) = V(r)$ in (5) and substituting (4) into (5), with $\widehat{p} = -i\hbar\nabla$ and $c = \hbar = 1$ (natural unit), we have

$$\begin{aligned}&-\left(\Delta - 2\alpha_{ML}\Delta^2\right)\psi(r,\theta,\varphi) \\ &-\begin{pmatrix} E^2 - M_o{}^2 \\ -(E+M_o)V(r)\end{pmatrix}\psi(r,\theta,\varphi) = 0\end{aligned} \tag{6}$$

and here we have set $V(r) \to (1/2)V(r)$.

Accordingly, it would be natural to scale the potential term in (6), so that the nonrelativistic energy is reproduced [18]. The new wave function that is used to get the approximate form of Klein-Gordon equation is given [11]:

$$\psi(r,\theta,\varphi) = (1 + 2\alpha_{ML}\Delta)\phi(r,\theta,\varphi) \tag{7}$$

Equation (7) is modification of (19) in [11]. The modification is proposed to eliminate quadratic of Laplacian factor such that we get (20) in [11]. By substituting (7) into (6), we obtain

$$\left[-\Delta - \frac{\begin{pmatrix} E^2-M_o{}^2 \\ -(E+M_o)V(r)\end{pmatrix}}{\left(1 + 2\alpha_{ML}\begin{pmatrix} E^2-M_o{}^2 \\ -(E+M_o) \\ V(r)\end{pmatrix}\right)}\right]\phi(r,\theta,\varphi) = 0 \tag{8}$$

Equation (8) is the minimal length formalism in Klein-Gordon equation within the approximate form. The component Δ^3 is eliminated due to the value of α_{ML}^2 which goes to zero, and value of α_{ML} is very small. Here, we have used the properties that Δ is scalar differential operator. If Δ operates to scalar fields ϕ at a point (r,θ,φ), $\Delta\phi$ will result in another scalar field. Here, $\Delta\phi$ is also called scalar Laplacian. Inserting (7) into (6) leads us to do the multiplication operation among Laplacian. By using the property that the multiplication operation is commutative for scalar differential operator with constant coefficient, then the multiplication operator among Laplacian also has commutative properties. Therefore, here we can eliminate Δ^2 when (7) is inserted into (6). To get simple solution of (8), binomial expansion is used for small α_{ML}; then (8) becomes

$$\begin{aligned}&\Delta\phi(r,\theta,\varphi) \\ &+\left[\begin{matrix} \left(E^2 - M_o{}^2\right) \\ -(E+M_o)V(r) \\ -2\alpha_{ML}\begin{pmatrix} E^2 - M_o{}^2 \\ -(E+M_o)V(r)\end{pmatrix}^2\end{matrix}\right]\phi(r,\theta,\varphi) = 0\end{aligned} \tag{9}$$

Equation (9) is obtained by setting $\alpha_{ML}{}^2$ which goes to zero, so $\alpha_{ML}{}^2$ is ignored. Applying spherical Laplacian operator, as

$$\begin{aligned}\Delta &= \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} \\ &+ \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\end{aligned} \tag{10}$$

into (9), we use variable separable method by setting the new wave function $\phi(r,\theta,\varphi) = R(r)\Theta(\theta)\Phi(\varphi)$, and we have a polar part and radial part of Klein-Gordon equation in the presence of minimal length. The polar part is given:

$$-\left[\begin{matrix}\dfrac{1}{\Theta(\theta)}\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\sin\theta\dfrac{\partial\Theta(\theta)}{\partial\theta} \\ +\dfrac{1}{\Phi(\varphi)}\dfrac{1}{\sin^2\theta}\dfrac{\partial^2\Phi(\varphi)}{\partial\varphi^2}\end{matrix}\right] = \lambda \tag{11}$$

and the radial part is as follows:

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial R(r)}{\partial r} + \left[\begin{matrix}\left(\begin{matrix}E^2 - M_o{}^2 \\ -(E+M_o)V(r)\end{matrix}\right) \\ -2\alpha_{ML}\left(\begin{matrix}\left(E^2 - M_o{}^2\right)^2 \\ -2\left(E^2 - M_o{}^2\right) \\ (E+M_o)V(r) \\ +(E+M_o)^2V^2(r)\end{matrix}\right)\end{matrix}\right]R(r) = \frac{\lambda}{r^2}R(r) \tag{12}$$

where λ is a constant of variable separable method which corresponds to angular momentum (L). By applying $R(r) = U_H(r)/r$ and $\lambda = L(L+1)$ into (12), it yields

$$\frac{d^2U_H(r)}{dr^2} - \frac{L(L+1)}{r^2}U_H(r) + \left[\begin{matrix}\left(\begin{matrix}E^2 - M_o{}^2 \\ -(E+M_o)V(r)\end{matrix}\right) \\ -2\alpha_{ML}\left(\begin{matrix}\left(E^2 - M_o{}^2\right)^2 \\ -2\left(E^2 - M_o{}^2\right)(E+M_o)V(r) \\ +(E+M_o)^2V^2(r)\end{matrix}\right)\end{matrix}\right]U_H(r) = 0 \tag{13}$$

Equation (13) is the minimal length formalism in Klein-Gordon equation with Hulthen potential in the form of one-dimensional Schrodinger-like equation.

3. Asymptotic Iteration Method

Asymptotic Iteration Method is method to solve the second-order differential equation in form [19–21]

$$y_n{}''(u) = \lambda_0(u)\,y_n{}'(u) + s_0(u)\,y_n(u) \tag{14}$$

where $\lambda_0(u) \neq 0$ and $s_0(u)$ are the coefficients of a differential equation and n is a quantum number. To obtain solution, we derive (14).

$$y_n{}^{z+1}(u) = \lambda_{z-1}(u)\,y_n{}'(u) + s_{z-1}(u)\,y_n(u) \tag{15}$$

and here

$$\begin{gathered}\lambda_z(u) = \lambda'_{z-1}(u) + s_{z-1}(u) + \lambda_0(u)\lambda_{z-1}(u) \\ s_z(u) = s'_{z-1}(u) + s_0(u)\lambda_{z-1}(u), \\ z = 1, 2, 3, \ldots\end{gathered} \tag{16}$$

The eigenvalue is obtained from the quantization condition which is given by

$$\Delta_z(u) = \lambda_z(u)\,s_{z-1}(u) - \lambda_{z-1}(u)\,s_z(u) = 0 \tag{17}$$

To obtain the wave function, (14) is reduced into the formalism, as follows

$$y_n{}''(u) = \left\{\begin{matrix}2\left(\dfrac{au^{N+1}}{1-bu^{N+2}} - \dfrac{t+1}{u}\right)y_n{}'(u) \\ -\dfrac{wu^N}{1-bu^{N+2}}y_n(u)\end{matrix}\right\} \tag{18}$$

Equation (18) is one-dimensional Schrodinger-like equation that has solution which is expressed in hypergeometric term, given as

$$\begin{gathered}y_n(u) = (-1)^n C'(N+2)^n(\sigma)_n \\ {}_2F_1\left(-n, \rho+n; \sigma; bu^{N+2}\right)\end{gathered} \tag{19}$$

where

$$\begin{gathered}(\sigma)_n = \frac{\Gamma(\sigma+n)}{\Gamma(\sigma)}; \\ \sigma = \frac{2t+N+3}{N+2}; \\ \rho = \frac{(2t+1)b+2a}{(N+2)b}\end{gathered} \tag{20}$$

C' is normalization constant and ${}_2F_1$ is a hypergeometric function. The unnormalized wave functions of Klein-Gordon equation are obtained by using (19)-(20) [19–21].

4. Hulthen Potential

The Hulthen potential is one of short range potentials in physics. The Hulthen potential has been used in particle physics, atomic physics, nuclear physics, solid-state physics, and chemical physics. The Hulthen wave functions have been used in solid-state physics problems [22]. The Hulthen-like wave functions have been found to investigate atomic problems [22]. The general Hulthen potential is given by [23]

$$V(r) = -V_H\omega_H\frac{e^{-2\omega_H r}}{1-e^{-2\omega_H r}} \tag{21}$$

where ω_H is a screening parameter and V_H is potential depth. The value of screening potential is 0.025 for low screening and 0.15 for high screening [22]. To get simple solution, (21) was changed in hyperbolic trigonometric term [23], given as

$$V(r) = \frac{V_H\omega_H}{2}\left(1 - \coth\omega_H r\right) \tag{22}$$

By setting $V_H = 7$ and $\omega_H = 0.1$, the visualization of Hulthen potential is expressed in Figure 1.

Figure 1 shows the visualization of Hulthen potential in r function. The Hulthen potential for different value of r is approximately from 0 until $0.05\mu/\mathrm{eV}$ (natural unit). The Hulthen potential has negative value in a very small value of r, while the Hulthen potential inclines to be constant for higher value of r.

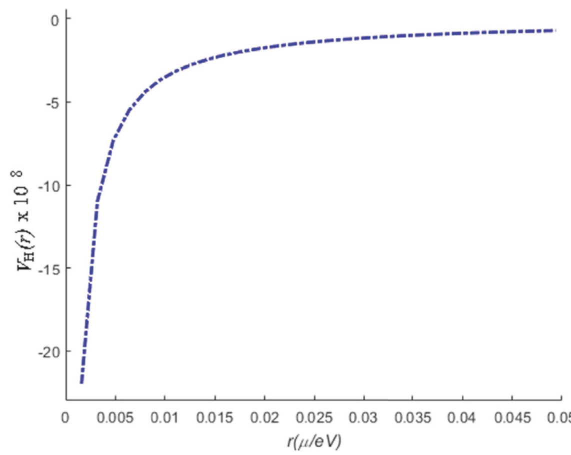


Figure 1: The visualization of Hulthen potential.

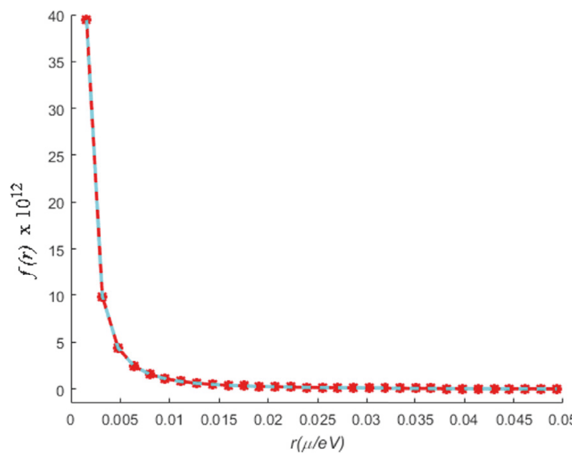


Figure 2: The visualization of approximation, with $\omega_H = 0.1$ and value of r from 0 until 0.05 μ/eV (natural unit).

5. Result and Discussion

Equation (13) can not be solved exactly unless we use the approximation to the function of $1/r^2$. The approximation of function $1/r^2$ is given as [23]

$$\frac{1}{r^2} = \frac{\omega_H^2}{\sinh^2 \omega_H r} = f(r) \tag{23}$$

for small value of ω_H or $\omega_H r \ll 1$. The visualization of that approximation is expressed in Figure 2.

Figure 2 shows the red line as a function of $1/r^2$ and light blue line as a function of $\omega_H^2/\sinh^2(\omega_H r)$. It is seen that the two lines overlap with each other; then the centrifugal term $1/r^2$ is approximated by $\omega_H^2/\sinh^2(\omega_H r)$ also as in [23].To obtain the exact solution of (13) the approximate term of $1/r^2$ in (23) is inserted into (13) and together with (22); then we get

$$\frac{d^2 U_H(r)}{dr^2} - \frac{\left(\omega_H^2 L(L+1) + \alpha_{ML}(E+M_o)^2\left((V_H\omega_H)^2/2\right)\right)}{\sinh^2\omega_H r} U_H(r) + \left[\begin{array}{c} + \left(\begin{array}{c} (E+M_o)\dfrac{V_H\omega_H}{2} + \alpha_{ML}(E+M_o)^2 (V_H\omega_H)^2 \\ -2\alpha_{ML}\left(E^2 - {M_o}^2\right)(E+M_o) V_H\omega_H \end{array} \right) \coth\omega_H r \\ + \left(\begin{array}{c} \left(E^2 - {M_o}^2\right) - (E+M_o)\dfrac{V_H\omega_H}{2} \\ +2\alpha_{ML}\left(E^2 - {M_o}^2\right)(E+M_o)V_H\omega_H \\ -\alpha_{ML}(E+M_o)^2 (V_H\omega_H)^2 - 2\alpha_{ML}\left(E^2 - {M_o}^2\right)^2 \end{array} \right) \end{array} \right] U_H(r) = 0 \tag{24}$$

Equation (24) is the Klein-Gordon equation with the minimal length for Hulthen potential which can be rewritten as

$$\frac{d^2U_H(r)}{dr^2} - \left[\frac{\nu_{PH}(\nu_{PH}-1)}{\sinh^2\omega_H r} - 2q_{PH}\coth\omega_H r + \kappa_{PH}^2\right]U_H(r) = 0 \tag{25}$$

with

$$\nu_{PH}(\nu_{PH}-1) = \left(\omega_H^2 L(L+1) + \alpha_{ML}(E+M_o)^2\frac{(V_H\omega_H)^2}{2}\right) \tag{26}$$

$$2q_{PH} = \begin{pmatrix} (E+M_o)\dfrac{(V_H\omega_H)}{2} + \alpha_{ML}(E+M_o)^2(V_H\omega_H)^2 \\ -2\alpha_{ML}\left(E^2-M_o{}^2\right)(E+M_o)(V_H\omega_H)\end{pmatrix} \tag{27}$$

$$-\kappa_{PH}^2 = \begin{pmatrix} \left(E^2-M_o{}^2\right) - (E+M_o)\dfrac{(V_H\omega_H)}{2} \\ +2\alpha_{ML}\left(E^2-M_o{}^2\right)(E+M_o)(V_H\omega_H) \\ -\alpha_{ML}(E+M_o)^2(V_H\omega_H)^2 - 2\alpha_{ML}\left(E^2-M_o{}^2\right)^2\end{pmatrix} \tag{28}$$

Equation (25) is second-order differential equation that will be reduced to hypergeometric differential equation type; by letting $\coth\omega_H r = 1-2z$, we get

$$z(1-z)\frac{d^2U_H(r)}{dz^2} + (1-2z)\frac{dU_H(r)}{dz} + \left[\nu_{PH}'(\nu_{PH}'-1) - \frac{-2q_{PH}'+\kappa_{PH}'^2}{4z} - \frac{2q_{PH}'+\kappa_{PH}'^2}{4(1-z)}\right]U_H(r) = 0 \tag{29}$$

and then we set

$$\begin{aligned} 2q_{PH}' &= \frac{2q_{PH}}{\omega_H^2}; \\ \kappa_{PH}'^2 &= \frac{\kappa_{PH}^2}{\omega_H^2} \\ \nu_{PH}'(\nu_{PH}'-1) &= \frac{\nu_{PH}(\nu_{PH}-1)}{\omega_H^2}; \\ -2q_{PH}'+\kappa_{PH}'^2 &= 4\alpha_{PH}^2; \\ 2q_{PH}'+\kappa_{PH}'^2 &= 4\beta_{PH}^2 \end{aligned} \tag{30}$$

in (29), so we have

$$z(1-z)\frac{d^2U_H(r)}{dz^2} + (1-2z)\frac{dU_H(r)}{dz} + \left[\nu_{PH}'(\nu_{PH}'-1) - \frac{4\alpha_{PH}^2}{4z} - \frac{4\beta_{PH}^2}{4(1-z)}\right]U_H(r) = 0 \tag{31}$$

By setting $U_H(r) = z^{\alpha_{PH}}(1-z)^{\beta_{PH}}f_H(z)$, then (31) is reduced to AIM-type differential equation that is similar to (14).

$$f_H''(z) + \left[\frac{(2\alpha_{PH}+1)-(2\alpha_{PH}+2\beta_{PH}+2)y}{z(1-z)}\right]f_H'(z) + \left[\frac{\nu_{PH}'(\nu_{PH}'-1)-(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)}{z(1-z)}\right]\cdot f_H(z) = 0 \tag{32}$$

By comparing (14) and (32), we have

$$\lambda_{0_{PH}} = \frac{-(2\alpha_{PH}+1)}{z} + \frac{(2\beta_{PH}+1)}{1-z} \tag{33}$$

$$s_{0_{PH}} = \left[\frac{(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)-\nu_{PH}'(\nu_{PH}'-1)}{z} + \frac{(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)-\nu_{PH}'(\nu_{PH}'-1)}{(1-z)}\right] \tag{34}$$

To obtain eigenvalue, we use (15)-(17) and (33)-(34), which yields

$$\nu_{PH}'(\nu_{PH}'-1) = (\alpha_{PH}+\beta_{PH}+n)(\alpha_{PH}+\beta_{PH}+(n+1)) \tag{35}$$

By inserting (27)-(28) and (30) into (35), we obtain the relativistic energy equation Klein-Gordon equation with minimal length for Hulthen potential, as follows:

$$\left(E^2-M_o{}^2\right) = \left\{\begin{matrix} -\omega_H^2\left[\dfrac{\left(\sqrt{\nu_{PH}'(\nu_{PH}'-1)+\dfrac{1}{4}}-n-\dfrac{1}{2}\right)^2}{} + \dfrac{\zeta_{PH}}{4\left(\omega_H^2\sqrt{\nu_{PH}'(\nu_{PH}'-1)+1/4}-n-1/2\right)^2}\right] \\ +(E+M_o)\dfrac{V_H}{2}+\tau_{PH}\end{matrix}\right\} \tag{36}$$

where

$$\nu'_{PH} = \frac{1}{2} + \sqrt{\frac{\left(\omega_H^2 L(L+1)+\alpha_{ML}(E+M_o)^2\left((V_H\omega_H)^2/2\right)\right)}{\omega_H^2}+\frac{1}{4}} \tag{37}$$

$$\zeta_{PH} = \begin{pmatrix}(E+M_o)\dfrac{(V_H\omega_H)}{2} + \alpha_{ML}(E+M_o)^2(V_H\omega_H)^2 \\ -2\alpha_{ML}\left(E^2-M_o{}^2\right)(E+M_o)(V_H\omega_H)\end{pmatrix} \tag{38}$$

Table 1: Relativistic energy with $V_H = 7$, $\omega_H = 0.1$, $M_o = 1$, and $L = 0$.

n	$E(eV)$			
	$\alpha_{ML} = 0$	$\alpha_{ML} = 0.001$	$\alpha_{ML} = 0.01$	$\alpha_{ML} = 0.1$
0	-1.9770	-1.9654	-1.8759	-1.5259
1	-1.8629	-1.8537	-1.7801	-1.4725
2	-1.7473	-1.7402	-1.6818	-1.4171
3	-1.6298	-1.6248	-1.5808	-1.3594
4	-1.5102	-1.5071	-1.4767	-1.2990
5	-1.3879	-1.3865	-1.3690	-1.2357

$$\tau_{PH} = \begin{bmatrix} \alpha_{ML}\left(E+M_o\right)^2\left(V_H\omega_H\right)^2 \\ +2\alpha_{ML}\left(E^2-M_o{}^2\right)^2 \\ -2\alpha_{ML}\left(E^2-M_o{}^2\right)\left(E+M_o\right)\left(V_H\omega_H\right) \end{bmatrix} \tag{39}$$

and n is quantum number. Equation (36) is relativistic energy equation of the minimal length formalism in Klein-Gordon equation with Hulthen potential. The relativistic energies were calculated numerically by using Matlab software. The results of relativistic energies are listed in Table 1.

Table 1 shows that the presence of minimal length and Hulthen potential in Klein-Gordon equation gives influence to the relativistic energy value. The relativistic energy value without minimal length parameter ($\alpha_{ML} = 0$) is lower than the relativistic energy value with minimal length parameter. Then, the relativistic energy value increases for the larger minimal length parameter and for the larger quantum number (n). The influence of Hulthen potential gives negative value in the relativistic energy value. If we set $\alpha_{ML} = 0$, without the presence of the minimal length parameter in relativistic energy equation (36), it was reduced to the relativistic energy equation which is in agreement with [5]. In [5] it was shown that the relativistic energy equation for the Klein-Gordon equation for Hulthen potential without the minimal length depends on the squared of quantum number (n).

To get the unnormalized wave function, we used (19)-(20), so we have

$$\begin{aligned} N_{PH} &= -1; \\ a_{PH} &= \left(\beta_{PH}+\frac{1}{2}\right); \\ b_{PH} &= 1; \\ t_{PH} &= \left(\alpha_{PH}-\frac{1}{2}\right); \\ \sigma_{PH} &= 2\alpha_{PH}+1; \\ \rho_{PH} &= 2\alpha_{PH}+2\beta_{H}+1 \end{aligned} \tag{40}$$

Inserting (40) into (19) yields

$$f_H(z) = \begin{bmatrix} (-1)^n C'(1)^n\left(2\alpha_{PH}+1\right)_n \\ {}_2F_1\left(-n, 2\alpha_{PH}+2\beta_{PH}+1+n, 2\alpha_{PH}+1, z\right) \end{bmatrix} \tag{41}$$

Equation (41) is substituted into $U_H(r) = z^{\alpha_{PH}}(1-z)^{\beta_{PH}} f_H(z)$, so we have the unnormalized wave function given as

$$U_H(r) = \begin{bmatrix} z^{\alpha_{PH}}(1-z)^{\beta_{PH}}(-1)^n C'(1)^n\left(2\alpha_{PH}+1\right)_n \\ {}_2F_1\left(-n, 2\alpha_{PH}+2\beta_{PH}+1+n, 2\alpha_{PH}+1, z\right) \end{bmatrix} \tag{42}$$

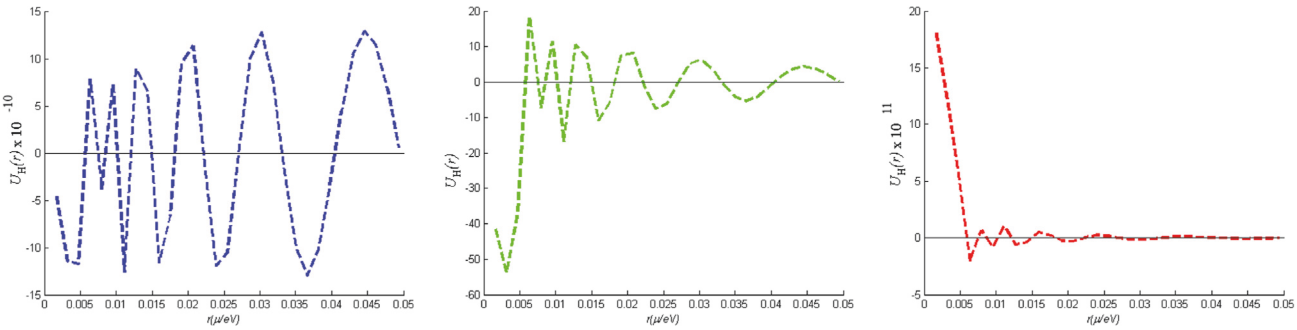
By applying $\coth\omega_H r = 1 - 2z$ in (42), we obtain the unnormalized wave function of minimal length formalism in Klein-Gordon equation, given as

$$U_{H_0}(z) = C'\left(\frac{1-\coth\omega_H r}{2}\right)^{\alpha_{PH}}\left(\frac{1+\coth\omega_H r}{2}\right)^{\beta_{PH}} \tag{43}$$

$$U_{H_1}(z) = \begin{Bmatrix} -C'\left(\frac{1-\coth\omega_H r}{2}\right)^{\alpha_{PH}}\left(\frac{1+\coth\omega_H r}{2}\right)^{\beta_{PH}}\left(2\alpha_{PH}+1\right) \\ \left[1+\frac{(-1)\left(2\alpha_{PH}+2\beta_{PH}+2\right)\left(\left(1-\coth\omega_H r\right)/2\right)}{\left(2\alpha_{PH}+1\right)1!}\right] \end{Bmatrix} \tag{44}$$

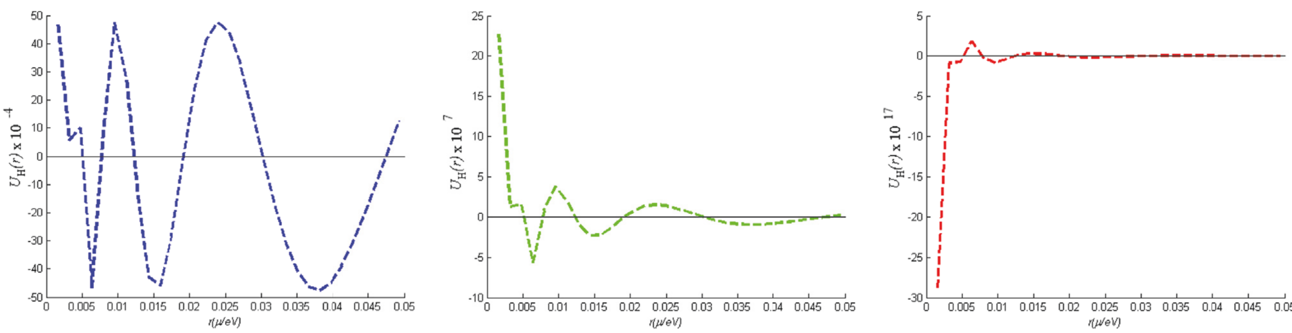
$$U_{H_2}(r) = \begin{Bmatrix} -C'\left(\frac{1-\coth\omega_H r}{2}\right)^{\alpha_{PH}}\left(\frac{1+\coth\omega_H r}{2}\right)^{\beta_{PH}}\begin{pmatrix}\left(2\alpha_{PH}+1\right) \\ \left(2\alpha_{PH}+2\right)\end{pmatrix} \\ \begin{bmatrix} 1+\frac{\begin{pmatrix}(-1)\left(2\alpha_{PH}+2\beta_{PH}+2\right) \\ \left(\left(1-\coth\omega_H r\right)/2\right)\end{pmatrix}}{\left(2\alpha_{PH}+1\right)1!} \\ +\frac{\begin{pmatrix}(-2)(-1)\left(2\alpha_{PH}+2\beta_{PH}+3\right) \\ \left(2\alpha_{PH}+2\beta_{PH}+4\right)\left(\left(1-\coth\omega_H r\right)/2\right)^2\end{pmatrix}}{\left(2\alpha_{PH}+1\right)\left(2\alpha_{PH}+2\right)2!} \end{bmatrix} \end{Bmatrix} \tag{45}$$

Equations (43), (44), and (45) are the unnormalized wave functions n=0 for ground state, n=1 for energy level 1, and n=2 for energy level 2, respectively. The visualization of the unnormalized wave functions is shown in Figures 3 and 4.



(a) (b) (c)

Figure 3: The unnormalized wave functions with $\alpha_{ML} = 0$ for (a) n=0, (b) n=1, and (c) n=2.



(a) (b) (c)

Figure 4: The unnormalized wave functions with $\alpha_{ML} = 0.1$ for (a) n=0, (b) n=1, and (c) n=2.

By inspecting Figures 3 and 4, we can see that the influence of minimal length parameter increases the amplitude of the unnormalized wave function.

6. Conclusion

The investigation of the minimal length formalism in the Klein-Gordon equation is obtained by approximate solution. The minimal length in Klein-Gordon equation for Hulthen potential is solved using Asymptotic Iteration Method. The Asymptotic Iteration Method is used to obtain the relativistic energy and unnormalized wave functions. The results show that the relativistic energies value increases for the larger minimal length parameter and for the larger quantum number (n). Then, the influence of minimal length parameter exerts effect in increasing the amplitude value of the unnormalized wave functions.

Data Availability

All data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Supplementary Materials

To solve the iteration equation and to do the numerical calculation, other software can be used such as Maple, Mathematica, and Octave. *(Supplementary Materials)*

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