

Research Article

A Formula for Eigenvalues of Jacobi Matrices with a Reflection Symmetry

S. B. Rutkevich 

Fakultät für Physik, Universität Duisburg-Essen, Duisburg 47048, Germany

Correspondence should be addressed to S. B. Rutkevich; sergei.rutkevich@uni-due.de

Received 11 July 2018; Accepted 1 October 2018; Published 16 October 2018

Academic Editor: Pavel Kurasov

Copyright © 2018 S. B. Rutkevich. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The spectral properties of two special classes of Jacobi operators are studied. For the first class represented by the $2M$ -dimensional real Jacobi matrices whose entries are symmetric with respect to the secondary diagonal, a new polynomial identity relating the eigenvalues of such matrices with their matrix entries is obtained. In the limit $M \rightarrow \infty$ this identity induces some requirements, which should satisfy the scattering data of the resulting infinite-dimensional Jacobi operator in the half-line, of which super- and subdiagonal matrix elements are equal to -1 . We obtain such requirements in the simplest case of the discrete Schrödinger operator acting in $l^2(\mathbb{N})$, which does not have bound and semibound states and whose potential has a compact support.

1. Introduction

The theory of tridiagonal (Jacobi) matrices has rich applications in different fields of physics and mathematics including the theory of Anderson localization [1–3], the thermodynamic Casimir effect [4, 5], orthogonal polynomials [6], and nonlinear integrable lattice models [7–9]. The spectral properties of Jacobi matrices are to much extent similar to the spectral properties of the continuous one-dimensional Sturm-Liouville operators. For a detailed review of the present stage of the theory of Jacobi operators see the monograph by G. Teschl [8].

In this paper we address the spectral properties of a special class of Jacobi matrices, which are symmetric with respect to both main and secondary diagonals. To be more specific, let us introduce the real Jacobi matrix, the Hamiltonian

$$H = \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 \\ a_1 & b_2 & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & a_{N-2} & b_{N-1} & a_{N-1} \\ 0 & 0 & 0 & a_{N-1} & b_N \end{pmatrix}, \quad (1)$$

which acts in the Hilbert space \mathbb{C}^N . Throughout this paper, the dimension of this space is supposed to be even,

$$N = 2M. \quad (2)$$

The (real) entries of the matrix H will be subject to the following constraints:

$$\begin{aligned} b_{N+1-j} &= b_j, & j &= 1, \dots, N, \\ a_{N-j} &= a_j, & j &= 1, \dots, N-1. \end{aligned} \quad (3)$$

Jacobi matrices satisfying this symmetry property were studied by Hochstadt [10].

Since the Hamiltonian H commutes with the reflection operator

$$R : \Psi(j) \rightarrow \Psi(N+1-j), \quad (4)$$

where $\{\Psi(j)\}_{j=1}^N \in \mathbb{C}^N$, the eigenvectors $\{\Psi_n\}_{n=1}^N$ of the matrix (1) can be classified by their parities $\kappa_n = \pm 1$,

$$H\Psi_n = \lambda_n \Psi_n, \quad (5)$$

$$R\Psi_n = \kappa_n \Psi_n. \quad (6)$$

There are exactly M eigenvectors $\Psi_m^{(ev)}$ of the matrix H , which are even under the action of the reflection (4), and

M eigenvectors $\Psi_m^{(od)}$, which are odd. We shall use different notation μ_m and ν_n for the corresponding eigenvalues,

$$\begin{aligned} H\Psi_m^{(ev)} &= \mu_m \Psi_m^{(ev)}, & \Psi_m^{(ev)}(N+1-j) &= \Psi_m^{(ev)}(j), \\ H\Psi_m^{(od)} &= \nu_m \Psi_m^{(od)}, & \Psi_m^{(od)}(N+1-j) &= -\Psi_m^{(od)}(j), \end{aligned} \quad (7)$$

with $j = 1, \dots, N, m = 1, \dots, M$.

As the main result, we prove the following.

Theorem 1. *Let H be the real Jacobi matrix (1) satisfying (2) and (3). Then, the eigenvalues $\{\mu_m\}_{m=1}^M$ and $\{\nu_m\}_{m=1}^M$ of the matrix H specified by relations (7) obey the equality*

$$\prod_{m=1}^M \prod_{n=1}^M (\mu_m - \nu_n) = (-1)^{M(M-1)/2} (2a_M)^M \prod_{j=1}^{M-1} a_j^{2j}. \quad (8)$$

Proceeding then to the limit $M \rightarrow \infty$, we obtain from (8) identities (60), (61) for the scattering data corresponding to the discrete Schrödinger operator (i.e., the Jacobi operator acting in $l^2(\mathbb{N})$, which is described by the infinite-dimensional matrix (1) with $a_j = -1$ and $b_j = 2 + \nu_j, j \in \mathbb{N}$) under certain assumptions on the potential $\{\nu_j\}_{j \in \mathbb{N}}$. These results are described in Section 4 and Theorem 5 therein.

A part of the results described here were given in the preprint [11].

2. Proof of Theorem 1

First, let us prove the theorem for the particular choice of the Jacobi matrix (1), which is specified by the relations

$$\begin{aligned} a_j &= -1, \\ b_{j'} &= 2 \\ \text{for all } j &= 1, \dots, N-1, \quad j' = 1, \dots, N. \end{aligned} \quad (9)$$

In this case, the solution of the eigenvalue problem (5) is well known,

$$\begin{aligned} \Psi_n(j) &= \sin(k_n j), \\ \lambda_n &= \omega(k_n), \\ k_n &= \frac{\pi n}{N+1}, \end{aligned} \quad (10)$$

with $n = 1, \dots, N$, and

$$\omega(p) = 4 \sin^2\left(\frac{p}{2}\right). \quad (11)$$

This yields

$$\begin{aligned} \mu_m &= 4 \sin^2\left(\frac{(2m-1)\pi}{2(2M+1)}\right), \\ \nu_m &= 4 \sin^2\left(\frac{2m\pi}{2(2M+1)}\right), \end{aligned} \quad (12)$$

and equality (8) reduces to the form

$$\begin{aligned} \prod_{m=1}^M \prod_{n=1}^M \left[4 \sin^2\left(\frac{(2m-1)\pi}{2(2M+1)}\right) - 4 \sin^2\left(\frac{2n\pi}{2(2M+1)}\right) \right] \\ = 2^M (-1)^{M(M+1)/2}. \end{aligned} \quad (13)$$

Proof of this equality is given in Appendix.

Turning to the eigenvalue problem for a general Jacobi matrix specified in the Theorem, we first rewrite it as a difference equation

$$\begin{aligned} a_j \Psi_n(j+1) + b_j \Psi_n(j) + a_{j-1} \Psi_n(j-1) &= \lambda_n \Psi_n(j), \\ j &= 1, \dots, N, \quad n = 1, \dots, N, \end{aligned} \quad (14)$$

subjected to constrains (3) and supplemented with the Dirichlet boundary conditions

$$\Psi_n(0) = \Psi_n(N+1) = 0. \quad (16)$$

Let us consider two associated eigenvalue problems for the even and odd states, which are restricted to the M -dimensional quotient vector space \mathbb{C}^M . We shall use the lower case ψ to denote vectors in this quotient space, $\{\psi(j)\}_{j=1}^M \in \mathbb{C}^M$. Of course, $\Psi(j) = \psi(j)$ for $j = 1, \dots, M$.

The vectors $\psi_m^{(ev)}(j), j = 1, \dots, M$, are the eigenstates of the tridiagonal $M \times M$ matrix $H^{(ev)}$:

$$H^{(ev)} = \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & \dots & 0 \\ a_1 & b_2 & a_2 & 0 & 0 & \dots & 0 \\ 0 & a_2 & b_3 & a_3 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & a_{M-2} & b_{M-1} & a_{M-1} \\ 0 & 0 & \dots & 0 & 0 & a_{M-1} & b_M + a_M \end{pmatrix}, \quad (17)$$

corresponding to eigenvalues $\mu_m, m = 1, \dots, M$. Similarly, the vectors $\psi_m^{(od)}(j), j = 1, \dots, M$, are the eigenstates of the tridiagonal $M \times M$ matrix $H^{(od)}$:

$$H^{(od)} = \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & \dots & 0 \\ a_1 & b_2 & a_2 & 0 & 0 & \dots & 0 \\ 0 & a_2 & b_3 & a_3 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & a_{M-2} & b_{M-1} & a_{M-1} \\ 0 & 0 & \dots & 0 & 0 & a_{M-1} & b_M - a_M \end{pmatrix}, \quad (18)$$

corresponding to eigenvalues $\nu_m, m = 1, \dots, M$. Note that the matrices $H^{(ev)}$ and $H^{(od)}$ are simply related

$$H^{(od)} - H^{(ev)} = -2a_M P, \quad (19)$$

with the projecting matrix $P_{m,m'} = \delta_{m,M}\delta_{m',M}$, $m, m' = 1, \dots, M$, and $\text{rank } P = 1$.

It is convenient now to allow parameters $\{a_j, b_j\}_{j=1}^M$ in the matrices (17), (18) to take complex values.

Lemma 2. *The matrices $H^{(od)}$ and $H^{(ev)}$ defined by (17), (18) have no common eigenvalues for arbitrary complex numbers $\{b_j\}_{j=1}^M$, if $a_j \neq 0$ for all $j = 1, \dots, M$.*

Proof. For a contradiction, we shall assume that the matrices $H^{(ev)}$ and $H^{(od)}$ have a common eigenvalue Λ .

So, let us suppose that

$$\begin{aligned} \sum_{j'=1}^M H_{j,j'}^{(ev)} x_{j'} &= \Lambda x_j, \\ \sum_{j'=1}^M H_{j,j'}^{(od)} y_{j'} &= \Lambda y_j, \end{aligned} \quad (20)$$

with nonzero vectors $\{x_j\}_{j=1}^M, \{y_j\}_{j=1}^M$. We get

$$\begin{aligned} \Lambda \sum_{j=1}^M y_j x_j &= \sum_{j=1}^M \sum_{j'=1}^M y_j H_{j,j'}^{(ev)} x_{j'} \\ &= \sum_{j=1}^M \sum_{j'=1}^M y_j \left(H_{j,j'}^{(od)} + 2a_M P_{j,j'} \right) x_{j'} \\ &= \sum_{j=1}^M \sum_{j'=1}^M y_j H_{j,j'}^{(od)} x_{j'} + 2a_M \sum_{j=1}^M \sum_{j'=1}^M y_j P_{j,j'} x_{j'} \\ &= 2a_M y_M x_M + \Lambda \sum_{j=1}^M y_j x_j. \end{aligned} \quad (21)$$

Here we have taken into account the fact that the matrix $H^{(od)}$ is symmetric. Therefore,

$$a_M y_M x_M = 0. \quad (22)$$

Since $a_M \neq 0$, this means that at least one of the numbers y_M and x_M is zero. However, if $y_M = 0$, we conclude immediately (really, if $0 = y_M \equiv \Psi(j = M) \equiv \Psi(j = M)$, then $\Psi(j = M + 1) = -\Psi(j = M) = 0$, since $\Psi(j) = -\Psi(2M + 1 - j)$ for all $j = 1, \dots, 2M$. And since the wave-function $\Psi(j)$ takes zero values at two neighbor sites $\Psi(M) = \Psi(M + 1) = 0$, and $a_j \neq 0$ for all $j = 1, \dots, M$, one can check, recursively, from (14) that $\Psi(M - 1) = 0, \Psi(M - 2) = 0, \dots, \Psi(1) = 0$, and, therefore, $\psi(j) = 0$ for all $j = 1, \dots, M$) that $y_m = 0$ for all $m = 1, \dots, M$. Therefore, Λ is not an eigenvalue of $H^{(od)}$. Similarly, if $x_M = 0$, we conclude that $x_m = 0$ for all $m = 1, \dots, M$, and, hence, Λ is not an eigenvalue of $H^{(ev)}$. This contradiction proves the lemma. \square

The eigenvalues $\{\mu_m\}_{m=1}^M$ and $\{\nu_m\}_{m=1}^M$ are the zeroes of the characteristic polynomials

$$P^{(ev)}(\mu) = \det(\mu - H^{(ev)}), \quad (23)$$

and

$$P^{(od)}(\nu) = \det(\nu - H^{(od)}), \quad (24)$$

respectively. It follows from Vieta's theorem that the symmetric polynomials of eigenvalues $\{\mu_m\}_{m=1}^M$, as well as the symmetric polynomials of eigenvalues $\{\nu_m\}_{m=1}^M$, can be written as polynomial functions of the variables $\{a_j\}_{j=1}^M$ and $\{b_j\}_{j=1}^M$. This is true, in particular for the function

$$\prod_{m=1}^M \prod_{n=1}^M (\mu_m - \nu_n) = \tilde{Q}(a, b), \quad (25)$$

since the product on the left-hand side is a symmetric polynomial of $\{\mu_m\}_{m=1}^M$, and it is also a symmetric polynomial of $\{\nu_m\}_{m=1}^M$.

However, the polynomial $\tilde{Q}(a, b)$ in the right-hand side of (25) must not depend on $\{b_j\}_{j=1}^M$, i.e.,

$$\tilde{Q}(a, b) \equiv Q(a). \quad (26)$$

To prove this, let us fix all a_j at some arbitrary nonzero complex values $a_j^{(0)} \in \mathbb{C}, a_j^{(0)} \neq 0$ for all $j = 1, \dots, M$. Then (25) takes the form

$$\prod_{m=1}^M \prod_{n=1}^M [\mu_m(b) - \nu_n(b)] = P(b), \quad (27)$$

$$P(b) \equiv \tilde{Q}(a^{(0)}, b), \quad (28)$$

where we have explicitly indicated the algebraic dependence of the eigenvalues μ_m and ν_m on the parameters b_j . One can easily see from (27) that the polynomial $P(b)$ has no zeroes. Really, if $P(b) = 0$ at some $b = b^{(0)}$, then at least one eigenvalue $\mu_{m_0}(b^{(0)})$ of the matrix $H^{(ev)}(b^{(0)})$ must coincide with some eigenvalue $\nu_{n_0}(b^{(0)})$ of the matrix $H^{(od)}(b^{(0)})$, as it is implied by equality (27). However, this is a contradiction with Lemma 2, which guaranties that matrices $H^{(ev)}(b^{(0)})$ and $H^{(od)}(b^{(0)})$ have no common eigenvalues. Since the polynomial $P(b)$ has no zeroes, it is just a constant, and we arrive at (26).

Thus, we have proven that

$$\prod_{m=1}^M \prod_{n=1}^M (\mu_m - \nu_n) = Q(a), \quad (29)$$

with some polynomial $Q(a)$. In order to find this polynomial explicitly, let us address some of its properties.

Lemma 3. (a) *The polynomial $Q(a)$ defined by (29) is homogeneous, and its degree equals to M^2 :*

$$Q(ta) = t^{M^2} Q(a). \quad (30)$$

(b) *The polynomial $Q(a)$ can be represented in the form*

$$Q(a) = a_1^2 a_2^4 \dots a_{M-1}^{2(M-1)} a_M^M Q_0(a), \quad (31)$$

where $Q_0(a)$ is also some polynomial of $\{a_j\}_{j=1}^M$.

Proof. The statement (a) follows immediately from (29). To prove (b), let us first note that $Q(a)$ vanishes, if one of the variables a_j takes the zero value. Really, if $a_j = 0$ for some j , the matrices $H^{(ev)}$ and $H^{(od)}$ defined by (17), (18) become block-diagonal containing two blocks of dimensions j and $M-j$. Furthermore, the j -dimensional blocks in their top-left corners are identical. Therefore, j eigenvalues of the matrix $H^{(ev)}$ merge with j eigenvalues of the matrix $H^{(od)}$ in this case,

$$\mu_m(a_j = 0) = \nu_m(a_j = 0) \equiv \Lambda_m, \quad \text{for } m = 1, \dots, j. \quad (32)$$

First, let us choose $j = 1$ and vary the parameter a_1 near the zero point $a_1 \rightarrow 0$ and keeping the rest $a_{j'}, j' = 2, \dots, M$ at some fixed nonzero values. Then, a simple perturbative analysis of equations

$$\begin{aligned} \det(\mu - H^{(ev)}) &= 0, \\ \det(\nu - H^{(od)}) &= 0, \end{aligned} \quad (33)$$

yields for small a_1 :

$$\begin{aligned} \mu_1(a_1) &= \Lambda_1 + O(a_1^2), \\ \nu_1(a_1) &= \Lambda_1 + O(a_1^2). \end{aligned} \quad (34)$$

Combining this result with (29), we conclude that the polynomial $Q(a)$ has at least the second order zero at a_1 .

Second, let us choose $j = 2$ and tune the parameter a_2 near the origin $a_2 \rightarrow 0$ keeping all others a_j nonzero. Then, for $j = 1, 2$, one can easily obtain from (32), (33)

$$\begin{aligned} \mu_j(a_2) &= \Lambda_j + O(a_2^2), \\ \nu_j(a_2) &= \Lambda_j + O(a_2^2). \end{aligned} \quad (35)$$

Therefore, two brackets $(\mu_1 - \nu_1)$ and $(\mu_2 - \nu_2)$ in the product on the left-hand side of (29) vanish at $a_2 \rightarrow 0$. Since the both brackets are of the order $O(a_2^2)$, the polynomial $Q(a)$ must have at least the fourth order zero at a_2 .

Finally, if a_j vanishes for some $j \in [1, M]$, while all others $a_{j'}$ remain nonzero, one can derive from (32), (33) for $m = 1, \dots, j$,

$$\begin{aligned} \mu_m(a_j) &= \begin{cases} \Lambda_m + O(a_j^2), & \text{if } j = 1, \dots, M-1, \\ \Lambda_m + O(a_j), & \text{if } j = M, \end{cases} \\ \nu_m(a_j) &= \begin{cases} \Lambda_m + O(a_j^2), & \text{if } j = 1, \dots, M-1, \\ \Lambda_m + O(a_j), & \text{if } j = M. \end{cases} \end{aligned} \quad (36)$$

Repeating for $j = 3, \dots, M$ the analysis described above leads to representation (31). \square

Combining (30) and (31), one concludes that the polynomial $Q_0(a)$ on the right-hand side of (31) is uniform, and its degree is zero. Therefore, $Q_0(a)$ does not depend on parameters a_j being just a constant, $Q_0(a) \equiv C_M$. To fix this

constant, we choose parameters $\{a_j, b_j\}_{j=1}^M$ according to (9). For such a choice of the Jacobi matrix, the product specified on the left-hand side of (29) is given by (13). Comparing the latter with (31), we find

$$C_M = (-1)^{M(M-1)/2} 2^M, \quad (37)$$

and arrive at equality (8).

Remark 4. If all $a_j \neq 0$ and the matrix (1) satisfying (3) is real, its eigenvalues λ_n are also real and nondegenerate. In this case, one can order them so that

$$\lambda_1 < \lambda_2 < \dots < \lambda_N. \quad (38)$$

It is possible to show that

$$\kappa_n = (-1)^n \text{sign } a_M, \quad \text{for } n = 1, \dots, 2M. \quad (39)$$

This follows from the remark after theorem 3 in [10] and Lemma 1.6 in [8].

3. Scattering Problem for the Discrete Schrödinger Operator

The main issue of this Section is to fix notations for later use. We briefly summarize some well-known facts of the scattering theory for the discrete Schrödinger operator. The scattering theory for general Jacobi operators, which is to some extent parallel to the scattering theory for the continuous Schrödinger equation (see, for example, [12]), is described in much details in the monograph [8].

Consider the discrete Schrödinger equation (14) in $l^2(\mathbb{N})$

$$(\mathbf{H}\psi)_j = \lambda\psi(j), \quad (40)$$

$$(\mathbf{H}\psi)_j = \nu_j\psi(j) + [2\psi(j) - \psi(j+1) - \psi(j-1)], \quad (41)$$

$$j = 1, 2, \dots, \infty, \quad (42)$$

with the potential $\nu_j \in \mathbb{R}$, supplemented with the Dirichlet boundary condition

$$\psi(0) = 0. \quad (43)$$

In order to simplify further analysis, only the potentials $V = \{\nu_j\}_{j \in \mathbb{N}}$ with a compact support will be considered, i.e.,

$$\nu_j = 0, \quad \text{if } j > J, \quad (44)$$

with some $J \in \mathbb{N}$. The latter requirement means that (40) looks as the free equation

$$2\psi(j) - \psi(j-1) - \psi(j+1) = \lambda\psi(j), \quad (45)$$

for $j \geq J+1$.

The Jacobi operator considered in the previous Section reduces to (41) at $a_j = -1$ and $b_j = 2 + \nu_j$.

The spectrum $\sigma[\mathbf{H}]$ of the discrete Schrödinger operator defined by (40)-(44) consists of the continuous part $\sigma_{cont}[\mathbf{H}] = [0, 4]$ and a finite number of discrete eigenvalues.

At a given λ , equations (40), (41) with omitted boundary condition (43) have two linearly independent solutions, and the general solution of (40), (41) can be written as their linear combination. Following [8], we introduce the Wronskian of two sequences $\{\psi_1(j)\}_{j \in \mathbb{N}}$, and $\{\psi_2(j)\}_{j \in \mathbb{N}}$,

$$\begin{aligned} W_j(\psi_1, \psi_2) &= -[\psi_1(j)\psi_2(j+1) - \psi_1(j+1)\psi_2(j)], \\ & \quad j = 0, 1, 2, \dots \end{aligned} \quad (46)$$

It does not depend on j , if $\psi_1(j)$ and $\psi_2(j)$ are solutions of (40), (41).

Let us turn now to the scattering problem associated with (40)-(44), which corresponds to the case $0 < \lambda < 4$. Instead of the parameter $\lambda \in \sigma_{\text{cont}}[\mathbf{H}]$, it is also convenient to use the momentum p and the related complex parameter $z = e^{ip}$:

$$\lambda = 2 - 2 \cos p = 2 - z - z^{-1}. \quad (47)$$

Three solutions of (40) are important for the scattering problem.

- (i) The *fundamental solution* $\varphi(j, p)$, which is fixed by the boundary conditions

$$\begin{aligned} \varphi(0, p) &= 0, \\ \varphi(1, p) &= 1. \end{aligned} \quad (48)$$

- (ii) Two *Jost solutions* $f(j, p)$, and $f(j, -p)$, which are determined by their behavior at large j and describe the out- and in-waves, respectively,

$$f(j, \pm p) = \exp(\pm ipj) = z^{\pm j}, \quad \text{at } j \geq J + 1. \quad (49)$$

Note that this formula is exact only in the considered case of the potential V with a compact support. In the general case, it must be replaced by the asymptotical formula $f(j, \pm p) \sim \exp(\pm ipj) = z^{\pm j}$, as $j \rightarrow \infty$. This comment also relates to (53) and (66).

The fundamental solution $\varphi(j, p)$ can be represented as a linear combination of two Jost solutions,

$$\begin{aligned} \varphi(j, p) &= \frac{i}{2 \sin p} [F(p) f(j, -p) - F(-p) f(j, p)], \\ & \quad 0 < p < \pi. \end{aligned} \quad (50)$$

The complex coefficient $F(p)$ in the above equation is known as the Jost function. It is determined by (50) for real momenta p in the interval $p \in (-\pi, \pi)$, where it satisfies the relation

$$F(-p) = [F(p)]^*, \quad (51)$$

and can be written as

$$F(p) = \exp[\sigma(p) - i\eta(p)], \quad (52)$$

with real $\sigma(p)$ and $\eta(p)$. At large $j \geq J + 1$, the fundamental solution behaves as

$$\varphi(j, p) = \frac{A(p)}{\sin p} \sin[pj + \eta(p)], \quad (53)$$

where $A(p) = \exp[\sigma(p)]$ is the scattering amplitude and $\eta(p)$ is the scattering phase. The latter can be defined in such a way that $\eta(-p) = -\eta(p)$ for $-\pi < p < \pi$.

Note that the Jost function $F(p)$ can be also represented as the Wronskian of the Jost and fundamental solutions

$$F(p) = W_j(\varphi(p), f(p)), \quad (54)$$

with arbitrary $j \in \mathbb{N}$.

The following exact representation holds for the Jost function $F(p)$ in terms of the fundamental solution $\varphi(j, p)$:

$$F(p) = 1 + \sum_{j=1}^J e^{ipj} v_j \varphi(j, p), \quad (55)$$

cf. (1.4.4) in [12] in the continuous case.

For $|z| = 1$, denote by $\hat{F}(z)$ the Jost function $F(p)$ expressed in terms of the complex parameter z : $F(p) = \hat{F}(z = e^{ip})$. The function $\hat{F}(z)$ can be analytically continued into the circle $|z| < 1$, where it has finite number of zeroes $\{\alpha_n\}_{n=1}^{\mathfrak{N}}$. These zeroes determine the discrete spectrum $\{\lambda_n\}_{n=1}^{\mathfrak{N}}$ of problem (40)-(43):

$$\lambda_n = 2 - \alpha_n - \alpha_n^{-1}, \quad \text{for } n = 1, \dots, \mathfrak{N}. \quad (56)$$

Of course, these eigenvalues are real in the boundary problem with a real potential.

Besides (44), two further restrictions will be imposed in the sequel on the potential V .

- (i) The Jost function $\hat{F}(z)$ associated with such a potential should not have zeroes inside the circle $|z| < 1$, i.e., $\mathfrak{N} = 0$. In other words, the spectrum $\sigma[\mathbf{H}]$ should be purely continuous.

- (ii) The Jost function $\hat{F}(z)$ should take nonzero values at $z = \pm 1$: $\hat{F}(1) \neq 0$, and $\hat{F}(-1) \neq 0$.

Conditions (i) and (ii) imply that the operator \mathbf{H} does not have bound and semibound states [12], respectively.

Since $\varphi(j, p)$ is a polynomial of the spectral parameter $\lambda = 2 - z - z^{-1}$ of the degree $j - 1$, the Jost function (55) expressed in the parameter z is a polynomial of the degree $2J - 1$:

$$\hat{F}(z) = 1 + \sum_{j=1}^{2J-1} c_j(V) z^j = \prod_{n=1}^{2J-1} \left[1 - \frac{z}{z_n(V)} \right], \quad (57)$$

where the real coefficients $c_j(V)$ polynomially depend on the potential v_j , $j = 1, \dots, J$. This indicates that the set of Jost polynomial (57) has the codimension $J - 1$ in the $2J$ -dimensional linear space of polynomials of the degree $2J - 1$. Due to the constrains (i), (ii), we get

$$|z_n(V)| > 1, \quad \text{for all } n = 1, \dots, 2J - 1. \quad (58)$$

Let us periodically continue the scattering phase $\eta(p)$ from the interval $(-\pi, \pi)$ to the whole real axis $p \in \mathbb{R}$. It follows from (58) that, for a potential V satisfying (44), (i), (ii), the scattering phase is a smooth odd 2π -periodical function in the whole real axis: $\eta(p) \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$.

The notation $\delta(\lambda)$ will be used for the scattering phase $\eta(p)$ expressed in terms of the spectral parameter λ : $\eta(p) = \delta(\lambda = 2 - 2 \cos p)$, for $0 \leq \lambda \leq 4$ and $0 \leq p \leq \pi$. Conditions (i), (ii) guarantee that

$$\delta(0) = \delta(4) = 0. \quad (59)$$

4. Identities on the Scattering Data for the Discrete Schrödinger Operator

In this Section we apply Theorem 1 to study some spectral properties of the discrete Schrödinger operator acting in $l^2(\mathbb{N})$, and prove the following

Theorem 5. *The scattering phase $\delta(\lambda)$ for the discrete Schrödinger operator $\mathbf{H} \in \text{End}[l^2(\mathbb{N})]$ which is defined by (41) with the Dirichlet boundary condition (43) and arbitrary potential $V = \{v_j\}_{j=1}^\infty$ obeying (44), (i), and (ii), must satisfy the equality*

$$\int_0^4 d\lambda \delta(\lambda) \frac{\lambda - 2}{\lambda(\lambda - 4)} + \frac{1}{\pi} \int_0^4 d\lambda_1 \delta(\lambda_1) \mathcal{P} \int_0^4 d\lambda_2 \frac{\delta'(\lambda_2)}{\lambda_2 - \lambda_1} = 0, \quad (60)$$

where $\mathcal{P} \int$ indicates the principal value integral.

Equality (60) can be rewritten as well in terms of the Jost function $\hat{F}(z)$:

$$\log \hat{F}(z = 1) + \log \hat{F}(z = -1) + \oint_{|z|=1} \frac{dz}{2\pi i} \log \left[\hat{F}\left(\frac{1}{z}\right) \right] \frac{d \log [\hat{F}(z)]}{dz} = 0, \quad (61)$$

where the integration path is counter-clockwise oriented.

Equality (61) can be also written in terms the zeroes $\{z_n(V)\}_{n=1}^{2J-1}$ of the Jost function $\hat{F}(z)$:

$$\prod_{1 \leq m < n \leq 2J-1} \left[1 - \frac{1}{z_n(V) z_m(V)} \right] = 1. \quad (62)$$

Proof. To prove (60), let us consider the discrete Schrödinger eigenvalue problem (14)-(16) in the finite interval $1 \leq j \leq N = 2M$, with $a_j = -1$, $M > J$, and with the even potential $V^{(M)} = \{v_j^{(M)} = b_j - 2\}_{j=1}^{2M}$, of which restriction to the interval $[1, M]$ coincides with that of the potential $V = \{v_j\}_{j=1}^\infty$:

$$v_j^{(M)} = \begin{cases} v_j, & \text{if } j \leq J, \\ 0, & \text{if } J < j \leq 2M - J, \\ v_{2M+1-j}, & \text{if } 2M - J < j \leq 2M. \end{cases} \quad (63)$$

As it was explained in the Introduction, the eigenfunctions $\Psi_l(j)$ of such a problem are either even, or odd with respect to the reflection (4); see (7). In both cases, the equality

$$[\Psi_l(M)]^2 = [\Psi_l(M+1)]^2 \quad (64)$$

must hold. On the other hand, the eigenfunction $\Psi_l(j)$ corresponding to the eigenvalue λ_l obeys the Dirichlet boundary condition at $j = 0$ and therefore is proportional to the fundamental solution $\varphi(j, p_l)$,

$$\Psi_l(j) = C_l \varphi(j, p_l), \quad (65)$$

where p_l and λ_l are related according to (47). Since the potential v_j is zero at the sites $j = M-2, M-1, M, M+1, M+2$ near the middle of the interval $[1, 2M]$, the fundamental solution $\varphi(j, p_l)$ has the form (53) at $j = M, M+1$. Substitution of (53) and (65) into (64) leads to the equation

$$\sin^2 [p_l M + \eta(p_l)] = \sin^2 [p_l (M+1) + \eta(p_l)], \quad (66)$$

which determines the spectrum $\lambda_l \equiv \omega(p_l)$ in terms of the scattering phase $\eta(p)$. The equivalent but more convenient form of (66) reads

$$(2M+1) p_l + 2\eta(p_l) = (2M+1) k_l, \quad (67)$$

where k_l were defined in (10).

For an arbitrary $M > J$, let us put $a_j = -1$ for all $j \in [0, M]$ in equality (8) and take the logarithm of the ratio of two versions of this equality written for the potential (63) and for the zero potential, respectively. As a result, we get

$$\sum_{m=1}^M S_m = 0, \quad (68)$$

where

$$S_m = \sum_{n=1}^M [\ln |\omega(p_{2n-1}) - \omega(p_{2m})| - \ln |\omega(k_{2n-1}) - \omega(k_{2m})|]. \quad (69)$$

In order to proceed to the large- M limit on the right-hand side, let us introduce three one-parametric sets of the C^∞ -functions $k(p; \epsilon)$, $p(k; \epsilon)$, and $F(k, k'; \epsilon)$, where ϵ is a small parameter lying in the interval $0 \leq \epsilon \leq \delta$ and δ is some fixed positive number such that

$$\delta < \frac{1}{2} \max_{p \in \mathbb{R}} |\eta'(p)|^{-1}. \quad (70)$$

The function $k(p; \epsilon)$ from the first set is defined in the real axis $p \in \mathbb{R}$ by the equation

$$k(p; \epsilon) = p + 2\epsilon \eta(p). \quad (71)$$

At a fixed $\epsilon \in [0, \delta]$, the function $k(p; \epsilon)$ monotonically increases in p being odd and quasiperiodic,

$$\begin{aligned} \partial_p k(p; \epsilon) &> 0, \\ k(-p; \epsilon) &= -k(p, \epsilon), \\ k(p + 2\pi; \epsilon) &= k(p; \epsilon) + 2\pi. \end{aligned} \quad (72)$$

For $\epsilon \in [0, \delta]$, the inverse function $p(k; \epsilon)$ is a well-defined odd quasiperiodic C^∞ -function of $k \in \mathbb{R}$, which also smoothly depends on the parameter ϵ . The Taylor expansion of $p(k, \epsilon)$ in ϵ reads as

$$p(k; \epsilon) = k - \epsilon \eta(k) + 4\epsilon^2 \eta(k) \eta'(k) + \sum_{j=3}^{\infty} \epsilon^j A_j(k), \quad (73)$$

where $A_j(k)$ are 2π -periodic odd bounded functions of k .

The function representing the third set

$$F(k, k'; \epsilon) = \ln \frac{\omega[p(k; \epsilon)] - \omega[p(k'; \epsilon)]}{\omega(k) - \omega(k')} \quad (74)$$

is 2π -periodic and even in both arguments $k, k' \in \mathbb{R}$ and $F(k, k'; \epsilon) \in C^\infty[\mathbb{R} \times \mathbb{R} \times [0, \delta]]$. Its Taylor expansion in ϵ

$$F(k, k'; \epsilon) = \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \partial_\epsilon^n F(k, k'; \epsilon) \Big|_{\epsilon=0} \quad (75)$$

converges at $0 \leq \epsilon \leq \delta$ uniformly in $(k, k') \in \mathbb{R}^2$. The two initial partial derivatives in this expansion read as

$$\partial_\epsilon F(k, k'; \epsilon) \Big|_{\epsilon=0} = 2 \frac{\omega'(k') \eta(k') - \omega'(k) \eta(k)}{\omega(k) - \omega(k')}, \quad (76)$$

$$\begin{aligned} \partial_\epsilon^2 F(k, k'; \epsilon) \Big|_{\epsilon=0} &= 4 \left\{ \frac{[\omega'(k) \eta^2(k)]' - [\omega'(k') \eta^2(k')]'}{\omega(k) - \omega(k')} \right. \\ &\quad \left. - \left[\frac{\omega'(k') \eta(k') - \omega'(k) \eta(k)}{\omega(k) - \omega(k')} \right]^2 \right\}. \end{aligned} \quad (77)$$

These and all higher derivatives $(\partial^n F(k, k'; \epsilon) / \partial \epsilon^n) \Big|_{\epsilon=0}$, $n = 3, \dots$, in the Taylor expansion (75) are the 2π -periodic $C^\infty[\mathbb{R}^2]$ -functions of their arguments $(k, k') \in \mathbb{R}^2$, which are therefore bounded in \mathbb{R}^2 together with all their partial derivatives in k and k' .

Let us now rewrite the sum (69) as

$$S_m = \sum_{n=1}^M F(k_{2n-1}, k_{2m}; \epsilon) \Big|_{\epsilon=(2M+1)^{-1}}, \quad (78)$$

and expand the right-hand side in ϵ to the second order,

$$\begin{aligned} S_m &= \sum_{n=1}^M \left[\frac{\partial_\epsilon F(k_{2n-1}, k_{2m}; \epsilon) \Big|_{\epsilon=0}}{2M+1} \right. \\ &\quad + \frac{\partial_\epsilon^2 F(k_{2n-1}, k_{2m}; \epsilon) \Big|_{\epsilon=0}}{2(2M+1)^2} \\ &\quad \left. + \delta_3 F(k_{2n-1}, k_{2m}; \epsilon) \Big|_{\epsilon=(2M+1)^{-1}} \right]. \end{aligned} \quad (79)$$

It follows from the above analysis that the correction term in the square brackets is uniformly bounded in n and m ,

$$|\delta_3 F(k_{2n-1}, k_{2m}; \epsilon)| < \epsilon^3 C, \quad (80)$$

with some positive C independent on n, m . Taking this into account, one obtains from (76)-(79)

$$S_m = S_m^{(0)} + S_m^{(1)} + O(M^{-2}), \quad (81)$$

where

$$S_m^{(0)} = \frac{1}{2M+1} \sum_{n=1}^M \partial_\epsilon F(k_{2n-1}, k_{2m}; \epsilon) \Big|_{\epsilon=0}, \quad (82)$$

$$S_m^{(1)} = \frac{X_m}{\pi(2M+1)}. \quad (83)$$

where

$$X_m = \frac{\pi}{2(2M+1)} \sum_{n=1}^M \partial_\epsilon^2 F(k_{2n-1}, k_{2m}; \epsilon) \Big|_{\epsilon=0}, \quad (84)$$

and the m -dependent correction term $O(M^{-2})$ is again uniformly bounded in m . Replacing in (84) the Riemann sum of the regular function of the momenta k_{2n-1} by the corresponding integral in the limit of large M , one obtains

$$X_m = \frac{1}{4} \int_0^\pi dq \partial_\epsilon^2 F(q, k_{2m}; \epsilon) \Big|_{\epsilon=0} + O(M^{-1}). \quad (85)$$

The m -dependent correction term $O(M^{-1})$ is uniformly bounded in m , since $\partial_\epsilon^2 F(q, k_{2m}; \epsilon) \Big|_{\epsilon=0} \in C^\infty[\mathbb{R}/(2\pi\mathbb{Z}) \times \mathbb{R}/(2\pi\mathbb{Z})]$.

Calculation of the large- M asymptotics of $S_m^{(0)}$ is more delicate. First, we extend summation in (82) in the index n from 1 till $2M+1$,

$$\begin{aligned} S_m^{(0)} &= \frac{2}{2M+1} \sum_{n=1}^M R_m(k_{2n-1}) \\ &= \frac{2}{2M+1} \left[-\frac{R_m(k_{2M+1})}{2} + \frac{1}{2} \sum_{n=1}^{2M+1} R_m(k_{2n-1}) \right], \end{aligned} \quad (86)$$

where

$$\begin{aligned} R_m(q) &= \frac{1}{2} \partial_\epsilon F(q, k_{2m}; \epsilon) \Big|_{\epsilon=0} \\ &= \frac{\omega'(k_{2m}) \eta(k_{2m}) - \omega'(q) \eta(q)}{\omega(q) - \omega(k_{2m})}. \end{aligned} \quad (87)$$

In (86) we have taken into account the reflection symmetry $R_m(q) = R_m(2\pi - q)$ of the function (87), which leads to $R_m(k_{2n-1}) = R_m(k_{2(2M+1-n)+1})$.

Since $k_{2M+1} = \pi$, $\eta(\pi) = 0$, and $\omega(\pi) = 4$, we get from (87)

$$R_m(k_{2M+1}) = \frac{\omega'(k_{2m}) \eta(k_{2m})}{4 - \omega(k_{2m})}. \quad (88)$$

The sum in the second line of (86) reads as

$$\sum_{n=1}^{2M+1} R_m(k_{2n-1}) = \sum_{n=1}^{2M+1} R_m\left(2\pi \frac{n-1/2}{2M+1}\right). \quad (89)$$

Since $R_m(q) \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$, this sum can be replaced with exponential accuracy by the integral at large $M \rightarrow \infty$:

$$\begin{aligned} \sum_{n=1}^{2M+1} R_m\left(2\pi \frac{n-1/2}{2M+1}\right) \\ = \frac{2M+1}{2\pi} \int_0^{2\pi} dq R_m(q) + o(M^{-\mu}), \end{aligned} \quad (90)$$

where μ is an arbitrary positive number; see formula 25.4.3 in [13]. Furthermore, since $\partial_\epsilon F(q, k; \epsilon)|_{\epsilon=0} \in C^\infty[\mathbb{R}/(2\pi\mathbb{Z}) \times \mathbb{R}/(2\pi\mathbb{Z})]$, the m -dependent correction term $o(M^{-\mu})$ is uniformly bounded in m .

Taking into account (87), the integral on the right-hand side of (90) can be written as

$$\begin{aligned} \int_0^{2\pi} dq R_m(q) \\ = \omega'(k_{2m}) \eta(k_{2m}) \mathcal{P} \int_0^{2\pi} \frac{dq}{\omega(q) - \omega(k_{2m})} \\ - \mathcal{P} \int_0^{2\pi} dq \frac{\omega'(q) \eta(q)}{\omega(q) - \omega(k_{2m})}. \end{aligned} \quad (91)$$

The principal value integral in the first term on the right-hand side vanishes

$$\begin{aligned} \mathcal{P} \int_0^{2\pi} \frac{dq}{\omega(q) - \omega(k_{2m})} = 2\mathcal{P} \int_0^\pi \frac{dq}{\omega(q) - \omega(k_{2m})} \\ = 0 \end{aligned} \quad (92)$$

due to the equality

$$\begin{aligned} \mathcal{P} \int_0^\pi \frac{dq}{[\omega(q) - \omega(k)]^\nu} \equiv \frac{1}{2} \\ \cdot \lim_{\epsilon \rightarrow +0} \left\{ \int_0^\pi \frac{dq}{[\omega(q+i\epsilon) - \omega(k)]^\nu} \right. \\ \left. + \int_0^\pi \frac{dq}{[\omega(q-i\epsilon) - \omega(k)]^\nu} \right\} = 0, \end{aligned} \quad (93)$$

which can be easily proved for $\omega(p) = 2 - 2 \cos p$, all $\nu \in \mathbb{N}$, and $0 < k < \pi$. The integral in the second line of (91) reduces after the change $\lambda = \omega(q)$ of the integration variable to the form

$$\mathcal{P} \int_0^{2\pi} dq \frac{\omega'(q) \eta(q)}{\omega(q) - \omega(k_{2m})} = 2I[\omega(k_{2m})], \quad (94)$$

where

$$I(\Lambda) = \mathcal{P} \int_0^4 d\lambda \frac{\delta(\lambda)}{\lambda - \Lambda}, \quad \text{with } 0 < \Lambda < 4. \quad (95)$$

Collecting (86)-(95), we get

$$\begin{aligned} S_m^{(0)} = -\frac{I[\omega(k_{2m})]}{\pi} - \frac{1}{2M+1} \frac{\omega'(k_{2m}) \eta(k_{2m})}{4 - \omega(k_{2m})} \\ + o(M^{-\mu}), \end{aligned} \quad (96)$$

where the m -dependent error term $o(M^{-\mu})$ is uniformly bounded in m . Thus, we obtain from (68) and (81) the equality

$$\lim_{M \rightarrow \infty} \sum_{m=1}^M S_m^{(0)} + \lim_{M \rightarrow \infty} \sum_{m=1}^M S_m^{(1)} = 0, \quad (97)$$

where $S_m^{(0)}$ and $S_m^{(1)}$ are given by (96), (83), and (85), respectively.

In the second term, we can replace with sufficient accuracy the summation in m by integration in the momentum k :

$$\begin{aligned} \sum_{m=1}^M S_m^{(1)} = \frac{1}{2\pi^2} \int_0^\pi dk \\ \cdot \int_0^\pi dq \frac{[\omega'(q) \eta^2(q)]' - [\omega'(k) \eta^2(k)]'}{\omega(q) - \omega(k)} \\ - \frac{1}{2\pi^2} \int_0^\pi dk \\ \cdot \int_0^\pi dq \left[\frac{\omega'(k) \eta(k) - \omega'(q) \eta(q)}{\omega(q) - \omega(k)} \right]^2 \\ + O(M^{-1}). \end{aligned} \quad (98)$$

The first integral on the right-hand side vanishes due to equality (93) with $0 < k < \pi$, and $\nu = 1$. The second line in (98) can be transformed as follows:

$$\begin{aligned} -\frac{1}{2\pi^2} \int_0^\pi dk \int_0^\pi dq \left[\frac{\omega'(k) \eta(k) - \omega'(q) \eta(q)}{\omega(q) - \omega(k)} \right]^2 \\ = -\frac{1}{\pi^2} \int_0^\pi dk [\omega'(k) \eta(k)]^2 \mathcal{P} \int_0^\pi \frac{dq}{[\omega(q) - \omega(k)]^2} \\ + \frac{1}{\pi^2} \int_0^\pi dk \mathcal{P} \int_0^\pi dq \frac{\omega'(k) \eta(k) \omega'(q) \eta(q)}{[\omega(q) - \omega(k)]^2}. \end{aligned} \quad (99)$$

The first term on the right-hand side vanishes due to equality (93) with $\nu = 2$. Then, substituting for $\lambda_1 = \omega(k)$ and $\lambda_2 = \omega(q)$ and integrating by parts, we obtain from the second term on the right-hand side of (99)

$$\lim_{M \rightarrow \infty} \sum_{m=1}^M S_m^{(1)} = \frac{1}{\pi^2} \int_0^4 d\lambda_1 \delta(\lambda_1) \mathcal{P} \int_0^4 d\lambda_2 \frac{\delta'(\lambda_2)}{\lambda_2 - \lambda_1}. \quad (100)$$

Let us turn now to calculation of the first term on the left-hand side of equality (97). At large M , one obtains

$$\begin{aligned} \sum_{m=1}^M S_m^{(0)} = -\frac{1}{2\pi} \int_0^\pi dk \frac{\omega'(k) \eta(k)}{4 - \omega(k)} - \frac{1}{\pi} \sum_{m=1}^M I[\omega(k_{2m})] \\ + O(M^{-1}). \end{aligned} \quad (101)$$

The first term on the right-hand side equals $I(4)/(2\pi)$. The large- M asymptotics of the sum on the right-hand side can be found as follows:

$$\begin{aligned} \sum_{m=1}^M I[\omega(k_{2m})] &= -\frac{I(0)}{2} + \frac{1}{2} \sum_{m=1}^{2M+1} I[\omega(k_{2m})] \\ &= -\frac{I(0)}{2} + \frac{2M+1}{4\pi} \int_0^{2\pi} dk I[\omega(k)] \\ &\quad + O(M^{-\mu}), \end{aligned} \quad (102)$$

where μ is an arbitrary positive number. In deriving (102) we have taken into account the fact that $I[\omega(k)]$ is the 2π -periodical function of p in \mathbb{R} , which is continuous with all its derivatives [i.e., $I[\omega(k)] \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$], and used formula 25.4.3 of the monograph [13] by Abramowitz and Stegun. The integral on the right-hand side vanishes due to equality (93) with $\nu = 1$:

$$\begin{aligned} \int_0^{2\pi} dk I[\omega(k)] &= \int_0^{2\pi} dk \mathcal{P} \int_0^4 d\lambda \frac{\delta(\lambda)}{\lambda - \omega(k)} \\ &= \int_0^4 d\lambda \delta(\lambda) \mathcal{P} \int_0^{2\pi} \frac{dk}{\lambda - \omega(k)} = 0. \end{aligned} \quad (103)$$

Collecting (101)-(103), we come to the simple formula

$$\lim_{M \rightarrow \infty} \sum_{m=1}^M S_m^{(0)} = \frac{I(0) + I(4)}{2\pi}. \quad (104)$$

From (95), (97), (100), and (104), we arrive at the final result (60).

Derivation of (61) from (60) is straightforward being based on (52) and exploits properties (i), (ii), which were supposed to hold for the Jost function $\widehat{F}(z)$ in Section 3.

The integrand of the integral on the left-hand side of (61) is meromorphic in the region $|z| > 1$ having there only simple poles at $z = z_n, n = 1, \dots, 2J-1$ with residues $\ln \widehat{F}(1/z_n)$. After straightforward application of the Cauchy's residue theorem, one finds from (61)

$$\ln \widehat{F}(1) + \ln \widehat{F}(-1) - \sum_{n=1}^{2J-1} \ln \widehat{F}\left(\frac{1}{z_n}\right) = 0. \quad (105)$$

Substitution of the right-hand side of (57) into this relation yields

$$\left[\prod_{1 \leq m < n \leq 2J-1} \left(1 - \frac{1}{z_n z_m}\right) \right]^2 = 1. \quad (106)$$

In order prove equality (62), it remains to show that the product in the square brackets on the left-hand side is positive. The latter statement follows immediately from the two properties of the zeroes z_n of the Jost function $\widehat{F}(z)$.

- (1) $|z_n| > 1$ for all $n = 1, \dots, 2J - 1$; see (58).

- (2) The numbers z_n either are real or else occur as complex conjugate pares in the set $\{z_n\}_{n=1}^{2J-1}$ of roots of the real polynomial (57). \square

Example 6. Let us check directly that equality (62) holds for $J = 2$. In this case, the Jost function (57) is the polynomial of the third degree,

$$\widehat{F}(z) = 1 + z(v_1 + v_2) + z^2 v_1 v_2 + z^3 v_2, \quad (107)$$

which is parametrized by two parameters v_1, v_2 . Let us rewrite (62) in the form

$$\frac{\sigma_3^2 - \sigma_3 \sigma_1 + \sigma_2 - 1}{\sigma_3^2} = 1 \quad (108)$$

in terms of the symmetric polynomials of the Jost function zeroes,

$$\begin{aligned} \sigma_1 &= z_1 + z_2 + z_3, \\ \sigma_2 &= z_1 z_2 + z_1 z_3 + z_2 z_3, \\ \sigma_3 &= z_1 z_2 z_3. \end{aligned} \quad (109)$$

Exploiting Vieta's theorem, one obtains from (107)

$$\begin{aligned} \sigma_1 &= -v_1, \\ \sigma_2 &= \frac{v_1 + v_2}{v_2}, \\ \sigma_3 &= -\frac{1}{v_2}. \end{aligned} \quad (110)$$

Upon substitution of the right-hand sides into (108), we convince ourselves that equality (108) holds for arbitrary complexes v_1 and v_2 .

5. Conclusions and Open Problems

We have studied some spectral properties of two classes of Jacobi operators.

The first class is represented by the real $2M$ -dimensional Jacobi matrices with the refraction symmetry (3). We have proved in Theorem 1 a new polynomial identity (8), which relates the eigenvalues of such Jacobi matrices with their matrix elements.

The second class of Jacobi operators considered in the present paper is represented by the discrete Schrödinger operators acting in $l^2(\mathbb{N})$ with purely continuous spectrum, of which potentials have a compact support. Note that the infinite-dimensional Jacobi matrix corresponding to the discrete Schrödinger operator has the super- and subdiagonal matrix elements $a_j = -1$ for all $j \in \mathbb{N}$. The scattering data of such operators must satisfy three identities (60), (61), and (62) which are proved in Theorem 5.

Obtained result could be extended and generalized in several directions.

We did not try to prove identities (60), (61) for the most general case of the discrete semiinfinite scattering problem. It

is natural to expect, however, that they should hold for some more general class of potentials, which vanish fast enough at infinity, though they do not have a compact support. On the other hand, in the case of the potentials with nonempty discrete spectrum and/or semibound states (the latter appear if the Jost function has zeroes at the end points of the continuous spectrum; see [12]), some modified forms of (60), (61) should also exist.

It should be noted also that no counterparts of Theorems 1 and 5 are known in the continuous Sturm-Liouville theory. Simple heuristic arguments exploiting the formal continuous limit of the discrete boundary problem (14)-(16) and identity (8) lead to the following tentative form of the extension of Theorem 1 to the continuous case.

Hypothesis. Let $\mathcal{H}^{(\pm)}$ be two Schrödinger operators

$$\mathcal{H}^{(\pm)} = -\partial_z^2 \pm v(z) \quad (111)$$

acting in “the appropriate space of function” defined in the interval $0 \leq z \leq \pi$, which satisfy the Dirichlet boundary conditions $\psi(0) = \psi(\pi) = 0$. Let the potential $v(z)$ in (111) be a real even function,

$$v(\pi - z) = v(z), \quad (112)$$

which belongs to some space \mathcal{L} of “good enough” potentials. The space \mathcal{L} should contain the piecewise continuous potentials defined on $[0, \pi]$.

Denote by $\{\mu_m^{(\pm)}, \nu_m^{(\pm)}\}_{m \in \mathbb{N}}$ the naturally ordered eigenvalues of the operators $\mathcal{H}^{(\pm)}$, which correspond to the eigenfunctions with different parities:

$$\begin{aligned} \mathcal{H}^{(\pm)} \psi_m^{(\pm, ev)} &= \mu_m^{(\pm)} \psi_m^{(\pm, ev)}, \\ \psi_m^{(\pm, ev)}(\pi - z) &= \psi_m^{(\pm, ev)}(z), \\ \mathcal{H}^{(\pm)} \psi_m^{(\pm, od)} &= \nu_m^{(\pm)} \psi_m^{(\pm, od)}, \\ \psi_m^{(\pm, od)}(\pi - z) &= -\psi_m^{(\pm, od)}(z), \end{aligned} \quad (113)$$

$$\mu_m^{(\pm)} < \mu_{m+1}^{(\pm)},$$

$$\nu_m^{(\pm)} < \nu_{m+1}^{(\pm)},$$

$$m = 1, \dots, \infty.$$

Then the following identity should hold:

$$\begin{aligned} \prod_{m=1}^{\infty} \prod_{n=1}^{\infty} \frac{\mu_m^{(+)} - \nu_n^{(+)}}{(2m)^2 - (2n-1)^2} \\ = \prod_{m=1}^{\infty} \prod_{n=1}^{\infty} \frac{(2m)^2 - (2n-1)^2}{\mu_m^{(-)} - \nu_n^{(-)}}. \end{aligned} \quad (114)$$

We have performed numerical check of formula (114) for the probe singular potential $v(z) = A\delta(z - \pi/2)$ with several values of the parameter A in the interval $1 \leq A \leq 5$ by cutting the infinite products in n and m in (114) at different values of $n_{max} = m_{max} = N$, with $10 \leq N \leq 100$. The results of our numerical calculations agree well with (114).

It would be interesting to find the precise form and the direct proof of the above statement.

Appendix

A. Proof of Equality (13)

Let us start from the following auxiliary.

Lemma A.1. *The following equality holds for all natural M and integer n :*

$$\begin{aligned} \prod_{m=1}^{2M+1} \left\{ 4 \sin^2 \left[\frac{(2m-1)\pi}{2(2M+1)} - \alpha \right] \right. \\ \left. - 4 \sin^2 \left[\frac{2n\pi}{2(2M+1)} \right] \right\} = 4 \cos^2 [\alpha(2M+1)]. \end{aligned} \quad (A.1)$$

Proof. Denote

$$\begin{aligned} g(\alpha) &= \prod_{m=1}^{2M+1} \left\{ 4 \sin^2 \left[\frac{(2m-1)\pi}{2(2M+1)} - \alpha \right] \right. \\ &\quad \left. - 4 \sin^2 \left[\frac{2n\pi}{2(2M+1)} \right] \right\}. \end{aligned} \quad (A.2)$$

Function $g(\alpha)$ is analytic in the complex α -plane and has the second order zeroes at the points

$$\alpha_l = \frac{\pi}{2(2M+1)} + \frac{\pi l}{2M+1}, \quad l = 0, \pm 1, \pm 2, \dots \quad (A.3)$$

It follows from (A.3) that the function

$$R(\alpha) = \frac{g(\alpha)}{4 \cos^2 [\alpha(2M+1)]} \quad (A.4)$$

is analytic and has no zeroes in the complex α -plane. Furthermore, this function is rational in the variable $z = e^{i\alpha}$ and approaches to 1 at $z \rightarrow \infty$ and at $z \rightarrow 0$. Therefore, $R(\alpha) = 1$. \square

Putting $\alpha = 0$ in (A.1) we find

$$\begin{aligned} \prod_{m=1}^{2M+1} \left\{ 4 \sin^2 \left[\frac{(2m-1)\pi}{2(2M+1)} \right] - 4 \sin^2 \left[\frac{2n\pi}{2(2M+1)} \right] \right\} \\ = 4. \end{aligned} \quad (A.5)$$

Since

$$\begin{aligned} \prod_{j=1}^{2M+1} \left\{ 4 \sin^2 \left[\frac{(2m-1)\pi}{2(2M+1)} \right] - 4 \sin^2 \left[\frac{2n\pi}{2(2M+1)} \right] \right\} \\ = \left[\prod_{m=1}^M \left\{ 4 \sin^2 \left[\frac{(2m-1)\pi}{2(2M+1)} \right] \right. \right. \\ \left. \left. - 4 \sin^2 \left[\frac{2n\pi}{2(2M+1)} \right] \right\} \right]^2 4 \left(1 - \sin^2 \left[\frac{2n\pi}{2(2M+1)} \right] \right), \end{aligned} \quad (A.6)$$

we get

$$\left[\prod_{m=1}^M \left\{ 4 \sin^2 \left[\frac{(2m-1)\pi}{2(2M+1)} \right] - 4 \sin^2 \left[\frac{2n\pi}{2(2M+1)} \right] \right\} \right]^2 = \left[\cos \frac{\pi n}{2M+1} \right]^{-2}, \quad (\text{A.7})$$

or

$$\prod_{m=1}^M \left\{ 4 \sin^2 \left[\frac{(2m-1)\pi}{2(2M+1)} \right] - 4 \sin^2 \left[\frac{2n\pi}{2(2M+1)} \right] \right\} = (-1)^n \left[\cos \frac{\pi n}{2M+1} \right]^{-1}. \quad (\text{A.8})$$

To fix the sign of the right-hand side of (A.8), we have taken into account that just the first n factors in the product on the left-hand side are negative at $n = 1, \dots, M$.

Thus,

$$\prod_{n=1}^M \prod_{m=1}^M \left[4 \sin^2 \frac{(2m-1)\pi}{2(2M+1)} - 4 \sin^2 \frac{2n\pi}{2(2M+1)} \right] = \prod_{n=1}^M \frac{(-1)^n}{\cos(\pi n / (2M+1))}. \quad (\text{A.9})$$

To determine the product on the right-hand side we use formula 1.392.1 in [14]:

$$\sin nx = 2^{n-1} \prod_{k=0}^{n-1} \sin \left(x + \frac{\pi k}{n} \right). \quad (\text{A.10})$$

For $n = 2M + 1$, $x = \pi/2$, we get from (A.10)

$$\prod_{k=0}^{2M} \cos \left(\frac{\pi k}{2M+1} \right) = 2^{-2M} (-1)^M, \quad (\text{A.11})$$

yielding

$$\prod_{k=1}^M \cos \left(\frac{\pi k}{2M+1} \right) = 2^{-M}. \quad (\text{A.12})$$

Substitution of (A.12) into (A.9) leads finally to (13).

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that they have no conflicts of interest.

Acknowledgments

The author is thankful to H. W. Diehl for many fruitful discussions. This work was funded by Deutsche Forschungsgemeinschaft (DFG), via Grant Ru 1506/1.

References

- [1] I. M. Lifshits, S. A. Gredeskul, and L. A. Pastur, *Introduction to the Theory of Disordered Systems*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1988.
- [2] V. A. Chulaevsky and Y. G. Sinai, "Anderson localization for the 1-D discrete Schrödinger operator with two-frequency potential," *Communications in Mathematical Physics*, vol. 125, no. 1, pp. 91–112, 1989.
- [3] M. Aizenman and S. Molchanov, "Localization at large disorder and at extreme energies: an elementary derivation," *Communications in Mathematical Physics*, vol. 157, no. 2, pp. 245–278, 1993.
- [4] H. W. Diehl, D. Grüneberg, M. Hasenbusch, A. Hucht, S. B. Rutkevich, and F. M. Schmidt, "Large- N approach to thermodynamic Casimir effects in slabs with free surfaces," *Physical Review E: Statistical, Nonlinear, and Soft Matter Physics*, vol. 89, no. 6, 2014.
- [5] S. B. Rutkevich and H. W. Diehl, "Inverse-scattering-theory approach to the exact $n \rightarrow \infty$ solutions of $O(n)\phi^4$ models on films and semi-infinite systems bounded by free surfaces," *Physical Review E: Statistical, Nonlinear, and Soft Matter Physics*, vol. 91, no. 6, Article ID 062114, 48 pages, 2015.
- [6] P. A. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach*, vol. 3 of *Courant Lecture Notes in Mathematics*, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, New York, NY, USA, 1999.
- [7] M. Toda, *Theory of Nonlinear Lattices*, vol. 20, Springer-Verlag, Berlin, Germany, 2nd edition, 1989.
- [8] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, vol. 72, The American Mathematical Society, Providence, RI, USA, 2000.
- [9] S. P. Novikov, "Riemann surfaces, operator fields, strings— analogues of the Fourier-Laurent bases," *Progress of Theoretical and Experimental Physics*, no. 102, pp. 293–300, 1990.
- [10] H. Hochstadt, "On the construction of a Jacobi matrix from spectral data," *Linear Algebra and Its Applications*, vol. 8, no. 5, pp. 435–446, 1974.
- [11] S. B. Rutkevich, "On the spectrum of the discrete 1d Schrödinger operator with an arbitrary even potential," arXiv1404.4325v1, 2014.
- [12] K. Chadan and P. C. Sabatier, *Inverse Problems in Quantum Scattering Theory*, Texts and Monographs in Physics, Springer, New York, NY, USA, 2nd edition, 1989.
- [13] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables*, Dover, New York, NY, USA, 1972.
- [14] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, London, 4th edition, 1980.



Hindawi

Submit your manuscripts at
www.hindawi.com

