Research Article

Moduli Space in Homological Mirror Symmetry

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We prove that the moduli space of the pseudo holomorphic curves in the A-model on a symplectic torus is homeomorphic to a moduli space of Feynman diagrams in the configuration space of the morphisms in the B-model on the corresponding elliptic curve. These moduli spaces determine the A_∞ structure of the both models.

1. Introduction

In the one-dimensional homological mirror symmetry (HMS) [1], the A-model on a symplectic torus corresponds to the B-model on an elliptic curve [2]. The objects, representing D-branes, are the Lagrangian submanifolds in the A-model and the complexes of the coherent sheaves in the B-model. In one dimension, any real one-dimensional submanifold is the Lagrangian and the line bundles are coherent sheaves. The morphisms, representing the open strings between the D-branes, are described by Abelian groups whose bases are given by intersecting points of the real one-dimensional submanifolds in the A-model and the maps between the complexes of the line bundles in the B-model. From these objects and morphisms, we can canonically construct the Fukaya category in the A-model and the derived category of coherent sheaves in the B-model. It is proved in [3–9] that the Fukaya category is equivalent as an A_∞ -category to the differential graded (DG) category (The DG-category is an A_∞ -category where m_d ($d \geq 3$) are trivial.) canonically extended from the derived category of coherent sheaves. Furthermore, the Fukaya category is also equivalent as an A_∞ -category to a nontrivial A_∞ -category extended from the DG-category by using homotopy operators [10–18]. In this extension, m_3 is explicitly constructed in the B-model [13], whereas m_d ($d \geq 4$) have not been explicitly constructed yet.

In this paper, we extend the DG-category in a different way, based on the topological string amplitudes. m_d are newly defined and explicitly constructed in the B-model. The A_∞ -category that consists of these m_d is shown to be equivalent

to the Fukaya category (Thus, the A_∞ -category defined in this paper is equivalent as an A_∞ -category to the A_∞ -category defined by the homotopy operators because they are equivalent to the Fukaya category as A_∞ categories.). In this construction, we find a moduli space necessary to define m_d that satisfy the A_∞ relations in the B-model. This moduli space is homeomorphic to the moduli space of the pseudo holomorphic curves in the A-model.

2. Topological String Amplitudes

The one-dimensional complex manifold in the B-model is an elliptic curve E_τ which is spanned by 1 and $\tau \in \mathbb{C}$. There are two ways to generalize the derived category of coherent sheaves on E_τ to a differential graded (DG) category. One way is based on Čech cohomology [3–9] and the other is based on Dolbeault cohomology [2, 10–16]. They are equivalent by Čech-Dolbeault isomorphism. We adopt Dolbeault cohomology. Coherent sheaves on E_τ are classified and constructed based on line bundles on E_τ [19]. We study only the line bundles. The extension to general coherent sheaves is straight forward and the way is written in [2, 13]. The objects representing $(d+1)$ D-branes are complexes of line bundles of degree n_i , $L(n_i, u_i)$ ($i = 0, \dots, d$), where u_i represent connections (gauge fields) over the D-branes. The morphisms representing open strings between $L(n_{i-1}, u_{i-1})$ and $L(n_i, u_i)$ are $H^{0,p}(E_\tau, L(n_{i-1,j}, u_{i-1,j}))$, where p represent the degrees of the grading ($p = 0, 1, n_{i-1,j} := n_i - n_{i-1}$, $u_{i-1,j} := u_i - u_{i-1}$).

When $n > 0$, $H^{0,1}(E_\tau, L(n, u)) = 0$ and the elements of $H^{0,0}(E_\tau, L(n, u))$ are the basis of global sections of $L(n, u)$:

$$\theta \left[\frac{m}{n} \right] (nz, n\tau) \in H^{0,0}(E_\tau, L(n, u)) \quad (1)$$

$$(m = 0, \dots, n-1),$$

where $\theta \left[\frac{a}{b} \right] (z, \tau)$ are theta functions with characteristics.

When $n < 0$, $H^{0,0}(E_\tau, L(n, u)) = 0$ and the elements of $H^{0,1}(E_\tau, L(n, u))$ are the basis of harmonic (0,1)-forms with values in the dual bundle $L(n, u)$:

$$\overline{\theta \left[\frac{m}{n} \right] (-nz, -n\tau) \exp(2\pi t(nz_2^2 + 2u_2z_2)) d\bar{z}} \quad (2)$$

$$\in H^{0,1}(E_\tau, L(n, u)),$$

where $\bar{\theta}$ is the complex conjugate of θ , $\tau = A + it$, $z = z_1 + \tau z_2$, $u = u_1 + \tau u_2$, and $A, t, z_1, z_2, u_1, u_2 \in \mathbf{R}$.

We are concerned with the $u = 0$ case for simplicity. One can easily introduce the connections. We simplify the expressions as

$$\theta_n[p](z, \tau) := \theta \left[\frac{p}{0} \right] (nz, n\tau). \quad (3)$$

In this case, the strings are represented by

$$\begin{aligned} \theta_n \left[\frac{m}{n} \right] (z, \tau) &\in H^{0,0}(E_\tau, L(n)) \quad (n > 0), \\ \theta_n^* \left[\frac{m}{n} \right] (z, \bar{z}, \tau, \bar{\tau}) d\bar{z} & \\ &:= \overline{\theta_{-n} \left[\frac{m}{n} \right] (z, \tau) \exp(2\pi t n z_2^2) d\bar{z}} \\ &\in H^{0,1}(E_\tau, L(n)) \quad (n < 0). \end{aligned} \quad (4)$$

Topological string amplitudes on E_τ are defined as [9, 15, 20]

$$\int_{E_\tau} \Omega \wedge \theta_n^* \left[\frac{m}{n} \right] d\bar{z} \theta_{n_{i_1-1, i_1}} \left[\frac{m_{i_1-1, i_1}}{n_{i_1-1, i_1}} \right] \cdots \theta_{n_{i_c-1, i_c}} \left[\frac{m_{i_c-1, i_c}}{n_{i_c-1, i_c}} \right] \psi_{n_{i_{c+1}-1, i_{c+1}}} \left[\frac{m_{i_{c+1}-1, i_{c+1}}}{n_{i_{c+1}-1, i_{c+1}}} \right] \cdots \psi_{n_{i_d-1, i_d}} \left[\frac{m_{i_d-1, i_d}}{n_{i_d-1, i_d}} \right], \quad (5)$$

where $n_{i_1-1, i_1}, \dots, n_{i_c-1, i_c} > 0$ and $n_{i_{c+1}-1, i_{c+1}}, \dots, n_{i_d-1, i_d}, n < 0$. $\Omega = dz$ is the holomorphic (1,0)-form, $\theta_n^*[m/n]d\bar{z} \in H^{0,1}(E_\tau, L(n))$, and $\theta_{n_{i_k-1, i_k}}[m_{i_k-1, i_k}/n_{i_k-1, i_k}] \in H^{0,0}(E_\tau, L(n_{i_k-1, i_k}))$ ($k = 1, \dots, c$). Here we explain what $\psi_n[m/n]$ is. In order to define topological string amplitudes of more than one $n < 0$ states, we need to deform the theory by those states because more than one (0,1)-form cannot enter the topological string amplitudes in one dimension. The derived category describes such a deformed theory. In the topological string theory, the deformation by $n < 0$ string states $O^{(0)} \in \Omega^{0,1}$ is given as follows. We define $O^{(1)}$ by $\tilde{d}O^{(0)} = \{Q, O^{(1)}\}$, where $\tilde{d}$ is a world sheet differential and Q is a BRST operator. Then, the deformation of the theory is to insert $\psi := \int_{\partial\Sigma} O^{(1)} \in \Omega^{0,0}$, where $\partial\Sigma$ is a world sheet boundary.

In our case, we define this deformation by an isomorphism $\psi: \Omega^{0,1} \rightarrow \Omega^{0,0}$ by

$$\begin{aligned} O^{(0)} &= \theta_n^* \left[\frac{m}{n} \right] (z, \bar{z}, \tau, \bar{\tau}) d\bar{z} \mapsto \\ \psi &= \theta_n \left[\frac{m}{n} \right] (z, \tau). \end{aligned} \quad (6)$$

That is, $\theta_n[m/n](z, \tau) \in \Omega^{0,0}$ represent string states, when not only $n > 0$ but also $n < 0$. This isomorphism will be justified later by mirror symmetry of the A_∞ structure and m_d . By using this isomorphism, (5) is written as

$$\begin{aligned} \int_{E_\tau} \Omega \wedge \theta_n^* \left[\frac{m}{n} \right] d\bar{z} \theta_{n_{0,1}} \left[\frac{m_{0,1}}{n_{0,1}} \right] \\ \cdot \theta_{n_{1,2}} \left[\frac{m_{1,2}}{n_{1,2}} \right] \cdots \theta_{n_{d-1,d}} \left[\frac{m_{d-1,d}}{n_{d-1,d}} \right]. \end{aligned} \quad (7)$$

On the other hand, (5) should also be written by using m_d in A_∞ -category [15, 20] like

$$\sim \int_{E_\tau} \Omega \wedge \theta_n^* \left[\frac{m}{n} \right] d\bar{z} m_d \left(\frac{m_{0,1}}{n_{0,1}}, \dots, \frac{m_{d-1,d}}{n_{d-1,d}} \right). \quad (8)$$

Therefore, m_d should be defined like

$$\begin{aligned} m_d \left(\frac{m_{0,1}}{n_{0,1}}, \dots, \frac{m_{d-1,d}}{n_{d-1,d}} \right) \\ \sim \theta_{n_{0,1}} \left[\frac{m_{0,1}}{n_{0,1}} \right] \cdots \theta_{n_{d-1,d}} \left[\frac{m_{d-1,d}}{n_{d-1,d}} \right]. \end{aligned} \quad (9)$$

We will define completely m_d that possesses A_∞ structure in the next section.

Here we discuss consistency of the integration over E_τ with the periodicity. Whereas the theta functions are invariant under $z \mapsto z + 1$, they are transformed under $z \mapsto z + \tau$ as

$$\begin{aligned} \theta_n \left[\frac{m}{n} \right] (z, \tau) &\mapsto e^{-\pi i n (2z + \tau)} \theta_n \left[\frac{m}{n} \right] (z, \tau) \\ \theta_n^* \left[\frac{m}{n} \right] (z, \bar{z}, \tau, \bar{\tau}) d\bar{z} & \\ \mapsto e^{-\pi i n (2z + \tau)} \theta_n^* \left[\frac{m}{n} \right] (z, \bar{z}, \tau, \bar{\tau}) d\bar{z}. \end{aligned} \quad (10)$$

Then, the integrand is transformed as

$$\theta_n^* \left[\frac{m}{n} \right] d\bar{z} \theta_{n_{0,1}} \left[\frac{m_{0,1}}{n_{0,1}} \right] \cdots \theta_{n_{d-1,d}} \left[\frac{m_{d-1,d}}{n_{d-1,d}} \right] \mapsto e^{-\pi i (2z+\tau)(n+n_{0,1}+\cdots+n_{d-1,d})} \theta_n^* \left[\frac{m}{n} \right] d\bar{z} \theta_{n_{0,1}} \left[\frac{m_{0,1}}{n_{0,1}} \right] \cdots \theta_{n_{d-1,d}} \left[\frac{m_{d-1,d}}{n_{d-1,d}} \right]. \quad (11)$$

Because this should be invariant for periodicity, n needs to be $-(n_{0,1} + \cdots + n_{d-1,d})$ and then $n_{0,1} + \cdots + n_{d-1,d} > 0$.

3. m_d and A_∞ Structure

In order to define m_d , we multiply theta functions with characteristics. They are defined by series as follows.

Definition 1 (theta functions with characteristics).

$$\begin{aligned} \theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau) \\ := \sum_{m \in \mathbb{Z}} \exp \left(\pi i (m+a)^2 \tau + 2\pi i (m+a)(z+b) \right), \end{aligned} \quad (12)$$

where $a \in \mathbb{R}/\mathbb{Z}$, $b \in \mathbb{C}$.

A product formula is given by the following.

Theorem 2.

$$\begin{aligned} \theta \begin{bmatrix} \frac{a_1}{n_1} \\ 0 \end{bmatrix} (z_1, n_1 \tau) \theta \begin{bmatrix} \frac{a_2}{n_2} \\ 0 \end{bmatrix} (z_2, n_2 \tau) \\ = \sum_{d \in \mathbb{Z}/(n_1+n_2)\mathbb{Z}} \theta \begin{bmatrix} \frac{n_2 a_1 - n_1 a_2 + n_1 n_2 d}{n_1 n_2 (n_1 + n_2)} \\ 0 \end{bmatrix} (n_2 z_1 \\ - n_1 z_2, n_1 n_2 (n_1 + n_2) \\ \cdot \tau) \theta \begin{bmatrix} \frac{a_1 + a_2 + n_1 d}{n_1 + n_2} \\ 0 \end{bmatrix} (z_1 + z_2, (n_1 + n_2) \tau), \end{aligned} \quad (13)$$

where $n_1, n_2 \in \mathbb{Z}$, $a_1, a_2 \in \mathbb{R}$, and $z_1, z_2, \tau \in \mathbb{C}$.

While this formula was proved as an addition formula when $n_1, n_2 \in \mathbb{N}$ in [21], it can also be proved as series when $n_1, n_2 \in \mathbb{Z}$ as follows.

Proof.

$$\begin{aligned} (\text{l.h.s.}) &= \sum_{m_1, m_2 \in \mathbb{Z}} \exp \left(\pi i n_1 \tau \left(m_1 + \frac{a_1}{n_1} \right)^2 + 2\pi i \left(m_1 \right. \right. \\ &\quad \left. \left. + \frac{a_1}{n_1} \right) z_1 + \pi i n_2 \tau \left(m_2 + \frac{a_2}{n_2} \right)^2 \right. \\ &\quad \left. + 2\pi i \left(m_2 + \frac{a_2}{n_2} \right) z_2 \right) \end{aligned}$$

$$\begin{aligned} &= \sum_{l, m \in \mathbb{Z}, d \in \mathbb{Z}/(n_1+n_2)\mathbb{Z}} \exp \pi i \left((n_1 + n_2) \tau l^2 \right. \\ &\quad \left. + 2((a_1 + a_2) \tau + z_1 + z_2) l + 2n_1 \tau l d + n_1 \tau d^2 \right. \\ &\quad \left. + (2a_1 \tau + 2z_1) d + n_1 \tau (n_1 + n_2)^2 m^2 \right. \\ &\quad \left. + 2n_1 (n_1 + n_2) \tau l m \right. \\ &\quad \left. + 2(a_1 \tau + z_1) (n_1 + n_2) m + 2n_1 (n_1 + n_2) \tau m d + \frac{a_1^2}{n_1} \tau \right. \\ &\quad \left. + \frac{a_2^2}{n_2} \tau + 2 \frac{a_1}{n_1} z_1 + 2 \frac{a_2}{n_2} z_2 \right), \end{aligned} \quad (14)$$

where we have set $m_1 = l + p$, $m_2 = l$ ($l, p \in \mathbb{Z}$), and $p = (n_1 + n_2)m + d$ ($m \in \mathbb{Z}$, $d \in \mathbb{Z}/(n_1 + n_2)\mathbb{Z}$). On the other hand,

$$\begin{aligned} (\text{r.h.s.}) &= \sum_{m, n \in \mathbb{Z}, d \in \mathbb{Z}/(n_1+n_2)\mathbb{Z}} \exp \left(\pi i n_1 n_2 (n_1 + n_2) \right. \\ &\quad \cdot \tau \left(m + \frac{n_2 a_1 - n_1 a_2 + n_1 n_2 d}{n_1 n_2 (n_1 + n_2)} \right)^2 \\ &\quad \left. + 2\pi i \left(m + \frac{n_2 a_1 - n_1 a_2 + n_1 n_2 d}{n_1 n_2 (n_1 + n_2)} \right) (n_2 z_1 - n_1 z_2) \right) \\ &\quad \cdot \pi i (n_1 + n_2) \tau \left(n + \frac{a_1 + a_2 + n_1 d}{n_1 + n_2} \right)^2 \\ &\quad \left. + 2\pi i \left(n + \frac{a_1 + a_2 + n_1 d}{n_1 + n_2} \right) (z_1 + z_2) \right). \end{aligned} \quad (15)$$

If we set $n = l + n_1 m$ ($l \in \mathbb{Z}$), (15) coincides with (14). $\square$

In a special case, $z_1 = n_1 z + u_1$ and $z_2 = n_2 z + u_2$ ($u_1, u_2 \in \mathbb{C}$), we obtain

$$\begin{aligned} \theta \begin{bmatrix} \frac{a_1}{n_1} \\ u_1 \end{bmatrix} (n_1 z, n_1 \tau) \theta \begin{bmatrix} \frac{a_2}{n_2} \\ u_2 \end{bmatrix} (n_2 z, n_2 \tau) \\ = \sum_{d \in \mathbb{Z}/(n_1+n_2)\mathbb{Z}} \theta \begin{bmatrix} \frac{n_2 a_1 - n_1 a_2 + n_1 n_2 d}{n_1 n_2 (n_1 + n_2)} \\ n_2 u_1 - n_1 u_2 \end{bmatrix} \\ \cdot (0, n_1 n_2 (n_1 + n_2) \\ \cdot \tau) \theta \begin{bmatrix} \frac{a_1 + a_2 + n_1 d}{n_1 + n_2} \\ u_1 + u_2 \end{bmatrix} ((n_1 + n_2) z, (n_1 + n_2) \tau). \end{aligned} \quad (16)$$

This product is expanded by the theta functions with complex coefficients. The coefficients are independent of z . We

simplify this formula. Because $(a_1 + a_2 + n_1 d)/(n_1 + n_2) \in \mathbf{R}/\mathbf{Z}$ on the last line, we can add $-((n_1 + n_2)/(n_1 + n_2))d$ and obtain

$$\begin{aligned}
& \theta \begin{bmatrix} a_1 \\ n_1 \\ u_1 \end{bmatrix} (n_1 z, n_1 \tau) \theta \begin{bmatrix} a_2 \\ n_2 \\ u_2 \end{bmatrix} (n_2 z, n_2 \tau) \\
&= \sum_{d \in \mathbf{Z}/(n_1+n_2)\mathbf{Z}} \theta \begin{bmatrix} n_2 a_1 - n_1 (a_2 - n_2 d) \\ n_1 n_2 (n_1 + n_2) \\ n_2 u_1 - n_1 u_2 \end{bmatrix} \\
&\cdot (0, n_1 n_2 (n_1 + n_2) \tau) \theta \begin{bmatrix} a_1 + (a_2 - n_2 d) \\ n_1 + n_2 \\ u_1 + u_2 \end{bmatrix} \\
&\cdot ((n_1 + n_2) z, (n_1 + n_2) \tau) \\
&= \sum_{m \in \mathbf{Z}, d \in \mathbf{Z}/(n_1+n_2)\mathbf{Z}} \exp \left(\pi i n_1 n_2 (n_1 + n_2) \tau \left(m + \frac{n_2 a_1 - n_1 (a_2 - n_2 d)}{n_1 n_2 (n_1 + n_2)} \right)^2 \right. \\
&\quad \left. + 2\pi i \left(m + \frac{n_2 a_1 - n_1 (a_2 - n_2 d)}{n_1 n_2 (n_1 + n_2)} \right) (n_2 u_1 - n_1 u_2) \right) \\
&\cdot \theta \begin{bmatrix} a_1 + (a_2 - n_2 d) \\ n_1 + n_2 \\ u_1 + u_2 \end{bmatrix} ((n_1 + n_2) z, (n_1 + n_2) \tau).
\end{aligned} \tag{17}$$

Because $(a_1 + (a_2 - n_2 d))/(n_1 + n_2) \in \mathbf{R}/\mathbf{Z}$ on the last line, we can add $-(n_2(n_1 + n_2)m/(n_1 + n_2))d$. By defining $\alpha := (n_1 + n_2)m + d$ ($\alpha \in \mathbf{Z}$), we obtain

Lemma 3.

$$\begin{aligned}
& \theta \begin{bmatrix} a_1 \\ n_1 \\ u_1 \end{bmatrix} (n_1 z, n_1 \tau) \theta \begin{bmatrix} a_2 \\ n_2 \\ u_2 \end{bmatrix} (n_2 z, n_2 \tau) \\
&= \sum_{\alpha \in \mathbf{Z}} \exp \left(\pi i \tau \frac{(n_2 a_1 - n_1 (a_2 - n_2 \alpha))^2}{n_1 n_2 (n_1 + n_2)} \right) \\
&\quad + 2\pi i \left(\frac{n_2 a_1 - n_1 (a_2 - n_2 \alpha)}{n_1 n_2 (n_1 + n_2)} \right) (n_2 u_1 - n_1 u_2) \\
&\quad \cdot \theta \begin{bmatrix} a_1 + (a_2 - n_2 \alpha) \\ n_1 + n_2 \\ u_1 + u_2 \end{bmatrix} ((n_1 + n_2) z, (n_1 + n_2) \tau).
\end{aligned} \tag{18}$$

Especially, when $u = 0$ we obtain

$$\begin{aligned}
& \theta_{n_1} \begin{bmatrix} a_1 \\ n_1 \end{bmatrix} (z, \tau) \theta_{n_2} \begin{bmatrix} a_2 \\ n_2 \end{bmatrix} (z, \tau) \\
&= \sum_{\alpha \in \mathbf{Z}} \exp \left(\pi i \tau \frac{(n_2 a_1 - n_1 (a_2 - n_2 \alpha))^2}{n_1 n_2 (n_1 + n_2)} \right) \\
&\quad \cdot \theta_{n_1+n_2} \begin{bmatrix} a_1 + (a_2 - n_2 \alpha) \\ n_1 + n_2 \end{bmatrix} (z, \tau).
\end{aligned} \tag{19}$$

From now on, we abbreviate $\theta_n[p](z, \tau)$ to $\theta_n[p]$ because (z, τ) does not vary. By using this formula, we obtain the following.

Theorem 4.

$$\begin{aligned}
& \theta_{n_{0,1}} \begin{bmatrix} a_{0,1} \\ n_{0,1} \end{bmatrix} \theta_{n_{1,2}} \begin{bmatrix} a_{1,2} \\ n_{1,2} \end{bmatrix} \cdots \theta_{n_{d-1,d}} \begin{bmatrix} a_{d-1,d} \\ n_{d-1,d} \end{bmatrix} \\
&= \sum_{\alpha_{1,2}, \dots, \alpha_{d-1,d} \in \mathbf{Z}} \exp \left(2\pi i \tau S_d \left(\frac{\tilde{a}_{0,1}}{n_{0,1}}, \dots, \frac{\tilde{a}_{d-1,d}}{n_{d-1,d}} \right) \right) \\
&\quad \cdot \theta_{n_{0,1} + \dots + n_{d-1,d}} \begin{bmatrix} \tilde{a}_{0,1} + \dots + \tilde{a}_{d-1,d} \\ n_{0,1} + \dots + n_{d-1,d} \end{bmatrix},
\end{aligned} \tag{20}$$

where

$$\tilde{a}_{i-1,i} := a_{i-1,i} - n_{i-1,i} \alpha_{i-1,i} \quad (i = 1, \dots, d)$$

$$\alpha_{0,1} \equiv 0$$

$$S_d(p_{0,1}, \dots, p_{d-1,d})$$

$$\begin{aligned}
&:= \frac{1}{2} \left(\left(\sum_{i=1}^d n_{i-1,i} p_{i-1,i} (p_{i-1,i} - 1) \right) \right. \\
&\quad \left. - n_{0,d} p_{0,d} (p_{0,d} - 1) \right)
\end{aligned}$$

$$\begin{aligned}
n_{0,d} &:= n_{0,1} + \cdots + n_{d-1,d} \\
p_{0,d} &:= \frac{1}{n_{0,d}} \sum_{i=1}^d n_{i-1,i} p_{i-1,i}.
\end{aligned} \tag{21}$$

Proof. We start with $d = 2$ case. Explicitly,

$$S_2 \left(\frac{\tilde{a}_{0,1}}{n_{0,1}}, \frac{\tilde{a}_{1,2}}{n_{1,2}} \right) = \frac{(n_{1,2}\tilde{a}_{0,1} - n_{0,1}\tilde{a}_{1,2})^2}{2n_{0,1}n_{1,2}(n_{0,1} + n_{1,2})}. \tag{22}$$

Then, Lemma 3 implies Theorem 4 is right when $d = 2$. For $2 \leq k \leq d-1$, explicitly,

$$\begin{aligned}
&S_{k+1} \left(\frac{\tilde{a}_{0,1}}{n_{0,1}}, \dots, \frac{\tilde{a}_{k,k+1}}{n_{k,k+1}} \right) - S_k \left(\frac{\tilde{a}_{0,1}}{n_{0,1}}, \dots, \frac{\tilde{a}_{k-1,k}}{n_{k-1,k}} \right) \\
&= \frac{(n_{k,k+1}(\tilde{a}_{0,1} + \cdots + \tilde{a}_{k-1,k}) - (n_{0,1} + \cdots + n_{k-1,k})\tilde{a}_{k,k+1})^2}{2(n_{0,1} + \cdots + n_{k-1,k})n_{k,k+1}(n_{0,1} + \cdots + n_{k-1,k} + n_{k,k+1})}.
\end{aligned} \tag{23}$$

Then, Lemma 3 implies

$$\begin{aligned}
&\theta_{n_{0,1} + \cdots + n_{k-1,k}} \left[\frac{\tilde{a}_{0,1} + \cdots + \tilde{a}_{k-1,k}}{n_{0,1} + \cdots + n_{k-1,k}} \right] \theta_{n_{k,k+1}} \left[\frac{\tilde{a}_{k,k+1}}{n_{k,k+1}} \right] \\
&= \sum_{\alpha_{k+1} \in \mathbb{Z}} \exp \left(2\pi i \tau \left(S_{k+1} \left(\frac{\tilde{a}_{0,1}}{n_{0,1}}, \dots, \frac{\tilde{a}_{k,k+1}}{n_{k,k+1}} \right) - S_k \left(\frac{\tilde{a}_{0,1}}{n_{0,1}}, \dots, \frac{\tilde{a}_{k-1,k}}{n_{k-1,k}} \right) \right) \right) \\
&\cdot \theta_{n_{0,1} + \cdots + n_{k,k+1}} \left[\frac{\tilde{a}_{0,1} + \cdots + \tilde{a}_{k,k+1}}{n_{0,1} + \cdots + n_{k,k+1}} \right].
\end{aligned} \tag{24}$$

Therefore,

$$\begin{aligned}
&\theta_{n_{0,1}} \left[\frac{a_{0,1}}{n_{0,1}} \right] \theta_{n_{1,2}} \left[\frac{a_{1,2}}{n_{1,2}} \right] \cdots \theta_{n_{d-1,d}} \left[\frac{a_{d-1,d}}{n_{d-1,d}} \right] \\
&= \sum_{\alpha_{1,2} \in \mathbb{Z}} \exp \left(2\pi i \tau S_2 \left(\frac{\tilde{a}_{0,1}}{n_{0,1}}, \frac{\tilde{a}_{1,2}}{n_{1,2}} \right) \right) \\
&\cdot \theta_{n_{0,1} + n_{1,2}} \left[\frac{\tilde{a}_{0,1} + \tilde{a}_{1,2}}{n_{0,1} + n_{1,2}} \right] \\
&\cdot \theta_{n_{2,3}} \left[\frac{a_{2,3}}{n_{2,3}} \right] \cdots \theta_{n_{d-1,d}} \left[\frac{a_{d-1,d}}{n_{d-1,d}} \right] \\
&= \sum_{\alpha_{1,2} \in \mathbb{Z}} \exp \left(2\pi i \tau S_2 \left(\frac{\tilde{a}_{0,1}}{n_{0,1}}, \frac{\tilde{a}_{1,2}}{n_{1,2}} \right) \right) \\
&\cdot \sum_{\alpha_{2,3} \in \mathbb{Z}} \exp \left(2\pi i \tau \left(S_3 \left(\frac{\tilde{a}_{0,1}}{n_{0,1}}, \frac{\tilde{a}_{1,2}}{n_{1,2}}, \frac{\tilde{a}_{2,3}}{n_{2,3}} \right) \right. \right. \\
&\quad \left. \left. - S_2 \left(\frac{\tilde{a}_{0,1}}{n_{0,1}}, \frac{\tilde{a}_{1,2}}{n_{1,2}} \right) \right) \right)
\end{aligned}$$

$$\begin{aligned}
&\cdot \theta_{n_{0,1} + n_{1,2} + n_{2,3}} \left[\frac{\tilde{a}_{0,1} + \tilde{a}_{1,2} + \tilde{a}_{2,3}}{n_{0,1} + n_{1,2} + n_{2,3}} \right] \theta_{n_{3,4}} \left[\frac{a_{3,4}}{n_{3,4}} \right] \\
&\cdots \theta_{n_{d-1,d}} \left[\frac{a_{d-1,d}}{n_{d-1,d}} \right] \\
&= \sum_{\alpha_{1,2}, \alpha_{2,3} \in \mathbb{Z}} \exp \left(2\pi i \tau S_3 \left(\frac{\tilde{a}_{0,1}}{n_{0,1}}, \frac{\tilde{a}_{1,2}}{n_{1,2}}, \frac{\tilde{a}_{2,3}}{n_{2,3}} \right) \right) \\
&\cdot \theta_{n_{0,1} + n_{1,2} + n_{2,3}} \left[\frac{\tilde{a}_{0,1} + \tilde{a}_{1,2} + \tilde{a}_{2,3}}{n_{0,1} + n_{1,2} + n_{2,3}} \right] \theta_{n_{3,4}} \left[\frac{a_{3,4}}{n_{3,4}} \right] \\
&\cdots \theta_{n_{d-1,d}} \left[\frac{a_{d-1,d}}{n_{d-1,d}} \right] = \cdots \\
&= \sum_{\alpha_{1,2}, \dots, \alpha_{d-1,d} \in \mathbb{Z}} \exp \left(2\pi i \tau S_d \left(\frac{\tilde{a}_{0,1}}{n_{0,1}}, \dots, \frac{\tilde{a}_{d-1,d}}{n_{d-1,d}} \right) \right) \\
&\cdot \theta_{n_{0,1} + \cdots + n_{d-1,d}} \left[\frac{\tilde{a}_{0,1} + \cdots + \tilde{a}_{d-1,d}}{n_{0,1} + \cdots + n_{d-1,d}} \right].
\end{aligned} \tag{25}$$

□

Next, we study A_∞ structure. We define an extended theta function $\tilde{\theta}_n[\tilde{p}]$ whose configuration space is a universal cover $\mathbf{R}(\ni \tilde{p})$ of the configuration space $\mathbf{R}/\mathbf{Z}(\ni p)$ of the theta function $\theta_n[p]$. That is, $\tilde{\theta}_n[a/n+1] \neq \tilde{\theta}_n[a/n]$ whereas $\theta_n[a/n+1] = \theta_n[a/n]$. The product is defined as follows.

Definition 5 (product of $\tilde{\theta}$).

$$\begin{aligned}
&\tilde{\theta}_{n_{0,1}} \left[\frac{a_{0,1}}{n_{0,1}} \right] (z, \tau) \tilde{\theta}_{n_{1,2}} \left[\frac{a_{1,2}}{n_{1,2}} \right] (z, \tau) \\
&:= \exp \left(\pi i \tau \frac{(n_{1,2}a_{0,1} - n_{0,1}a_{1,2})^2}{n_{0,1}n_{1,2}(n_{0,1} + n_{1,2})} \right) \\
&\cdot \tilde{\theta}_{n_{0,1} + n_{1,2}} \left[\frac{a_{0,1} + a_{1,2}}{n_{0,1} + n_{1,2}} \right] (z, \tau).
\end{aligned} \tag{26}$$

This definition leads to the following.

Lemma 6.

$$\begin{aligned}
&\sum_{\alpha \in \mathbb{Z}} \tilde{\theta}_{n_{0,1}} \left[\frac{a_{0,1}}{n_{0,1}} \right] (z, \tau) \tilde{\theta}_{n_{1,2}} \left[\frac{a_{1,2}}{n_{1,2}} - \alpha \right] (z, \tau) \\
&= \sum_{\alpha \in \mathbb{Z}} \exp \left(\pi i \tau \frac{(n_{1,2}a_{0,1} - n_{0,1}(a_{1,2} - n_{1,2}\alpha))^2}{n_{0,1}n_{1,2}(n_{0,1} + n_{1,2})} \right) \\
&\cdot \tilde{\theta}_{n_{0,1} + n_{1,2}} \left[\frac{a_{0,1} + (a_{1,2} - n_{1,2}\alpha)}{n_{0,1} + n_{1,2}} \right] (z, \tau).
\end{aligned} \tag{27}$$

Proof. Clear. □

FIGURE 1: Three-point interaction.

The coefficients in this formula coincide with those of Lemma 3. Furthermore, the definition leads to the following.

Lemma 7 (local product in the configuration space).

$$\tilde{\theta}_{n_{0,1}}[\tilde{p}](z, \tau) \tilde{\theta}_{n_{1,2}}[\tilde{p}](z, \tau) = \tilde{\theta}_{n_{0,1}+n_{1,2}}[\tilde{p}](z, \tau). \quad (28)$$

Proof.

$$\begin{aligned} (\text{l.h.s.}) &= \exp\left(\pi i \tau \frac{(n_{1,2}n_{0,1}\tilde{p} - n_{0,1}n_{1,2}\tilde{p}')^2}{n_{0,1}n_{1,2}(n_{0,1} + n_{1,2})}\right) \\ &\cdot \tilde{\theta}_{n_{0,1}+n_{1,2}}\left[\frac{n_{0,1}\tilde{p} + n_{1,2}\tilde{p}'}{n_{0,1} + n_{1,2}}\right](z, \tau) = (\text{r.h.s.}). \end{aligned} \quad (29)$$

□

Next, we define a propagator $\tilde{G}_n[\tilde{p}_0, \tilde{p}](z, \tau)$ in the configuration space as follows.

Definition 8 (propagator in the configuration space).

$$\tilde{G}_n[\tilde{p}_0, \tilde{p}](z, \tau) \tilde{\theta}_n[\tilde{p}](z, \tau) := \tilde{\theta}_n[\tilde{p}_0](z, \tau). \quad (30)$$

As a result, Feynman diagrams appear in the configuration space (Figure 1):

$$\begin{aligned} &\tilde{\theta}_{n_{0,1}}\left[\frac{a_{0,1}}{n_{0,1}}\right] \tilde{\theta}_{n_{1,2}}\left[\frac{a_{1,2}}{n_{1,2}}\right] = \tilde{G}_{n_{0,1}}\left[\frac{a_{0,1}}{n_{0,1}}, \tilde{p}\right] \\ &\cdot \tilde{G}_{n_{1,2}}\left[\frac{a_{1,2}}{n_{1,2}}, \tilde{p}\right] \tilde{\theta}_{n_{0,1}}[\tilde{p}] \tilde{\theta}_{n_{1,2}}[\tilde{p}] \\ &= \tilde{G}_{n_{0,1}}\left[\frac{a_{0,1}}{n_{0,1}}, \tilde{p}\right] \tilde{G}_{n_{1,2}}\left[\frac{a_{1,2}}{n_{1,2}}, \tilde{p}\right] \tilde{\theta}_{n_{0,1}+n_{1,2}}[\tilde{p}]. \\ &= \tilde{G}_{n_{0,1}}\left[\frac{a_{0,1}}{n_{0,1}}, \tilde{p}\right] \tilde{G}_{n_{1,2}}\left[\frac{a_{1,2}}{n_{1,2}}, \tilde{p}\right] \tilde{G}_{n_{0,1}+n_{1,2}}[\tilde{p}, \tilde{p}'] \\ &\cdot \tilde{\theta}_{n_{0,1}+n_{1,2}}[\tilde{p}']. \end{aligned} \quad (31)$$

When external states $\tilde{\theta}_n[m/n]$ propagate from m/n to $\tilde{p}$, we can parametrize $\tilde{p} = (m + v)/n$. If two external states $\tilde{\theta}_{n_{0,1}}[m_{0,1}/n_{0,1}]$ and $\tilde{\theta}_{n_{1,2}}[m_{1,2}/n_{1,2}]$ propagate to the same point $\tilde{p} = (m_{0,1} + v_{0,1})/n_{0,1} = (m_{1,2} + v_{1,2})/n_{1,2}$ and interact locally, Lemma 7 leads to

FIGURE 2: $\tilde{p}_{0,1}$.

$$\begin{aligned} &\tilde{\theta}_{n_{0,1}}\left[\frac{m_{0,1} + v_{0,1}}{n_{0,1}}\right] \tilde{\theta}_{n_{1,2}}\left[\frac{m_{1,2} + v_{1,2}}{n_{1,2}}\right] \\ &= \tilde{\theta}_{n_{0,1}+n_{1,2}}\left[\frac{m_{0,1} + m_{1,2} + v_{0,1} + v_{1,2}}{n_{0,1} + n_{1,2}}\right], \end{aligned} \quad (32)$$

because $\tilde{p} = (m_{0,1} + m_{1,2} + v_{0,1} + v_{1,2})/(n_{0,1} + n_{1,2})$. That is, n , m , and v are preserved. Then, we define a canonical form of $\tilde{\theta}$, including internal states, as $\tilde{\theta}_n[(m+v)/n]$, where $n, m \in \mathbf{Z}$ and $v \in \mathbf{R}$. As a result, m represent a kind of preserved numbers of string states. $\Delta\tilde{p} = \Delta v/n = (v' - v)/n$ represents how long the state propagates in an expression $\tilde{G}_n[(m+v)/n, (m+v')/n]$.

We define the direction of the propagator $\tilde{G}_n[(m+v)/n, (m+v')/n]$ as the same as the direction of v and v' (v and v' should have the same direction.). Because incoming states propagate from m/n to $(m+v)/n$, $\Delta\tilde{p} = (m+v)/n - m/n = v/n$. Coincidence of the signs of $\Delta\tilde{p}$ and v implies that $n > 0$ or $\Delta\tilde{p} = 0$. That is, incoming states with $n < 0$ cannot propagate. Similarly, outgoing states with $n > 0$ cannot propagate.

Strings between (i-1)-th and i-th D-branes can interact only with strings between (i-2)-th and (i-1)-th D-branes and strings between i-th and (i+1)-th D-branes. Therefore, we need to demand that the ordering of incoming states is noncommutative in the Feynman diagrams.

We consider the moduli space $\mathcal{F}(\tilde{p}_{0,d}; \tilde{p}_{0,1}, \tilde{p}_{1,2}, \dots, \tilde{p}_{d-1,d})$ of the Feynman diagrams that satisfy the above conditions, for incoming states $\tilde{\theta}_{n_{0,1}}[\tilde{p}_{0,1}]$, $\tilde{\theta}_{n_{1,2}}[\tilde{p}_{1,2}]$, $\dots$, $\tilde{\theta}_{n_{d-1,d}}[\tilde{p}_{d-1,d}]$, where $\tilde{p}_{i-1,i} = \tilde{m}_{i-1,i}/n_{i-1,i} = m_{i-1,i}/n_{i-1,i} - \alpha_{i-1,i}$ ($\alpha_{i-1,i} \in \mathbf{Z}$, $i = 1, \dots, d$), and outgoing states $\tilde{\theta}_{n_{0,d}}[\tilde{p}_{0,d}]$, where $\tilde{p}_{0,d} = \tilde{m}_{0,d}/n_{0,d} = \sum_{i=1}^d \tilde{m}_{i-1,i}/n_{0,d}$. We also consider the zero- and one-dimensional subspaces of the moduli space: $\mathcal{F}_0(\tilde{p}_{0,d}; \tilde{p}_{0,1}, \tilde{p}_{1,2}, \dots, \tilde{p}_{d-1,d})$ and $\mathcal{F}_1(\tilde{p}_{0,d}; \tilde{p}_{0,1}, \tilde{p}_{1,2}, \dots, \tilde{p}_{d-1,d})$, respectively. Then, we obtain the following theorem.

Theorem 9 (correct Feynman diagram). *If $\mathcal{F}_0(\tilde{p}_{0,d}; \tilde{p}_{0,1}, \tilde{p}_{1,2}, \dots, \tilde{p}_{d-1,d}) \neq \emptyset$, $\mathcal{F}_0(\tilde{p}_{0,d}; \tilde{p}_{0,1}, \tilde{p}_{1,2}, \dots, \tilde{p}_{d-1,d}) = \{\tilde{\delta}\}$, that is, $\mathcal{F}_0(\tilde{p}_{0,d}; \tilde{p}_{0,1}, \tilde{p}_{1,2}, \dots, \tilde{p}_{d-1,d})$, consists of only one element $\tilde{\delta}$. Then, $\tilde{\delta} \in \mathcal{F}_0(\tilde{p}_{0,d}; \tilde{p}_{0,1}, \tilde{p}_{1,2}, \dots, \tilde{p}_{d-1,d})$ determines a correlation function $\langle \tilde{p}_{0,d}; \tilde{p}_{0,1}, \tilde{p}_{1,2}, \dots, \tilde{p}_{d-1,d} \rangle_{\tilde{\delta}}$, which satisfies*

$$\begin{aligned} &\sum_{\alpha_{1,2}, \dots, \alpha_{d-1,d} \in \mathbf{Z}} \langle \tilde{p}_{0,d}; \tilde{p}_{0,1}, \tilde{p}_{1,2}, \dots, \tilde{p}_{d-1,d} \rangle_{\tilde{\delta}} \theta_{n_{0,d}}[\tilde{p}_{0,d}] \\ &= \theta_{n_{0,1}}[\tilde{p}_{0,1}] \theta_{n_{1,2}}[\tilde{p}_{1,2}] \cdots \theta_{n_{d-1,d}}[\tilde{p}_{d-1,d}]. \end{aligned} \quad (33)$$

Proof. First, we classify the Feynman diagrams. There is no internal vertex because $\dim(\mathcal{F}_0) = 0$. Then, the vertices are external only: $\tilde{p}_{0,d}, \tilde{p}_{0,1}, \tilde{p}_{1,2}, \dots, \tilde{p}_{d-1,d}$. $\tilde{p}_{0,1}$ is at the end of the diagrams because of the nearest neighbour interaction (Figure 2). $n_{0,1} > 0$ because the string needs to propagate.

FIGURE 3: The external vertices that cannot propagate.

FIGURE 4: End point when $n_{0,d} < 0$. FIGURE 5: End point when $n_{0,d} > 0$.

Because of the three point interactions, $\tilde{p}_{i-1,i}$ not at the end of the diagrams cannot propagate, namely, $n_{i-1,i} < 0$ (Figure 3). Let us consider the outgoing vertex $\tilde{p}_{0,d}$. When $n_{0,d} < 0$, a vertex propagate to $\tilde{p}_{0,d}$ (Figure 4). When $n_{0,d} > 0$, because no vertex propagates to $\tilde{p}_{0,d}$, the last interaction must be as in (Figure 5). As a result, we obtain only (i) (Figure 6) when $n_{0,d} < 0$ and (ii) (Figure 7) when $n_{0,d} > 0$. Therefore, we have shown that $\mathcal{F}_0(\tilde{p}_{0,d}; \tilde{p}_{0,1}, \tilde{p}_{1,2}, \dots, \tilde{p}_{d-1,d})$ consists of only one element $\tilde{\delta}$ if $\mathcal{F}_0(\tilde{p}_{0,d}; \tilde{p}_{0,1}, \tilde{p}_{1,2}, \dots, \tilde{p}_{d-1,d}) \neq \emptyset$.

Next, we calculate the correlation function determined by $\tilde{\delta}$.

(i) When $n_{0,d} < 0$, from the diagram in Figure 6,

$$\begin{aligned}
 & \sum_{\alpha_{1,2}, \dots, \alpha_{d-1,d} \in \mathbb{Z}} \langle \tilde{p}_{0,d}; \tilde{p}_{0,1}, \tilde{p}_{1,2}, \dots, \tilde{p}_{d-1,d} \rangle_{\tilde{\delta}} \\
 & \cdot \tilde{\theta}_{n_{0,1} + \dots + n_{d-1,d}} [\tilde{p}_{0,d}] \\
 & = \sum_{\alpha_{1,2}, \dots, \alpha_{d-1,d} \in \mathbb{Z}} \tilde{G}_{n_{0,1}} \left[\frac{\tilde{m}_{0,1}}{n_{0,1}}, \frac{\tilde{m}_{1,2}}{n_{1,2}} \right] \\
 & \cdot \tilde{G}_{n_{0,1} + n_{1,2}} \left[\frac{\tilde{m}_{1,2}}{n_{1,2}}, \frac{\tilde{m}_{2,3}}{n_{2,3}} \right] \\
 & \cdots \tilde{G}_{n_{0,1} + n_{1,2} + \dots + n_{d-2,d-1}} \left[\frac{\tilde{m}_{d-2,d-1}}{n_{d-2,d-1}}, \frac{\tilde{m}_{d-1,d}}{n_{d-1,d}} \right] \\
 & \cdot \tilde{G}_{n_{0,1} + n_{1,2} + \dots + n_{d-1,d}} \left[\frac{\tilde{m}_{d-1,d}}{n_{d-1,d}}, \frac{\tilde{m}_{0,d}}{n_{0,d}} \right] \\
 & \cdot \tilde{\theta}_{n_{0,1} + n_{1,2} + \dots + n_{d-1,d}} \left[\frac{\tilde{m}_{0,d}}{n_{0,d}} \right] \\
 & = \sum_{\alpha_{1,2}, \dots, \alpha_{d-1,d} \in \mathbb{Z}} \tilde{G}_{n_{0,1}} \left[\frac{\tilde{m}_{0,1}}{n_{0,1}}, \frac{\tilde{m}_{1,2}}{n_{1,2}} \right] \\
 & \cdot \tilde{G}_{n_{0,1} + n_{1,2}} \left[\frac{\tilde{m}_{1,2}}{n_{1,2}}, \frac{\tilde{m}_{2,3}}{n_{2,3}} \right]
 \end{aligned}$$

FIGURE 6: When $n_{0,d} < 0$. FIGURE 7: When $n_{0,d} > 0$.

$$\begin{aligned}
 & \cdots \tilde{G}_{n_{0,1} + n_{1,2} + \dots + n_{d-2,d-1}} \left[\frac{\tilde{m}_{d-2,d-1}}{n_{d-2,d-1}}, \frac{\tilde{m}_{d-1,d}}{n_{d-1,d}} \right] \\
 & \cdot \tilde{\theta}_{n_{0,1} + n_{1,2} + \dots + n_{d-1,d}} \left[\frac{\tilde{m}_{d-1,d}}{n_{d-1,d}} \right] \\
 & = \sum_{\alpha_{1,2}, \dots, \alpha_{d-1,d} \in \mathbb{Z}} \tilde{G}_{n_{0,1}} \left[\frac{\tilde{m}_{0,1}}{n_{0,1}}, \frac{\tilde{m}_{1,2}}{n_{1,2}} \right] \\
 & \cdot \tilde{G}_{n_{0,1} + n_{1,2}} \left[\frac{\tilde{m}_{1,2}}{n_{1,2}}, \frac{\tilde{m}_{2,3}}{n_{2,3}} \right] \\
 & \cdots \tilde{G}_{n_{0,1} + n_{1,2} + \dots + n_{d-2,d-1}} \left[\frac{\tilde{m}_{d-2,d-1}}{n_{d-2,d-1}}, \frac{\tilde{m}_{d-1,d}}{n_{d-1,d}} \right] \\
 & \cdot \tilde{\theta}_{n_{0,1} + n_{1,2} + \dots + n_{d-2,d-1}} \left[\frac{\tilde{m}_{d-1,d}}{n_{d-1,d}} \right] \tilde{\theta}_{n_{d-1,d}} \left[\frac{\tilde{m}_{d-1,d}}{n_{d-1,d}} \right] \\
 & \cdots \\
 & = \sum_{\alpha_{1,2}, \dots, \alpha_{d-1,d} \in \mathbb{Z}} \tilde{\theta}_{n_{0,1}} \left[\frac{\tilde{m}_{0,1}}{n_{0,1}} \right] \tilde{\theta}_{n_{1,2}} \left[\frac{\tilde{m}_{1,2}}{n_{1,2}} \right] \\
 & \cdots \tilde{\theta}_{n_{d-1,d}} \left[\frac{\tilde{m}_{d-1,d}}{n_{d-1,d}} \right].
 \end{aligned}$$

(34)

From Lemma 6,

$$\begin{aligned}
 & \sum_{\alpha_{1,2}, \dots, \alpha_{d-1,d} \in \mathbb{Z}} \langle \tilde{p}_{0,d}; \tilde{p}_{0,1}, \tilde{p}_{1,2}, \dots, \tilde{p}_{d-1,d} \rangle_{\tilde{\delta}} \\
 & \cdot \theta_{n_{0,1} + \dots + n_{d-1,d}} [\tilde{p}_{0,d}] = \theta_{n_{0,1}} [p_{0,1}] \theta_{n_{1,2}} [p_{1,2}] \\
 & \cdots \theta_{n_{d-1,d}} [p_{d-1,d}].
 \end{aligned} \tag{35}$$

(ii) When $n_{0,d} > 0$, from the diagram in Figure 7,

$$\begin{aligned}
 & \sum_{\alpha_{1,2}, \dots, \alpha_{d-1,d} \in \mathbb{Z}} \langle \tilde{p}_{0,d}; \tilde{p}_{0,1}, \tilde{p}_{1,2}, \dots, \tilde{p}_{d-1,d} \rangle_{\tilde{\delta}} \\
 & \cdot \tilde{\theta}_{n_{0,1} + \dots + n_{d-1,d}} [\tilde{p}_{0,d}]
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{\alpha_{1,2}, \dots, \alpha_{d-1,d} \in \mathbb{Z}} \bar{G}_{n_{0,1}} \left[\frac{\bar{m}_{0,1}}{n_{0,1}}, \frac{\bar{m}_{1,2}}{n_{1,2}} \right] \\
&\cdot \bar{G}_{n_{0,1}+n_{1,2}} \left[\frac{\bar{m}_{1,2}}{n_{1,2}}, \frac{\bar{m}_{2,3}}{n_{2,3}} \right] \\
&\cdots \bar{G}_{n_{0,1}+n_{1,2}+\dots+n_{d-2,d-1}} \left[\frac{\bar{m}_{d-2,d-1}}{n_{d-2,d-1}}, \frac{\bar{m}_{0,d}}{n_{0,d}} \right] \\
&\cdot \bar{\theta}_{n_{0,1}+n_{1,2}+\dots+n_{d-2,d-1}} \left[\frac{\bar{m}_{0,d}}{n_{0,d}} \right] \bar{G}_{n_{d-1,d}} \left[\frac{\bar{m}_{d-1,d}}{n_{d-1,d}}, \frac{\bar{m}_{0,d}}{n_{0,d}} \right] \\
&\cdot \bar{\theta}_{n_{d-1,d}} \left[\frac{\bar{m}_{0,d}}{n_{0,d}} \right] \\
&= \sum_{\alpha_{1,2}, \dots, \alpha_{d-1,d} \in \mathbb{Z}} \bar{\theta}_{n_{0,1}} \left[\frac{\bar{m}_{0,1}}{n_{0,1}} \right] \bar{\theta}_{n_{1,2}} \left[\frac{\bar{m}_{1,2}}{n_{1,2}} \right] \\
&\cdots \bar{\theta}_{n_{d-2,d-1}} \left[\frac{\bar{m}_{d-2,d-1}}{n_{d-2,d-1}} \right] \bar{\theta}_{n_{d-1,d}} \left[\frac{\bar{m}_{d-1,d}}{n_{d-1,d}} \right].
\end{aligned} \tag{36}$$

From Lemma 6,

$$\begin{aligned}
&\sum_{\alpha_{1,2}, \dots, \alpha_{d-1,d} \in \mathbb{Z}} \langle \bar{p}_{0,d}; \bar{p}_{0,1}, \bar{p}_{1,2}, \dots, \bar{p}_{d-1,d} \rangle_{\bar{\delta}} \\
&\cdot \theta_{n_{0,1}+\dots+n_{d-1,d}} [\bar{p}_{0,d}] = \theta_{n_{0,1}} [p_{0,1}] \theta_{n_{1,2}} [p_{1,2}] \\
&\cdots \theta_{n_{d-1,d}} [p_{d-1,d}].
\end{aligned} \tag{37}$$

□

By using these Feynman diagrams, we define m_d .

Definition 10 (m_d).

$$\begin{aligned}
&m_d(p_{0,1}, p_{1,2}, \dots, p_{d-1,d}) \\
&:= \sum_{\alpha_{1,2}, \dots, \alpha_{d-1,d} \in \mathbb{Z}} \sum_{\bar{\delta} \in \mathcal{F}_0(\bar{p}_{0,d}; \bar{p}_{0,1}, \bar{p}_{1,2}, \dots, \bar{p}_{d-1,d})} (-1)^{|\bar{\delta}|} \\
&\cdot \langle \bar{p}_{0,d}; \bar{p}_{0,1}, \bar{p}_{1,2}, \dots, \bar{p}_{d-1,d} \rangle_{\bar{\delta}} \theta_{n_{0,d}} [\bar{p}_{0,d}]
\end{aligned} \tag{38}$$

$|\bar{\delta}| \in \mathbb{Z}$ should be determined so that m_d satisfy the A_∞ relation. We have given a general definition including more than one-dimensional case, whereas $\bar{\delta}$ is unique in this one-dimensional situation. Although m_d are essentially products of the theta functions as one can see in Theorem 9, the naturally defined directions of the propagators and noncommutativity of the external vertices restrict the states of the external vertices. As a result, we obtain the following theorems.

Theorem 11. *The degree of m_d is $2-d$.*

Proof. Because p in $H^{0,p}(E_\tau, L(n_{i-1,i}, u_{i-1,i}))$ represent the degrees of the open string states, the states with $n_{i-1,i} > 0$ and $n_{i-1,i} < 0$ have degrees 0 and 1, respectively. In the proof of Theorem 9, the Feynman diagrams are classified into two kinds (Figures 6 and 7). In the diagrams in Figure 6, $n_{0,d} < 0$, $n_{0,1} > 0$, and $n_{i-1,i} < 0$ ($i = 2, \dots, d$). If we compare the degrees of the both sides of the formula in Definition 10, we obtain $0 + (d-1) + (\text{the degree of } m_d) = 1$. Thus, the degree of m_d is $2-d$. Because $n_{0,d} > 0$, $n_{0,1} > 0$, $n_{i-1,i} < 0$ ($i = 2, \dots, d-1$), and $n_{d-1,d} > 0$ in the diagrams in Figure 7, $0 + (d-2) + 0 + (\text{the degree of } m_d) = 0$. Thus, the degree of m_d is $2-d$. □

Theorem 12 (A_∞ relation).

$$\begin{aligned}
&\sum_{1 \leq p \leq d, 0 \leq q \leq d-p} (-1)^{\deg(p_{0,1})+\dots+\deg(p_{q-1,q})-q} m_{d-p+1}(p_{0,1}, \\
&\dots, p_{q-1,q}, m_p(p_{q,q+1}, \dots, p_{p+q-1,p+q}), p_{p+q,p+q+1}, \dots, \\
&p_{d-1,d}) = 0,
\end{aligned} \tag{39}$$

where $\deg(p_{i-1,i})$ ($i = 1, \dots, d$) represent the degrees of the string states.

Proof.

$$\begin{aligned}
&m_{d-p+1}(p_{0,1}, \dots, p_{q-1,q}, m_p(p_{q,q+1}, \dots, p_{p+q-1,p+q}), \\
&p_{p+q,p+q+1}, \dots, p_{d-1,d}) \\
&= \sum_{\alpha_{q+1,q+2}, \dots, \alpha_{p+q-1,p+q} \in \mathbb{Z}} \sum_{\bar{\delta}_1 \in \mathcal{F}_0(\bar{r}; \bar{p}_{q,q+1}, \dots, \bar{p}_{p+q-1,p+q})} (-1)^{|\bar{\delta}_1|} \langle \bar{r}; \bar{p}_{q,q+1}, \dots, \bar{p}_{p+q-1,p+q} \rangle_{\bar{\delta}_1} \\
&m_{d-p+1}(p_{0,1}, \dots, p_{q-1,q}, \bar{r}, p_{p+q,p+q+1}, \dots, p_{d-1,d}) \\
&= \sum_{\alpha_{0,1}, \dots, \alpha_{q-1,q}, \alpha_{q+1,q+2}, \dots, \alpha_{d-1,d} \in \mathbb{Z}} \sum_{\bar{\delta}_1 \in \mathcal{F}_0(\bar{r}; \bar{p}_{q,q+1}, \dots, \bar{p}_{p+q-1,p+q})} \sum_{\bar{\delta}_2 \in \mathcal{F}_0(\bar{p}_{0,d}; \bar{p}_{0,1}, \dots, \bar{p}_{q-1,q}, \bar{r}, \bar{p}_{p+q,p+q+1}, \dots, \bar{p}_{d-1,d})} (-1)^{|\bar{\delta}_1|+|\bar{\delta}_2|} \langle \bar{r}; \bar{p}_{q,q+1}, \dots, \bar{p}_{p+q-1,p+q} \rangle_{\bar{\delta}_1} \\
&\langle \bar{p}_{0,d}; \bar{p}_{0,1}, \dots, \bar{p}_{q-1,q}, \bar{r}, \bar{p}_{p+q,p+q+1}, \dots, \bar{p}_{d-1,d} \rangle_{\bar{\delta}_2} \theta_{n_{0,1}+\dots+n_{d-1,d}} [\bar{p}_{0,d}].
\end{aligned} \tag{40}$$

FIGURE 8: $\tilde{\delta}_1 \otimes \tilde{\delta}_2 \in \partial \tilde{\mathcal{F}}_1$.

When $\tilde{r}$ take arbitrary values, the diagrams in Figure 8 are elements of $\tilde{\mathcal{F}}_1(\tilde{p}_{0,d}; \tilde{p}_{0,1}, \tilde{p}_{1,2}, \dots, \tilde{p}_{d-1,d})$, which is the one-dimensional subspace of $\tilde{\mathcal{F}}(\tilde{p}_{0,d}; \tilde{p}_{0,1}, \tilde{p}_{1,2}, \dots, \tilde{p}_{d-1,d})$, where $\tilde{\mathcal{F}}(\tilde{p}_{0,d}; \tilde{p}_{0,1}, \tilde{p}_{1,2}, \dots, \tilde{p}_{d-1,d})$ is a closure of $\mathcal{F}(\tilde{p}_{0,d}; \tilde{p}_{0,1}, \tilde{p}_{1,2}, \dots, \tilde{p}_{d-1,d})$. Especially, $\tilde{\delta}_1 \otimes \tilde{\delta}_2$ are the diagrams in Figure 8 with $\tilde{r} = (n_{q,q+1}\tilde{p}_{q,q+1} + \dots + n_{q+p-1,q+p}\tilde{p}_{q+p-1,q+p}) / (n_{q,q+1} + \dots + n_{q+p-1,q+p})$, which are elements of $\partial \tilde{\mathcal{F}}_1(\tilde{p}_{0,d}; \tilde{p}_{0,1}, \tilde{p}_{1,2}, \dots, \tilde{p}_{d-1,d})$, where $\partial \tilde{\mathcal{F}}_1(\tilde{p}_{0,d}; \tilde{p}_{0,1}, \tilde{p}_{1,2}, \dots, \tilde{p}_{d-1,d})$ is a boundary of $\tilde{\mathcal{F}}_1(\tilde{p}_{0,d}; \tilde{p}_{0,1}, \tilde{p}_{1,2}, \dots, \tilde{p}_{d-1,d})$. By choosing $|\tilde{\delta}|$ appropriately, (40) becomes 0 because the contributions from the boundary of the one-dimensional space cancel with each other as in the same mechanism in the A-model [6, 9, 22]. $\square$

Here we compare m_d in the B-model and A-model.

Theorem 13 (mirror symmetry). *m_d in the B-model in Definition 10 coincides with m_d in the A-model by identifying $\theta_n[\tilde{p}]$ with $[\tilde{p}]$ (This identification is a quasi-isomorphism [6, 9]).*

Proof. The formula in Definition 10 can be regarded as m_d in the A-model by replacing $\theta_n[\tilde{p}]$ with $[\tilde{p}]$. The following is the definition of the formula in Definition 10 in the A-model [6, 9]. In the A-model, the i -th D-brane (Lagrangian submanifold) $\tilde{L}_i$ ($i = 1, \dots, d$) is represented by a line with slope $-n_i$ on the universal cover of the symplectic torus TB . The intersection of $\tilde{L}_{i-1}$ and $\tilde{L}_i$ is represented by $\tilde{y} = \tilde{p}_{i-1,i}$ in $(\tilde{y}, \tilde{x})$ coordinates that is the universal cover of (y, x) coordinates of TB ($0 \leq y, x < 1$). $[\tilde{p}_{i-1,i}]$ represents the string state with $n_{i-1,i}$ between the D-branes $\tilde{L}_{i-1}$ and $\tilde{L}_i$. $[\tilde{p}_{i-1,i}]$ corresponds to $\theta_{n_{i-1,i}}[\tilde{p}_{i-1,i}]$ in the B-model. The intersection

of L_d and L_0 is determined as $\tilde{p}_{0,d} = (1/n_{0,d}) \sum_{i=1}^d n_{i-1,i} \tilde{p}_{i-1,i}$. $[\tilde{p}_{0,d}]$ is a basis of the $m_d(p_{0,1}, p_{1,2}, \dots, p_{d-1,d})$, which corresponds to $\theta_{n_{0,d}}[\tilde{p}_{0,d}]$ in the B-model. The area of the convex surrounded by $L_0, \dots, L_d$ is determined as $S_d(\tilde{p}_{0,1}, \dots, \tilde{p}_{d-1,d}) = (1/2)(\sum_{i=1}^d n_{i-1,i} \tilde{p}_{i-1,i} (\tilde{p}_{i-1,i} - 1) - n_{0,d} \tilde{p}_{0,d} (\tilde{p}_{0,d} - 1))$. $\exp(2\pi i \tau S_d(\tilde{p}_{0,1}, \dots, \tilde{p}_{d-1,d}))$ is the coefficient of the $m_d(p_{0,1}, p_{1,2}, \dots, p_{d-1,d})$, which is equal to the coefficient of the $m_d(p_{0,1}, p_{1,2}, \dots, p_{d-1,d})$ in the B-model as one can see in Theorems 4 and 9.

We are going to show that the Feynman diagrams, which determine the A_∞ structure of the B-model, coincide with the tropical Morse trees, which determine the A_∞ structure of the A-model. That is, both the A_∞ structures coincide. The tropical Morse trees $\tilde{\phi}$ are continuous maps that satisfy the following conditions, (The condition (6) in [9] determines not the tropical Morse trees but the length of the metric ribbon trees from the tropical Morse trees.) from the metric ribbon trees S to the universal cover $B (\ni \tilde{y})$ of $\mathbf{R}/\mathbf{Z} (\ni y)$ [6, 9].

- (0) We attach $n_{i-1,i}$ to the i -th external incoming edge and attach the sum of all numbers labelling the edges coming into a vertex to the edge coming out of the vertex. That is, the numbers n are preserved at vertices. We define an affine displacement vector v on each edge.
- (1) The coordinates of the external incoming and outgoing vertices are $\tilde{y} = \tilde{p}_{i-1,i}$ and $\tilde{y} = \tilde{p}_{0,d}$, respectively.
- (2) All the edges and vertices of the metric ribbon trees map to the edges and vertices in the universal cover, respectively.
- (3) $v = 0$ at the external vertices.

- (4) $\Delta v = n\Delta \tilde{v}$ on the edges. The directions of v and the edges coincide.
- (5) v are preserved at the vertices.

The conditions (0), ..., (5) coincide with the properties of the Feynman diagrams in the B-model: (1) is satisfied in the B-model by Theorem 9. (2) is automatically satisfied in the B-model. (0), (3), and (5) follow from the definition of the canonical form of the $\tilde{\theta}$. (4) coincides with the condition that $\Delta v = n\Delta \tilde{v}$ and directions of the propagators are the same as those of v . $\square$

Because the Feynman diagrams in the B-model coincide with the tropical Morse trees in the A-model, $\mathcal{F}(p_{0,d}; p_{0,1}, p_{1,2}, \dots, p_{d-1,d}) = S^{\text{trop}}(p_{0,d}; p_{0,1}, p_{1,2}, \dots, p_{d-1,d})$, which is the moduli space of the tropical Morse trees. It is shown in [6, 9] that $S^{\text{trop}}(p_{0,d}; p_{0,1}, p_{1,2}, \dots, p_{d-1,d})$ is bijective (The bijection map is given as follows [6, 9]: if the tropical Morse trees are decomposed to the edges, the corresponding pseudo holomorphic curves are also decomposed and, thus, the moduli spaces are decomposed as $\mathcal{S}_e^{\text{trop}} = \bigcup_e \mathcal{S}_e^{\text{trop}}$ and $\mathcal{M}/\text{PSL}(2, \mathbf{R}) = \bigcup_e \mathcal{M}_e/\text{PSL}(2, \mathbf{R})$. If the decomposed tropical Morse trees are denoted as $\mathcal{S}_e^{\text{trop}} \ni \tilde{\phi} : S_e \rightarrow B$ defined by $s \mapsto \tilde{\phi}(s)$ ($0 \leq s \leq 1$), the decomposed pseudo holomorphic curves are denoted as $\mathcal{M}_e/\text{PSL}(2, \mathbf{R}) \ni R_{\tilde{\phi}} : D_e \rightarrow TB$ defined by $(s, t) \mapsto (\tilde{\phi}(s), -n_e \tilde{\phi}(s) - t(d/ds)\tilde{\phi}(s))$ ($0 \leq t \leq 1$), where $(d/ds)\tilde{\phi}(s) = v(s)$. Then, the map between these moduli spaces is given by $F : \mathcal{S}_e^{\text{trop}} \rightarrow \mathcal{M}_e/\text{PSL}(2, \mathbf{R})$ defined by $\tilde{\phi} \mapsto R_{\tilde{\phi}}$, that is, $\tilde{\phi}(s) \mapsto (\tilde{\phi}(s), -n_e \tilde{\phi}(s) - t(d/ds)\tilde{\phi}(s))$.) to the moduli space $\mathcal{M}(p_{0,d}; p_{0,1}, p_{1,2}, \dots, p_{d-1,d})/\text{PSL}(2, \mathbf{R})$ of the pseudo holomorphic maps from a disk with $d + 1$ marked points to the symplectic torus TB in the A-model in one dimension. If a topology, for example Gromov topology [23], is defined on $\mathcal{M}(p_{0,d}; p_{0,1}, p_{1,2}, \dots, p_{d-1,d})/\text{PSL}(2, \mathbf{R})$, a topology is uniquely induced on $S^{\text{trop}}(p_{0,d}; p_{0,1}, p_{1,2}, \dots, p_{d-1,d})$ by the bijective map (For example, if $\mathcal{M}_e/\text{PSL}(2, \mathbf{R})$ is equipped with the topology of the functional space, equivalently, compact-open topology, the same topology is induced to $\mathcal{S}_e^{\text{trop}}$ and $\mathcal{S}_e^{\text{trop}} \cong \mathcal{M}_e/\text{PSL}(2, \mathbf{R})$. By patching them together, $\mathcal{S}^{\text{trop}} \cong \mathcal{M}/\text{PSL}(2, \mathbf{R})$.) As a result, we obtain the following.

Theorem 14 (topological space in homological mirror symmetry). $\mathcal{F}(p_{0,d}; p_{0,1}, p_{1,2}, \dots, p_{d-1,d})$ in the B-model is homeomorphic to $\mathcal{M}(p_{0,d}; p_{0,1}, p_{1,2}, \dots, p_{d-1,d})/\text{PSL}(2, \mathbf{R})$ in the A-model.

4. Conclusion and Discussion

In this paper, we have defined and explicitly constructed m_d that form A_∞ -category of the B-model on an elliptic curve. The way to define m_d is summarized as follows. The morphisms in the DG-category of this model are represented by theta functions with characteristics. From the products of the two theta functions, we can derive Feynman rules in the space of the characteristics, in other words, the configuration space of the morphisms. m_d are defined by the products of d theta functions with the characteristics in the case that the

moduli space of the Feynman diagrams is restricted to zero-dimensional. This restriction is necessary for m_d to satisfy the A_∞ relations. This m_d algebra coincides with the m_d algebra in the Fukaya category of the A-model on the corresponding symplectic torus. Thus, the A_∞ -category formed by these m_d is equivalent to the Fukaya category as an A_∞ -category. We have also shown that the moduli space of the Feynman diagrams in the B-model is homeomorphic to the moduli space of the pseudo holomorphic curves in the A-model.

The way to define m_d is naturally generalized in the case of the B-model on $2n$ -dimensional complex manifolds: we derive Feynman rules in the configuration space of the morphisms from the products of the two morphisms in the DG-category of the B-model. Then we define m_d by the products of d morphisms with the configurations in the case that the moduli space of the Feynman diagrams is restricted to zero-dimensional. We conjecture that these m_d satisfy the A_∞ relations and form an A_∞ -category, which is equivalent to the Fukaya category of the A-model on the corresponding $2n$ -dimensional symplectic manifold, as an A_∞ -category. We also conjecture that the moduli space of the Feynman diagrams in the B-model on the $2n$ -dimensional complex manifolds is homeomorphic to the moduli space of the pseudo holomorphic curves in the A-model on the corresponding $2n$ -dimensional symplectic manifold. These moduli spaces determine the A_∞ structure of the both models.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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