Research Article

Optimal System and Group Invariant Solutions of the Whitham-Broer-Kaup System

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From the nonlocal symmetries of the Whitham-Broer-Kaup system, an eight-dimensional Lie algebra is found and the corresponding one-dimensional optimal system is constructed to provide an inequivalent classification. Six types of inequivalent group invariant solutions are demonstrated, some of which reflect the interactions between soliton and other nonlinear waves.

1. Introduction

For a given nonlinear evolution equation, whether it is integrable or not, the Lie group theory is one of the most effective methods for seeking exact and analytic solutions. Due to the Lie point symmetries, one can obtain new solutions from the old one through the symmetry group and acquire the group invariant solutions by the similar reductions. With the development of symmetry theory, a variety of nonlocal symmetries have been intensively investigated in the literature, which may come from the Darboux transformation [1–5], Bäcklund transformation [6, 7], and Painlevé analysis [8–10]. Using these kinds of nonlocal symmetries, one can obtain new analytic solutions, especially the exact interaction solutions between soliton and cnoidal periodic waves.

Another interesting and important problem about the group invariant solutions is to find those inequivalent branches of solutions, which leads to the concept of the optimal system. A method of constructing optimal system was firstly established by Ovsiannikov [11], using a global matrix for the adjoint transformation. Hereafter, the method has received extensive development by authors in [12–19] and others. Chou and Qu [20–22] offer many numerical invariants to address the inequivalences among the elements in the optimal system. More recently, a direct and systematic algorithm that can guarantee both the comprehensiveness and the inequivalence of the optimal system is proposed in [23] to find one-dimensional optimal system.

The Whitham-Broer-Kaup (WBK) system [24–26]

\begin{align*}
u_t + uu_x + v_x &= 0, \quad (1) \\
v_t + u_{xxx} + u_x + (u v)_x &= 0 \quad (2)
\end{align*}

is derived from the Hamiltonian theory of water waves and has been said to be “the oldest, simplest and most widely known set of equations...”. Here $v(x,t)$ is the amplitude of a surface wave, propagating along the $x$-axis with the horizontal velocity $u(x,t)$. A good understanding of the WBK system is very helpful for coastal and civil engineers to apply the nonlinear water wave model in a harbor and coastal design. The multisoliton solutions and the scattering problem are presented by Kaup [26]. A large class of almost periodic solutions are found algebrogeometrically by Matveev and Yavor [27]. Some soliton excitations and periodic wave solutions without dispersion relation are obtained in [28, 29]. The classical symmetries and corresponding one-dimensional optimal system of (1)-(2) are performed in [30]. Recently, Zhou and Lu have obtained the nonlocal symmetries of (1)-(2) in [31]. So a question arises: can we construct the one-dimensional optimal system based on the nonlocal symmetries? The purpose of this paper is to answer this question. Thus, Section 2 is devoted to constructing the corresponding one-dimensional optimal system. To show the interactions between soliton and other nonlinear waves, six types of inequivalent group invariant solutions are demonstrated in Section 3. The last section contains a summary.
Here let us give a brief account of the nonlocal symmetries, which are somewhat different from [31], where six dependent variables were considered. We point out that five dependent variables are just enough to constitute a closed prolonged system. By direct calculations, we know that if the function \( \phi \) satisfies

\[
C_t + 2C_{tx} - CC_x + S_x = 0
\]

with

\[
C = \frac{\phi_t}{\phi_x},
\]

\[
S = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2} \left( \frac{\phi_{xx}}{\phi_x} \right)^2,
\]

then

\[
u = -\frac{\phi_{xx} + \phi_t}{\phi_x},
\]

\[
\sigma^u = 2\phi_x,
\]

\[
\sigma^v = 2\phi_{xx}.
\]

That is to say, if \( \{u, v\} \) satisfy (1)-(2), so do \( \{\n(x, t, u, v, \phi, g_1, g_2) + O(\epsilon^2),
\]

\[
t \rightarrow t + \epsilon T(x, t, u, v, \phi, g_1, g_2) + O(\epsilon^2),
\]

\[
u \rightarrow v + \epsilon V(x, t, u, v, \phi, g_1, g_2) + O(\epsilon^2),
\]

\[
\phi \rightarrow \phi + \epsilon \Phi(x, t, u, v, \phi, g_1, g_2) + O(\epsilon^2),
\]

\[
g_1 \rightarrow g_1 + \epsilon G_1(x, t, u, v, \phi, g_1, g_2) + O(\epsilon^2),
\]

\[
g_2 \rightarrow g_2 + \epsilon G_2(x, t, u, v, \phi, g_1, g_2) + O(\epsilon^2)
\]

and the vector field associated with the above group of transformations

\[
K = \frac{X}{\partial x} + T \frac{\partial}{\partial t} + U \frac{\partial}{\partial u} + V \frac{\partial}{\partial v} + \Phi \frac{\partial}{\partial \phi} + G_1 \frac{\partial}{\partial g_1} + G_2 \frac{\partial}{\partial g_2}.
\]

Applying the standard Lie symmetry approach [12–19] to the prolonged system, we have

\[
X = \frac{\partial}{\partial x} + \frac{\partial}{\partial t} + C_2 t + C_6,
\]

\[
T = \frac{\partial}{\partial t} + C_2 t + C_6,
\]

\[
U = \frac{\partial}{\partial u} + C_4 g_1 + 2C_6 g_2,
\]

\[
V = \frac{\partial}{\partial v} + C_4 (1 + 2t) + C_4 + 2C_6 g_2,
\]

\[
\Phi = -C_6 \phi^2 + C_2 \phi + C_6,
\]

\[
G_1 = -C_6 g_1 - C_6 g_2 + 2C_6 g_1 \phi + C_7 g_1,
\]

\[
G_2 = -2C_6 g_2 - 2C_6 g_2 - 2C_6 (g_1^2 + \phi g_2) + C_7 g_2.
\]

2. One-Dimensional Optimal System

For the original WBK system (1)-(2), its Lie point symmetries are provided in the following:

\[
\theta = C_1 \delta_1 + C_2 \delta_2 + C_3 \delta_3 + C_4 \delta_4
\]
Thus we can obtain an eight-dimensional Lie algebra represented by the following eight generators:

\[ K_1 = tx \frac{\partial}{\partial x} + t^2 \frac{\partial}{\partial t} + (x - tu) \frac{\partial}{\partial u} + \left( 1 - 2t - 2tv \right) \frac{\partial}{\partial v} - tg_1 \frac{\partial}{\partial g_1} - 2tg_2 \frac{\partial}{\partial g_2}, \]

\[ K_2 = x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u} - 2(1 + v) \frac{\partial}{\partial v} - g_1 \frac{\partial}{\partial g_1} - 2g_2 \frac{\partial}{\partial g_2}, \]

\[ K_3 = \frac{\partial}{\partial t}, \]

\[ K_4 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \]

\[ K_5 = \frac{\partial}{\partial u}, \]

\[ K_6 = 2g_1 \frac{\partial}{\partial u} + 2g_2 \frac{\partial}{\partial v} - \phi^2 \frac{\partial}{\partial \phi} - 2g_1 \phi \frac{\partial}{\partial g_1} - 2 \left( g_1^2 + \phi g_2 \right) \frac{\partial}{\partial g_2}, \]

\[ K_7 = \phi \frac{\partial}{\partial \phi} + g_1 \frac{\partial}{\partial g_1} + g_2 \frac{\partial}{\partial g_2}, \]

\[ K_8 = \frac{\partial}{\partial \phi}. \]

Clearly seen from (15), all \( K_1 - K_8 \) are point symmetries to the eight equations of the prolonged system, while \( K_9 \) is a nonlocal symmetry of the WBK system (1)-(2). The corresponding transformation groups of (15) are obtained directly by solving the initial value problems, saying

\[ \mathcal{G}_1 : (x, t, u, v, \phi, g_1, g_2) \rightarrow \left( \frac{x}{1 - et}, \frac{t}{1 - et}, u + \epsilon (x - ut), v + (1 - 2t - 2tv) \epsilon \right), \]

\[ \mathcal{G}_2 : (x, t, u, v, \phi, g_1, g_2) \rightarrow \left( \frac{x}{1 + e^2}, xe^\phi, e^\phi, g_1, g_2, \frac{g_2}{1 + e^2} \right), \]

\[ \mathcal{G}_3 : (x, t, u, v, \phi, g_1, g_2) \rightarrow \left( x + e, t + u + e, v + (1 + e) \phi, g_1, g_2 \right), \]

\[ \mathcal{G}_4 : (x, t, u, v, \phi, g_1, g_2) \rightarrow \left( x + e, t + u + e, v + (1 + e) \phi, g_1, g_2 \right), \]

\[ \mathcal{G}_5 : (x, t, u, v, \phi, g_1, g_2) \rightarrow \left( x + e, t, u, v, \phi, g_1, g_2 \right), \]

\[ \mathcal{G}_6 : (x, t, u, v, \phi, g_1, g_2) \rightarrow \left( x + e, t, u, v, \phi, g_1, g_2 \right), \]

\[ \mathcal{G}_7 : \left( x, t, u, v, \phi, g_1, g_2 \right) \rightarrow \left( x, t, u, v, \phi e^\epsilon, g_1 e^\epsilon, g_2 e^\epsilon \right), \]

\[ \mathcal{G}_8 : \left( x, t, u, v, \phi, g_1, g_2 \right) \rightarrow \left( x, t, u, v, \phi + \epsilon, g_1, g_2 \right). \]

The general Lie point symmetries of the prolonged system read

\[ \sigma_u = (c_1 t + c_2 x + c_4 t + c_5) u_x + \left( c_1 t^2 + 2c_2 t + c_3 \right) u_t - \left[ c_1 (x - tu) - c_2 u + c_4 + 2c_6 g_1 \right], \]

\[ \sigma_v = (c_1 t + c_2 x + c_4 t + c_5) v_x + \left( c_1 t^2 + 2c_2 t + c_3 \right) v_t - \left[ c_1 (1 - 2t - 2tv) - 2c_2 (1 + v) + c_4 + 2c_6 g_2 \right], \]

\[ \sigma_\phi = (c_1 t + c_2 x + c_4 t + c_5) \phi_x + \left( c_1 t^2 + 2c_2 t + c_3 \right) \phi_t \]

\[ - \left[ -c_6 \phi^2 + c_5 \phi + c_4 \right], \]

\[ \sigma_{g_1} = (c_1 t + c_2 x + c_4 t + c_5) g_{1x} + \left( c_1 t^2 + 2c_2 t + c_3 \right) g_{1t} - \left[ -c_1 t g_1 - c_2 g_1 - 2c_6 g_1 \phi + c_5 g_1 \right], \]

\[ \sigma_{g_2} = \left( c_1 t + c_2 x + c_4 t + c_5 \right) g_{2x} + \left( c_1 t^2 + 2c_2 t + c_3 \right) g_{2t} \]

\[ - \left[ -2c_1 t g_2 - 2c_6 g_2 - c_4 g_2 + \phi g_2 \right]. \]

A meaningful and interesting job is to list those inequivalent group invariant solutions, which can be reduced to give a classification of the corresponding Lie algebra. We concentrate on constructing an optimal system of the eight-dimensional Lie algebra spanned by the basis \( \{K_1, K_2, \ldots, K_8\} \) in (15). Denote the symmetry algebra generated by \( \{K_1, K_2, \ldots, K_8\} \) as \( \mathcal{G} \) and the corresponding group as \( G \). The commutation relations between \( K_j \) and \( K_j \) are shown in Table 1, where the entry in row \( i \) and column \( j \) represents \( [K_i, K_j] = K_j K_i - K_i K_j \). Then the adjoint representation relations between \( K_j \) and \( K_j \), that is,

\[ Ad_{exp(\epsilon K_j)} \left( K_j \right) = e^{-\epsilon K_j} K_j e^{\epsilon K_j} \]

\[ = K_j - \epsilon \left[ K_j, K_j \right] + \frac{\epsilon^2}{2!} \left[ K_j, \left[ K_j, K_j \right] \right] \]

\[ = \cdots \],

are directly provided in Table 2 with the help of Table 1.

A real function \( \phi \) on the Lie algebra \( \mathcal{G} \) is called an invariant if \( \phi(Ad_{\epsilon K_j}(K)) = \phi(K) \) for all \( K \in \mathcal{G} \) and all \( \epsilon \in G \). If two vectors \( Y \) and \( \Psi \) are equivalent to each other under the adjoint action, it is necessary that \( \phi(Y) = \phi(\Psi) \) for any invariant \( \phi \). To compute the invariants of \( Y = \sum_{j=1}^{8} a_j K_j \), applying any \( \Psi = \sum_{j=1}^{8} b_j K_j \) to it yields

\[ Ad_{exp(\epsilon \Psi)} \left( a_j K_1 + \cdots + a_8 K_8 \right) \]

\[ = \epsilon \left( \Theta_1 K_1 + \cdots + \Theta_8 K_8 \right) + O(\epsilon^2) \]

\[ \left( x, t, u, v, \phi, g_1, g_2 \right) \rightarrow \left( x, t, u, v, \phi e^\epsilon, g_1 e^\epsilon, g_2 e^\epsilon \right). \]
with

\[
\begin{align*}
\Theta_1 &= 2a_1b_2 - 2b_1a_2, \\
\Theta_2 &= a_1b_3 - b_1a_3, \\
\Theta_3 &= 2a_2b_3 - b_2a_3, \\
\Theta_4 &= a_1b_5 - b_1a_5 + a_2b_1 - b_2a_1, \\
\Theta_5 &= a_2b_5 - b_2a_5 - a_1b_1 + a_3b_2 - b_3a_2, \\
\Theta_6 &= a_6b_7 - a_7b_6, \\
\Theta_7 &= 2a_8b_6 - 2a_6b_8, \\
\Theta_8 &= a_5b_7 - a_7b_5.
\end{align*}
\] (20)

For any \(b_i (i = 1 \ldots 8)\), it requires

\[
\Theta_1 \frac{\partial \phi}{\partial a_1} + \Theta_2 \frac{\partial \phi}{\partial a_2} + \cdots + \Theta_8 \frac{\partial \phi}{\partial a_8} = 0. \tag{21}
\]

Extracting the coefficients of all \(b_i\) in (21) and setting them to be zeros, two basic invariants of \(G\) are obtained, saying

\[
\Delta_1 = a_2^2 + 4a_6a_8, \tag{22}
\]
\[
\Delta_2 = a_5^2 - a_1a_3.
\]

Moreover, applying the separate adjoint actions of \(K_1, K_2, \ldots, K_8\) to \(Y = \sum_{i=1}^{8} a_iK_i\), respectively, and with the help of Table 2, the corresponding adjoint matrices \(A_1, A_2, \ldots, A_8\) are obtained:
For each case, select a corresponding representative element to construct the optimal system of degrees of

\[ \begin{align*}
\Lambda = \sum_{i=1}^{8} a_i K_i = \lambda\sum_{i=1}^{8} \bar{a}_i \bar{K}_i, \text{ shown by} \\
(\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4, \bar{a}_5, \bar{a}_6, \bar{a}_7, \bar{a}_8) \\
= (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) A.
\end{align*} \]

Now, by virtue of (26) and all the invariants, one can start to construct the optimal system of \( \mathcal{G} \) step by step. Since the degrees of \( \Delta_1 \) and \( \Delta_2 \) are both two, we just need to consider the following five cases:

\[ \begin{align*}
[\Delta_1 = 1, \Delta_2 = c] ; \\
[\Delta_1 = -1, \Delta_2 = c] ; \\
[\Delta_1 = 0, \Delta_2 = 1] ; \\
[\Delta_1 = 0, \Delta_2 = -1] ; \\
[\Delta_1 = 0, \Delta_2 = 0] .
\end{align*} \]

For each case, select a corresponding representative element in the simplest form named \( \tilde{Y} = \sum_{i=1}^{8} \tilde{a}_i \tilde{K}_i \) and solve (26).

Then the general adjoint transformation matrix \( A \) is selected as

\[ A = A_1 A_2 A_3 A_4 A_5 A_6 A_7 A_8 \]

\[ = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} . \]

That is to say, after the adjoint action of

\[ Ad_{exp(e_i K_i)} A d_{exp(e_i K_i)} A d_{exp(e_i K_i)} A d_{exp(e_i K_i)} A d_{exp(e_i K_i)} A d_{exp(e_i K_i)} A d_{exp(e_i K_i)} A d_{exp(e_i K_i)} A d_{exp(e_i K_i)} A d_{exp(e_i K_i)}, \]

\( Y = \sum_{i=1}^{8} a_i K_i \) is transformed into \( \tilde{Y} = \sum_{i=1}^{8} \tilde{a}_i \tilde{K}_i \), shown by

\[ \begin{align*}
\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4, \tilde{a}_5, \tilde{a}_6, \tilde{a}_7, \tilde{a}_8 \end{align*} \]

\[ = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) A. \]

If (26) has the solution with respect to \( e_1, \cdots e_8 \), it signifies that the selected representative element is right. If the chosen representative element makes (26) be incompatible, we need to adopt a new proper one. Repeat the process until all the cases in (27) are finished. Omitting the tedious but regular operational processes, the one-dimensional optimal system of eight-dimensional Lie algebra (15) is found to be

\[ \begin{align*}
\omega_1 = K_1 - cK_3 + K_6 + \frac{1}{4}K_8, \\
\omega_2 = K_1 - cK_3 + K_6 - \frac{1}{4}K_8, \\
\omega_3 = K_1 - cK_3 - K_6 + \frac{1}{4}K_8, \\
\omega_4 = K_1 + K_3 + K_6 + \frac{1}{4}K_8, \\
( c \in \mathbb{R} ) \end{align*} \]
Table 1: Commutator table of $G$.

<table>
<thead>
<tr>
<th>$K_1$</th>
<th>$K_2$</th>
<th>$K_3$</th>
<th>$K_4$</th>
<th>$K_5$</th>
<th>$K_6$</th>
<th>$K_7$</th>
<th>$K_8$</th>
</tr>
</thead>
<tbody>
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<td>$-2K_1$</td>
<td>$-K_3$</td>
<td>0</td>
<td>$-K_4$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$K_2$</td>
<td>2$K_1$</td>
<td>0</td>
<td>$-2K_3$</td>
<td>$K_4$</td>
<td>$-K_5$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$K_3$</td>
<td>$K_2$</td>
<td>$2K_4$</td>
<td>0</td>
<td>$K_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$K_4$</td>
<td>0</td>
<td>$-K_4$</td>
<td>$-K_5$</td>
<td>0</td>
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<td>0</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
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<td>2$K_7$</td>
<td>0</td>
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<td>0</td>
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<td>0</td>
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<tr>
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<td>0</td>
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<td>0</td>
<td>0</td>
<td>$-2K_7$</td>
<td>$K_8$</td>
</tr>
</tbody>
</table>

Table 2: Adjoint representation table of $G$.

<table>
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<tr>
<th>$Ad$</th>
<th>$K_1$</th>
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<th>$K_6$</th>
<th>$K_7$</th>
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</tr>
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<td>$K_3 + eK_2 + e^2K_1$</td>
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<td>$K_6$</td>
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<td>$K_2 - eK_3$</td>
<td>$K_3 - eK_2$</td>
<td>$K_4 - eK_3$</td>
<td>$K_5 - eK_4$</td>
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<td>$K_8$</td>
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<td>$K_3 + eK_2$</td>
<td>$K_4 + eK_3$</td>
<td>$K_5 + eK_4$</td>
<td>$K_6$</td>
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<tr>
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<td>$K_2$</td>
<td>$K_3$</td>
<td>$K_4$</td>
<td>$K_5$</td>
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<td>$K_7$</td>
<td>$e^2K_8$</td>
</tr>
<tr>
<td>$K_8$</td>
<td>$K_1$</td>
<td>$K_2$</td>
<td>$K_3$</td>
<td>$K_4$</td>
<td>$K_5$</td>
<td>$K_6 + 2eK_7 - e^2K_8$</td>
<td>$K_7$</td>
<td>$e^2K_8$</td>
</tr>
</tbody>
</table>

(28)

3. Group Invariant Solutions

For any given subgroup, an original nonlinear system can be reduced to a system with fewer independent variables, which may be easily solved to provide group invariant solutions. As Olver said, since there is almost always an infinite amount of such subgroups, it is usually not feasible to list all possible group invariant solutions to the system. In this section, some types of inequivalent group invariant solutions which correspond to the elements in the optimal system (28) are listed.
For $\omega_i$ with $c = 0$, we consider an equivalent case $K_6 + K_8 + K_3 + \alpha K_5$, where $\alpha$ is an arbitrary constant. The group invariant solution is

$$\phi = \sqrt{2} \tanh \left( \frac{\sqrt{2}}{2} t + \frac{\sqrt{2}}{2} p \right),$$

$$u = -\sqrt{2} G_1 \tanh \left( \frac{\sqrt{2}}{2} t + \frac{\sqrt{2}}{2} p \right) + U,$$

$$v = -G_1^2 \tanh^2 \left( \frac{\sqrt{2}}{2} t + \frac{\sqrt{2}}{2} p \right) + \left( 2G_1^2 + \sqrt{2}G_2 \right) \tanh \left( \frac{\sqrt{2}}{2} t + \frac{\sqrt{2}}{2} p \right) + V,$$

$$g_1 = -\frac{2G_1}{1 + \cosh \left( \sqrt{2}t + \sqrt{2}p \right)},$$

$$g_2 = \frac{2G_2 + 4\sqrt{2}G_1^2 \left( 1 + e^{\sqrt{2}(t+p)} \right)^{-1}}{1 + \cosh \left( \sqrt{2}t + \sqrt{2}p \right)}.$$

Here, $U, V, P, G_1$, and $G_2$ are all group invariant functions of $X = x - at$. Substituting the group invariant solutions (29) into the prolonged system leads to the symmetry reduction equations for $U, V, P, G_1$, and $G_2$:

$$P' = -G_1,$$

$$G_2 = -G_1' - \sqrt{2}G_1^3,$$

$$U = -\frac{G_1' - \alpha G_1 - 1}{G_1},$$

$$V = -\frac{1}{2} t^2 + \alpha U + c_0$$

with

$$G_1'' = \frac{3 G_1'^2}{2 G_1} - \frac{2G_1'}{G_1} + G_1^3 - \alpha_0 G_1 + \frac{1}{2G_1},$$

$$\alpha_0 = 1 + \frac{1}{2} \alpha^2 + c_0.$$

Here, and in the latter of the paper, the primes $', ''$ on the functions with only one independent variable denote derivatives. Thus by the transformation

$$G_1 = \frac{R}{\sqrt{2} R''},$$

$$R \equiv R \left( \sqrt{2}X \right),$$

(31) becomes

$$R''' = \frac{R''^2}{2R'} - \frac{R^2}{2R'} \frac{\sqrt{2}}{2} \alpha_0 R'.$$

For different values of $\alpha_0$, (33) is exactly solvable. For example, if we take $\alpha_0 = 3/2$, this equation has a solution

$$R = c_1 e^{\sqrt{2}X} + c_2 e^{-X} + c_3 e^{\sqrt{2}X} + \frac{c_3 c_1}{2c_1} e^{\sqrt{2}X}.$$

Then, by solving (30), one can directly obtain an exact solution of the WBK system (1)-(2):

$$u = \left( 2c_1 c_2 e^X + c_1^2 e^{-X} + c_1 c_2 e^{\sqrt{2}X} + c_2 c_1 e^{-\sqrt{2}X} \right) \tanh \left( \frac{\sqrt{2}}{2} t + \frac{1}{2} \ln \frac{c_1 e^{\sqrt{2}X} + \sqrt{2}c_2 e^{-\sqrt{2}X} + c_4}{c_1 e^{\sqrt{2}X} - \sqrt{2}c_2 e^{\sqrt{2}X} - c_4} \right) + \frac{2 \sqrt{2} c_2 c_1 (1 + \alpha) e^X - \sqrt{2} c_1^2 (\alpha - 1) e^{-X} + 2c_1 c_2 (\alpha + \sqrt{2}) e^{\sqrt{2}X} - 2c_1 c_3 (\sqrt{2} - \alpha) e^{-\sqrt{2}X}}{2 \sqrt{2} c_2 c_1 e^X - \sqrt{2} c_1^2 e^{-X} + 2c_1 c_2 e^{\sqrt{2}X} - 2c_1 c_3 e^{-\sqrt{2}X}},$$

$$v = u_x - 1.$$

(2) One equivalent case of $\omega_i$ is $K_6 + K_8 + K_3 - K_4$ and the group invariant solution in this case is

$$\phi = \sqrt{2} \tanh \left( \frac{\sqrt{2}}{2} t + \frac{\sqrt{2}}{2} p \right),$$

$$u = -\sqrt{2} G_1 \tanh \left( \frac{\sqrt{2}}{2} t + \frac{\sqrt{2}}{2} p \right) + U + t + P,$$

$$v = -G_1^2 \tanh^2 \left( \frac{\sqrt{2}}{2} t + \frac{\sqrt{2}}{2} p \right) + \left( 2G_1^2 + \sqrt{2}G_2 \right) \tanh \left( \frac{\sqrt{2}}{2} t + \frac{\sqrt{2}}{2} p \right) + V,$$

$$g_1 = -\frac{2G_1}{1 + \cosh \left( \sqrt{2}t + \sqrt{2}p \right)},$$

$$g_2 = \frac{2G_2 + 4\sqrt{2}G_1^2 \left( 1 + e^{\sqrt{2}(t+p)} \right)}{\left( 1 + e^{\sqrt{2}(t+p)} \right) \left( 1 + \cosh \left( \sqrt{2}t + \sqrt{2}p \right) \right)}.$$
Here $U, V, P, G_1$, and $G_2$ are all group invariant functions of $Z(= x - t^2/2)$, which should satisfy

\[ p = -G_1, \]
\[ G_2 = -G'_1 - \sqrt{2}G_1^2, \]
\[ U = \frac{1}{2}G_1^2 - P, \]
\[ V = U_x + G_2^2 - G_1 - 1. \]

And

\[ G''_1 = \frac{3}{2} \frac{G_1'^2}{G_1} - 2G_1' \frac{G_1'}{G_1} + (X - c_0) G_1 + \frac{1}{2G_1}. \]  

(37)

Thus by means of the transformation

\[ G_1 = \frac{W}{\sqrt{2}W'}, \]
\[ W \equiv W (Y), \]
\[ Y = \sqrt{2}X, \]

we can transform (38) into

\[ W''' = \frac{W'''}{2W'} - \frac{W''}{2W'} + \frac{1}{2} \left( \sqrt{2}c_0 - \sqrt{2}Y \right) W'. \]  

(40)

On differentiation, (40) becomes linear and of the fourth order:

\[ W'''' = -\frac{\sqrt{2}}{2} W' + \left( \sqrt{2}c_0 - \sqrt{2}Y \right) W'' - W. \]  

(41)

Thus, the general solution of (38) is a rational function of the constants of integration [32].

**Remark 1.** It is known that the solitary wave is a very important phenomenon in the nature and physics. In fact, these solitary waves must interact with other waves or spread on some nonconstant background. Here, the forms of solutions (29) and (36) directly embody this kind of interactions between the soliton and other nonlinear waves.

(3) For $\omega_{10}$ and $\omega_{11}$, it is not difficult to obtain the following group invariant solutions:

\[ u = \pm \left[ \frac{\sqrt{2}}{2t} \tanh \left( \frac{\sqrt{2} x + c_1 t + c_2}{t} \right) + \frac{x + c_2}{t} \right], \]  

(42)

\[ v = u_x - 1. \]

(4) Instead of studying $\omega_{15}$, consider an equivalent case: $v_5 - v_4 + v_6$. The corresponding group invariant solution of the WBK system reads

\[ u = -t + U (X) - \frac{G_1 (X)}{t + 2P (X)}, \]
\[ v = V (X) - \frac{2 - U (X) G_1 (X)}{t + 2P (X)} - \frac{G_1 (X)^2}{2 (t + 2P (X))^2}, \]
\[ X = x + \frac{1}{2}t^2, \]

(43)

where

\[ U (X) = \frac{\sqrt{7} \left( c_1 \text{Ai} (1, X_1) + \text{Bi} (1, X_1) \right)}{c_1 \text{Ai} (X_1) + \text{Bi} (X_1)}, \]
\[ V (X) = X - 1 + \frac{\sqrt{7}}{2} \left( \frac{c_1 \text{Ai} (1, X_1) + \text{Bi} (1, X_1)}{c_1 \text{Ai} (X_1) + \text{Bi} (X_1)} \right)^2, \]
\[ G_1 (X) = 2X - 2\sqrt{2} \left( \frac{c_1 \text{Ai} (1, X_1) + \text{Bi} (1, X_1)}{c_1 \text{Ai} (X_1) + \text{Bi} (X_1)} \right)^2 + c_2, \]
\[ P (X) = -\frac{1}{4} x^2 \]
\[ + \frac{\sqrt{2}}{2} \int_{X_1}^X \frac{c_1 \text{Ai} (1, Y) + \text{Bi} (1, Y)}{c_1 \text{Ai} (Y) + \text{Bi} (Y)} \, dX, \]
\[ X_1 = \frac{X}{\sqrt{2}} \]

Here, $\text{Ai}(x)$ and $\text{Bi}(x)$ are the Airy wave functions, which are linearly independent solutions for $w'' - xw = 0$.

(5) For $\omega_{19}$, it is equivalent to $K_1 + \alpha K_2 + K_3$, with $\alpha$ being an arbitrary constant. The group invariant solution is presented as

\[ u = U (X) = \frac{G_1 (X)}{t + 2P (X)}, \]
\[ v = V (X) - \frac{G_1 (X)^2}{2 (t + 2P (X))^2}, \]
\[ X = x - \alpha t \]

with

\[ U (X) = \alpha + 2c_1 \tanh \left( c_x X_1 + c_2 \right), \]
\[ V (X) = 2c_1 \text{sech}^2 \left( c_x X_1 + c_2 \right) - 1, \]
\[ P (X) = \frac{X}{4c_1} + \frac{2c_2 - c_3}{4c_1} \left( 1 + e^{2c_1 X + 2c_2} \right) + c_4, \]
\[ G_1 (X) = \frac{2c_2 X + \sinh \left( 2c_2 X + 2c_3 \right) - c_3}{c_1 \left( 1 + \cosh \left( 2c_1 X + 2c_2 \right) \right)}, \]
\[ G_2 (X) = \frac{4 + 4 \cosh \left( 2c_1 X + 2c_2 \right) + 2 \left( c_3 - 2c_1 X \right) \sinh \left( 2c_1 X + 2c_3 \right)}{(1 + \cosh \left( 2c_1 X + 2c_2 \right))^2} \]

(46)

(6) By solving $\omega_{21}$, one can easily give a simple exact solution:

\[ u = \frac{2}{x - c_1 t + 2c_2}, \]
\[ v = -\frac{2}{(x - c_1 t + 2c_2)^2} - 1. \]  

(47)

**4. Summary**

In the beginning of [31], the authors also discussed the nonlocal symmetry (6) of the WBK system. To realize the localization of the nonlocal symmetry, they referred to six...
dependent variables. In fact, we point that five dependent variables are enough in this paper. Based on a Bäcklund transformation, [31] constructed some special exact solutions, which have no direct connection with the given nonlocal symmetry. The purpose of our article is to give a classification of the eight-dimensional Lie algebra (15) and then to find out some inequivalent group invariant solutions with respect to the nonlocal symmetry (6).

In this paper, we firstly localize the nonlocal symmetries of the (1+1)-dimensional WBK system (1)-(2) to Lie point symmetries by prolonging the original system to a larger one. Then, all the Lie point symmetries of the whole prolonged system are presented, which constitute an eight-dimensional Lie algebra (15). The one-dimensional optimal system (28) is constructed to give a classification of the elements in the Lie algebra. The adjoint transformation matrix and all the invariants of the Lie algebra are also displayed. Finally, we demonstrate six kinds of inequivalent group invariant solutions, some of which reflect the interactions between soliton and other nonlinear waves.

Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that there are no conflicts of interest regarding the publication of this paper.

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