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Research Article

Optimal System and Group Invariant Solutions of the Whitham-Broer-Kaup System

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From the nonlocal symmetries of the Whitham-Broer-Kaup system, an eight-dimensional Lie algebra is found and the corresponding one-dimensional optimal system is constructed to provide an inequivalent classification. Six types of inequivalent group invariant solutions are demonstrated, some of which reflect the interactions between soliton and other nonlinear waves.

1. Introduction

For a given nonlinear evolution equation, whether it is integrable or not, the Lie group theory is one of the most effective methods for seeking exact and analytic solutions. Due to the Lie point symmetries, one can obtain new solutions from the old one through the symmetry group and acquire the group invariant solutions by the similar reductions. With the development of symmetry theory, a variety of nonlocal symmetries have been intensely investigated in the literature, which may come from the Darboux transformation [1–5], Bäcklund transformation [6, 7], and Painlevé analysis [8–10]. Using these kinds of nonlocal symmetries, one can obtain new analytic solutions, especially the exact interaction solutions between soliton and cnoidal periodic waves.

Another interesting and important problem about the group invariant solutions is to find those inequivalent branches of solutions, which leads to the concept of the optimal system. A method of constructing optimal system was firstly established by Ovsiannikov [11], using a global matrix for the adjoint transformation. Hereafter, the method has received extensive development by authors in [12–19] and others. Chou and Qu [20–22] offer many numerical invariants to address the inequivalences among the elements in the optimal system. More recently, a direct and systematic algorithm that can guarantee both the comprehensiveness and the inequivalence of the optimal system is proposed in [23] to find one-dimensional optimal system.

The Whitham-Broer-Kaup (WBK) system [24–26]

$$u_t + uu_x + v_x = 0, \tag{1}$$

$$v_t + u_{xxx} + u_x + (uv)_x = 0 \tag{2}$$

is derived from the Hamiltonian theory of water waves and has been said to be "the oldest, simplest and most widely known set of equations...". Here $v(x,t)$ is the amplitude of a surface wave, propagating along the x-axis with the horizontal velocity $u(x,t)$. A good understanding of the WBK system is very helpful for coastal and civil engineers to apply the nonlinear water wave model in a harbor and coastal design. The multisoliton solutions and the scattering problem are presented by Kaup [26]. A large class of almost periodic solutions are found algebrogeometrically by Matveev and Yavor [27]. Some soliton excitations and periodic waves solutions without dispersion relation are obtained in [28, 29]. The classical symmetries and corresponding one-dimensional optimal system of (1)-(2) are performed in [30]. Recently, Zhou and Lu have obtained the nonlocal symmetries of (1)-(2) in [31]. So a question arises: can we construct the one-dimensional optimal system based on the nonlocal symmetries? The purpose of this paper is to answer this question. Thus, Section 2 is devoted to constructing the corresponding one-dimensional optimal system. To show the interactions between soliton and other nonlinear waves, six types of inequivalent group invariant solutions are demonstrated in Section 3. The last section contains a summary.

Here let us give a brief account of the nonlocal symmetries, which are somewhat different from [31], where six dependent variables were considered. We point out that five dependent variables are just enough to constitute a closed prolonged system. By direct calculations, we know that if the function ϕ satisfies

$$C_t + 2C_{xx} - CC_x + S_x = 0 \tag{3}$$

with

$$\begin{aligned} C &\equiv \frac{\phi_t}{\phi_x}, \\ S &\equiv \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2, \end{aligned} \tag{4}$$

then

$$\begin{aligned} u &= -\frac{\phi_{xx} + \phi_t}{\phi_x}, \\ v &= -1 - S - C_x - \frac{1}{2}\frac{\phi_{xx}^2}{\phi_x^2} \end{aligned} \tag{5}$$

is just the solution of the WBK system (1)-(2). Thus one can verify that the WBK system (1)-(2) possesses the following nonlocal symmetries:

$$\begin{aligned} \sigma^u &= 2\phi_x, \\ \sigma^v &= 2\phi_{xx}. \end{aligned} \tag{6}$$

That is to say, if $\{u, v\}$ satisfy (1)-(2), so do $\{\overline{u}(\epsilon), \overline{v}(\epsilon)\} = \{u, v\} + \epsilon\{\sigma^u, \sigma^v\}$. Because (6) is nonlocal for $\{u, v\}$, we need to extend the original system by introducing two other new dependent variables:

$$\begin{aligned} g_1 &= \phi_x, \\ g_2 &= g_{1x}. \end{aligned} \tag{7}$$

Furthermore, the compatibility conditions of $\phi_{xt} = \phi_{tx}$ and $g_{1xt} = g_{1tx}$ require

$$u_x = v + 1, \tag{8}$$

$$g_{2t} + \left(g_{2x} + ug_2 + vg_1\right)_x + g_2 = 0 \tag{9}$$

In the following, we investigate the prolonged system including five dependent variables and eight equations (i.e., (1), (2), (5), (7), (8), and (9)), instead of the two original equations, (1) and (2).

2. One-Dimensional Optimal System

For the original WBK system (1)-(2), its Lie point symmetries are provided in the following:

$$\vartheta = C_1\vartheta_1 + C_2\vartheta_2 + C_3\vartheta_3 + C_4\vartheta_4 \tag{10}$$

with

$$\begin{aligned} \vartheta_1 &= x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - 2(1+v)\frac{\partial}{\partial v}, \\ \vartheta_2 &= \frac{\partial}{\partial t}, \\ \vartheta_3 &= t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \\ \vartheta_4 &= \frac{\partial}{\partial x}. \end{aligned} \tag{11}$$

For the whole prolonged system, we need to consider the following one-parameter Lie group of infinitesimal transformations with a small parameter $\epsilon \ll 1$,

$$\begin{aligned} x &\longrightarrow x + \epsilon X(x, t, u, v, \phi, g_1, g_2) + O(\epsilon^2), \\ t &\longrightarrow t + \epsilon T(x, t, u, v, \phi, g_1, g_2) + O(\epsilon^2), \\ u &\longrightarrow u + \epsilon U(x, t, u, v, \phi, g_1, g_2) + O(\epsilon^2), \\ v &\longrightarrow v + \epsilon V(x, t, u, v, \phi, g_1, g_2) + O(\epsilon^2), \\ \phi &\longrightarrow \phi + \epsilon \Phi(x, t, u, v, \phi, g_1, g_2) + O(\epsilon^2), \\ g_1 &\longrightarrow g_1 + \epsilon G_1(x, t, u, v, \phi, g_1, g_2) + O(\epsilon^2), \\ g_2 &\longrightarrow g_2 + \epsilon G_2(x, t, u, v, \phi, g_1, g_2) + O(\epsilon^2) \end{aligned} \tag{12}$$

and the vector field associated with the above group of transformations

$$\begin{aligned} K = {} & X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + V\frac{\partial}{\partial v} + \Phi\frac{\partial}{\partial \phi} + G_1\frac{\partial}{\partial g_1} \\ & + G_2\frac{\partial}{\partial g_2}. \end{aligned} \tag{13}$$

Applying the standard Lie symmetry approach [12–19] to the prolonged system, we have

$$\begin{aligned} X &= c_1 tx + c_2 x + c_4 t + c_5, \\ T &= c_1 t^2 + 2c_2 t + c_3, \\ U &= c_1(x - tu) - c_2 u + c_4 + 2c_6 g_1, \\ V &= c_1(1 - 2t - 2tv) - 2c_2(1 + v) + c_4 + 2c_6 g_2, \\ \Phi &= -c_6\phi^2 + c_7\phi + c_8, \\ G_1 &= -c_1 t g_1 - c_2 g_1 - 2c_6 g_1\phi + c_7 g_1, \\ G_2 &= -2c_1 t g_2 - 2c_2 g_2 - 2c_6\left(g_1^2 + \phi g_2\right) + c_7 g_2. \end{aligned} \tag{14}$$

Thus we can obtain an eight-dimensional Lie algebra represented by the following eight generators:

$$\begin{aligned}
K_1 &= tx\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} + (x - tu)\frac{\partial}{\partial u} + (1 - 2t - 2tv)\frac{\partial}{\partial v} \\
&\quad - tg_1\frac{\partial}{\partial g_1} - 2tg_2\frac{\partial}{\partial g_2}, \\
K_2 &= x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - 2(1 + v)\frac{\partial}{\partial v} - g_1\frac{\partial}{\partial g_1} \\
&\quad - 2g_2\frac{\partial}{\partial g_2}, \\
K_3 &= \frac{\partial}{\partial t}, \\
K_4 &= t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \\
K_5 &= \frac{\partial}{\partial x}, \\
K_6 &= 2g_1\frac{\partial}{\partial u} + 2g_2\frac{\partial}{\partial v} - \phi^2\frac{\partial}{\partial \phi} - 2g_1\phi\frac{\partial}{\partial g_1} \\
&\quad - 2\left(g_1^2 + \phi g_2\right)\frac{\partial}{\partial g_2}, \\
K_7 &= \phi\frac{\partial}{\partial \phi} + g_1\frac{\partial}{\partial g_1} + g_2\frac{\partial}{\partial g_2}, \\
K_8 &= \frac{\partial}{\partial \phi}.
\end{aligned} \tag{15}$$

Clearly seen from (15), all $K_1 - K_8$ are point symmetries to the eight equations of the prolonged system, while K_6 is a nonlocal symmetry of the WBK system (1)-(2). The corresponding transformation groups of (15) are obtained directly by solving the initial value problems, saying

$$\begin{aligned}
&\mathcal{G}_1 : (x, t, u, v, \phi, g_1, g_2) \longrightarrow \Big(\frac{x}{1 - \epsilon t}, \frac{t}{1 - \epsilon t}, u \\
&\quad + \epsilon(x - ut), v + (1 - 2t - 2vt)\epsilon \\
&\quad + t(t + vt - 1)\epsilon^2, \phi, g_1(1 - \epsilon t), g_2(1 - \epsilon t)^2\Big), \\
&\mathcal{G}_2 : (x, t, u, v, \phi, g_1, g_2) \longrightarrow \Big(xe^{\epsilon}, te^{2\epsilon}, ue^{-\epsilon}, -1 \\
&\quad + (v + 1)e^{-2\epsilon}, \phi, g_1e^{-\epsilon}, g_2e^{-2\epsilon}\Big), \\
&\mathcal{G}_3 : (x, t, u, v, \phi, g_1, g_2) \longrightarrow (x, t + \epsilon, u, v, \phi, g_1, g_2), \\
&\mathcal{G}_4 : (x, t, u, v, \phi, g_1, g_2) \longrightarrow (x + \epsilon t, t, u + \epsilon, v, \phi, g_1, g_2), \\
&\mathcal{G}_5 : (x, t, u, v, \phi, g_1, g_2) \longrightarrow (x + \epsilon, t, u, v, \phi, g_1, g_2), \\
&\mathcal{G}_6 : (x, t, u, v, \phi, g_1, g_2) \longrightarrow \Bigg(x, t, u + \frac{2\epsilon g_1}{1 + \epsilon\phi}, v \\
&\quad + \frac{2\epsilon g_2}{1 + \epsilon\phi} - \frac{2\epsilon^2 g_1^2}{(1 + \epsilon\phi)^2}, \frac{\phi}{1 + \epsilon\phi}, \frac{g_1}{(1 + \epsilon\phi)^2}, \frac{g_2}{(1 + \epsilon\phi)^2} \\
&\quad - \frac{2g_1^2\epsilon}{(1 + \epsilon\phi)^3}\Bigg), \\
&\mathcal{G}_7 : (x, t, u, v, \phi, g_1, g_2) \longrightarrow (x, t, u, v, \phi e^{\epsilon}, g_1e^{\epsilon}, g_2e^{\epsilon}), \\
&\mathcal{G}_8 : (x, t, u, v, \phi, g_1, g_2) \longrightarrow (x, t, u, v, \phi + \epsilon, g_1, g_2).
\end{aligned} \tag{16}$$

The general Lie point symmetries of the prolonged system read

$$\begin{aligned}
\sigma_u &= (c_1tx + c_2x + c_4t + c_5)u_x + \left(c_1t^2 + 2c_2t + c_3\right)u_t \\
&\quad - \left[c_1(x - tu) - c_2u + c_4 + 2c_6g_1\right], \\
\sigma_v &= (c_1tx + c_2x + c_4t + c_5)v_x + \left(c_1t^2 + 2c_2t + c_3\right)v_t \\
&\quad - \left[c_1(1 - 2t - 2tv) - 2c_2(1 + v) + c_4 + 2c_6g_2\right], \\
\sigma_\phi &= (c_1tx + c_2x + c_4t + c_5)\phi_x + \left(c_1t^2 + 2c_2t + c_3\right)\phi_t \\
&\quad - \left[-c_6\phi^2 + c_7\phi + c_8\right], \\
\sigma_{g_1} &= (c_1tx + c_2x + c_4t + c_5)g_{1x} + \left(c_1t^2 + 2c_2t + c_3\right)g_{1t} \\
&\quad - \left[-c_1tg_1 - c_2g_1 - 2c_6g_1\phi + c_7g_1\right], \\
\sigma_{g_2} &= (c_1tx + c_2x + c_4t + c_5)g_{2x} + \left(c_1t^2 + 2c_2t + c_3\right)g_{2t} \\
&\quad - \left[-2c_1tg_2 - 2c_2g_2 - 2c_6\left(g_1^2 + \phi g_2\right) + c_7g_2\right].
\end{aligned} \tag{17}$$

A meaningful and interesting job is to list those inequivalent group invariant solutions, which can be reduced to give a classification of the corresponding Lie algebra. We concentrate on constructing an optimal system of the eight-dimensional Lie algebra spanned by the basis $\{K_1, K_2, \ldots, K_8\}$ in (15). Denote the symmetry algebra generated by $\{K_1, K_2, \ldots, K_8\}$ as $\mathcal{G}$ and the corresponding group as G. The commutation relations between K_i and K_j are shown in Table 1, where the entry in row i and column j represents $[K_i, K_j] = K_iK_j - K_jK_i$. Then the adjoint representation relations between K_i and K_j, that is,

$$\begin{aligned}
Ad_{\exp(\epsilon K_i)}\left(K_j\right) &= e^{-\epsilon K_i}K_je^{\epsilon K_i} \\
&= K_j - \epsilon\left[K_i, K_j\right] + \frac{1}{2!}\epsilon^2\left[K_i, \left[K_i, K_j\right]\right] \\
&\quad - \cdots,
\end{aligned} \tag{18}$$

are directly provided in Table 2 with the help of Table 1.

A real function ϕ on the Lie algebra $\mathcal{G}$ is called an invariant if $\phi(Ad_g(K)) = \phi(K)$ for all $K \in \mathcal{G}$ and all $g \in G$. If two vectors Υ and Ψ are equivalent to each other under the adjoint action, it is necessary that $\phi(\Upsilon) = \phi(\Psi)$ for any invariant ϕ. To compute the invariants of $\Upsilon = \sum_{i=1}^{8} a_iK_i$, applying any $\Psi = \sum_{j=1}^{8} b_jK_j$ to it yields

$$\begin{aligned}
Ad_{\exp(\epsilon\Psi)}(\Upsilon) &= \left(a_1K_1 + \cdots + a_8K_8\right) \\
&\quad - \epsilon\left(\Theta_1K_1 + \ldots + \Theta_8K_8\right) + O\left(\epsilon^2\right)
\end{aligned} \tag{19}$$

with

$$\begin{aligned}
\Theta_1 &= 2a_1b_2 - 2b_1a_2,\\
\Theta_2 &= a_1b_3 - b_1a_3,\\
\Theta_3 &= 2a_2b_3 - 2b_2a_3,\\
\Theta_4 &= a_1b_5 - a_5b_1 + a_4b_2 - a_2b_4,\\
\Theta_5 &= a_2b_5 - a_5b_2 + a_4b_3 - a_3b_4.\\
\Theta_6 &= a_6b_7 - a_7b_6,\\
\Theta_7 &= 2a_8b_6 - 2a_6b_8,\\
\Theta_8 &= a_7b_8 - a_8b_7.
\end{aligned} \tag{20}$$

For any $b_i (i = 1 \cdots 8)$, it requires

$$\Theta_1 \frac{\partial \phi}{\partial a_1} + \Theta_2 \frac{\partial \phi}{\partial a_2} + \cdots + \Theta_8 \frac{\partial \phi}{\partial a_8} = 0. \tag{21}$$

Extracting the coefficients of all b_i in (21) and setting them to be zeros, two basic invariants of $\mathcal{G}$ are obtained, saying

$$\begin{aligned}
\Delta_1 &= a_7^2 + 4a_6a_8,\\
\Delta_2 &= a_2^2 - a_1a_3.
\end{aligned} \tag{22}$$

Moreover, applying the separate adjoint actions of K_1, $K_2, \ldots, K_8$ to $\Upsilon = \sum_{i=1}^{8} a_iK_i$, respectively, and with the help of Table 2, the corresponding adjoint matrices $A_1, A_2, \ldots, A_8$ are obtained:

$$A_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
2\epsilon_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
\epsilon_1^2 & \epsilon_1 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \epsilon_1 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

$$A_2 = \begin{pmatrix}
e^{-2\epsilon_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & e^{2\epsilon_2} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & e^{-2\epsilon_2} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & e^{\epsilon_2} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

$$A_3 = \begin{pmatrix}
1 & -\epsilon_3 & \epsilon_3^2 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & -2\epsilon_3 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & -\epsilon_3 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

$$A_4 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & \epsilon_4 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & \epsilon_4 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

$$A_5 = \begin{pmatrix}
1 & 0 & 0 & -\epsilon_5 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & -\epsilon_5 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

$$A_6 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \epsilon_6 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & -\epsilon_6^2 & -2\epsilon_6 & 1
\end{pmatrix}.$$

$$A_7 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-\epsilon_7} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{\epsilon_7} \end{pmatrix}, \quad A_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2\epsilon_8 & -\epsilon_8^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\epsilon_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{23}$$

Then the general adjoint transformation matrix A is selected as

$$A = A_6 A_7 A_8 A_4 A_5 A_1 A_2 A_3$$
$$= \begin{pmatrix} e^{-2\epsilon_2} & -\epsilon_3 e^{-2\epsilon_2} & \epsilon_3^2 e^{-2\epsilon_2} & -\epsilon_5 e^{-\epsilon_2} & \epsilon_3\epsilon_5 e^{-\epsilon_2} & 0 & 01 & 0 \\ 2\epsilon_1 e^{-2\epsilon_2} & 1-2\epsilon_1\epsilon_3 e^{-2\epsilon_2} & 2\epsilon_3\left(\epsilon_1\epsilon_3 e^{2-\epsilon_2}-1\right) & \left(\epsilon_4-\epsilon_1\epsilon_5\right)e^{-\epsilon_2} & \left(\epsilon_1\epsilon_5-\epsilon_4\right)\epsilon_3 e^{-\epsilon_2}-\epsilon_5 e^{\epsilon_2} & 0 & 0 & 0 \\ \epsilon_1^2 e^{-2\epsilon_2} & \epsilon_1\left(1-\epsilon_1\epsilon_3 e^{-2\epsilon_2}\right) & \left(e^{\epsilon_2}-\epsilon_1\epsilon_3 e^{-\epsilon_2}\right)^2 & \epsilon_1\epsilon_4 e^{-\epsilon_2} & \epsilon_4\left(e^{\epsilon_2}-\epsilon_1\epsilon_3 e^{-\epsilon_2}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-\epsilon_2} & -\epsilon_3 e^{-\epsilon_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_1 e^{-\epsilon_2} & e^{\epsilon_2}-\epsilon_1\epsilon_3 e^{-\epsilon_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-\epsilon_7} & 2\epsilon_8 e^{-\epsilon_7} & -\epsilon_8^2 e^{-\epsilon_7} \\ 0 & 0 & 0 & 0 & 0 & \epsilon_6 e^{-\epsilon_7} & 1+2\epsilon_6\epsilon_8 e^{-\epsilon_7} & -\epsilon_8\left(1+\epsilon_6\epsilon_8 e^{-\epsilon_7}\right) \\ 0 & 0 & 0 & 0 & 0 & -\epsilon_6^2 e^{-\epsilon_7} & -2\epsilon_6\left(1+\epsilon_6\epsilon_8 e^{-\epsilon_7}\right) & \left(\epsilon_6\epsilon_8 e^{-(1/2)\epsilon_7}+e^{(1/2)\epsilon_7}\right)^2 \end{pmatrix}. \tag{24}$$

That is to say, after the adjoint action of

$$Ad_{\exp(\epsilon_6 K_6)} Ad_{\exp(\epsilon_7 K_7)} Ad_{\exp(\epsilon_8 K_8)} Ad_{\exp(\epsilon_4 K_4)} Ad_{\exp(\epsilon_5 K_5)} Ad_{\exp(\epsilon_1 K_1)} Ad_{\exp(\epsilon_2 K_2)} Ad_{\exp(\epsilon_3 K_3)}, \tag{25}$$

$\Upsilon = \sum_{i=1}^{8} a_i K_i$ is transformed into $\widetilde{\Upsilon} = \sum_{i=1}^{8} \widetilde{a}_i K_i$, shown by

$$\begin{aligned} &\left(\widetilde{a}_1, \widetilde{a}_2, \widetilde{a}_3, \widetilde{a}_4, \widetilde{a}_5, \widetilde{a}_6, \widetilde{a}_7, \widetilde{a}_8\right) \\ &\quad = \left(a_1, a_2, a_3, \mathrm{a}_4, a_5, a_6, a_7, a_8\right) A. \end{aligned} \tag{26}$$

Now, by virtue of (26) and all the invariants, one can start to construct the optimal system of $\mathscr{G}$ step by step. Since the degrees of Δ_1 and Δ_2 are both two, we just need to consider the following five cases:

$$\begin{aligned} &\left\{\Delta_1 = 1, \Delta_2 = c\right\}; \\ &\left\{\Delta_1 = -1, \Delta_2 = c\right\}; \\ &\left\{\Delta_1 = 0, \Delta_2 = 1\right\}; \\ &\left\{\Delta_1 = 0, \Delta_2 = -1\right\}; \\ &\left\{\Delta_1 = 0, \Delta_2 = 0\right\}. \end{aligned} \tag{27}$$

For each case, select a corresponding representative element in the simplest form named $\widetilde{\Upsilon} = \sum_{i=1}^{n} \widetilde{a}_i K_i$ and solve (26). If (26) has the solution with respect to $\epsilon_1, \cdots \epsilon_8$, it signifies that the selected representative element is right. If the chosen representative element makes (26) be incompatible, we need to adopt a new proper one. Repeat the process until all the cases in (27) are finished. Omitting the tedious but regular operational processes, the one-dimensional optimal system of eight-dimensional Lie algebra (15) is found to be

$$\begin{aligned} \omega_1 &= K_1 - cK_3 + K_6 + \frac{1}{4}K_8, \\ \omega_2 &= K_1 - cK_3 + K_6 - \frac{1}{4}K_8, \\ \omega_3 &= K_1 - cK_3 - K_6 + \frac{1}{4}K_8, \\ & \qquad\qquad (c \in \mathbb{R}) \\ \omega_4 &= K_1 + K_5 + K_6 + \frac{1}{4}K_8, \end{aligned}$$

Table 1: Commutator table of $\mathcal{G}$.

	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8
K_1	0	$-2K_1$	$-K_2$	0	$-K_4$	0	0	0
K_2	$2K_1$	0	$-2K_3$	K_4	$-K_5$	0	0	0
K_3	K_2	$2K_3$	0	K_5	0	0	0	0
K_4	0	$-K_4$	$-K_5$	0	0	0	0	0
K_5	K_4	K_5	0	0	0	0	0	0
K_6	0	0	0	0	0	0	$-K_6$	$2K_7$
K_7	0	0	0	0	0	K_6	0	$-K_8$
K_8	0	0	0	0	0	$-2K_7$	K_8	0

Table 2: Adjoint representation table of $\mathcal{G}$.

Ad	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8
K_1	K_1	$K_2+2\epsilon K_1$	$K_3+\epsilon K_2+\epsilon^2 K_1$	K_4	$K_5+\epsilon K_4$	K_6	K_7	K_8
K_2	$e^{-2\epsilon}K_1$	K_2	$e^{2\epsilon}K_3$	$e^{-\epsilon}K_4$	$e^{\epsilon}K_5$	K_6	K_7	K_8
K_3	$K_1-\epsilon K_2+\epsilon^2 K_3$	$K_2-2\epsilon K_3$	K_3	$K_4-\epsilon K_5$	K_5	K_6	K_7	K_8
K_4	K_1	$K_2+\epsilon K_4$	$K_3+\epsilon K_5$	K_4	K_5	K_6	K_7	K_8
K_5	$K_1-\epsilon K_4$	$K_2-\epsilon K_5$	K_3	K_4	K_5	K_6	K_7	K_8
K_6	K_1	K_2	K_3	K_4	K_5	K_6	$K_7+\epsilon K_6$	$K_8-2\epsilon K_7-\epsilon^2 K_6$
K_7	K_1	K_2	K_3	K_4	K_5	$e^{-\epsilon}K_6$	K_7	$e^{\epsilon}K_8$
K_8	K_1	K_2	K_3	K_4	K_5	$K_6+2\epsilon K_7-\epsilon^2 K_8$	$K_7-\epsilon K_8$	K_8

$$
\begin{aligned}
\omega_5 &= K_1 - K_5 + K_6 + \frac{1}{4}K_8,\\
\omega_6 &= K_1 + K_5 + K_6 - \frac{1}{4}K_8,\\
\omega_7 &= K_1 - K_5 + K_6 - \frac{1}{4}K_8,\\
\omega_8 &= K_1 + K_5 - K_6 + \frac{1}{4}K_8,\\
\omega_9 &= K_1 - K_5 - K_6 + \frac{1}{4}K_8,\\
\omega_{10} &= K_4 + K_6 + \frac{1}{4}K_8,\\
\omega_{11} &= K_4 + K_6 - \frac{1}{4}K_8,\\
\omega_{12} &= K_1 + K_3 + K_6,\\
\omega_{13} &= K_1 + K_3 - K_6,\\
\omega_{14} &= K_1 - K_3 + K_6,\\
\omega_{15} &= K_1 + K_5 + K_6,\\
\omega_{16} &= K_1 + K_5 - K_6,\\
\omega_{17} &= K_1 - K_5 + K_6,\\
\omega_{18} &= K_1 - K_5 - K_6,\\
\omega_{19} &= K_1 + K_6,\\
\omega_{20} &= K_1 - K_6,\\
\omega_{21} &= K_4 + K_6,\\
\omega_{22} &= K_1 + K_3,\\
\omega_{23} &= K_1 - K_3,\\
\omega_{24} &= K_1 + K_5,\\
\omega_{25} &= K_1 - K_5,\\
\omega_{26} &= K_1,\\
\omega_{27} &= K_4,\\
\omega_{28} &= K_6,\\
\omega_{29} &= K_6 + \frac{1}{4}K_8,\\
\omega_{30} &= K_6 - \frac{1}{4}K_8.
\end{aligned}
\tag{28}
$$

3. Group Invariant Solutions

For any given subgroup, an original nonlinear system can be reduced to a system with fewer independent variables, which may be easily solved to provide group invariant solutions. As Olver said, since there is almost always an infinite amount of such subgroups, it is usually not feasible to list all possible group invariant solutions to the system. In this section, some types of inequivalent group invariant solutions which correspond to the elements in the optimal system (28) are listed.

(1) For ω_1 with $c = 0$, we consider an equivalent case $K_6 + K_8 + K_3 + \alpha K_5$, where α is an arbitrary constant. The group invariant solution is

$$\begin{aligned}
\phi &= \sqrt{2}\tanh\left(\frac{\sqrt{2}}{2}t + \frac{\sqrt{2}}{2}P\right),\\
u &= -\sqrt{2}G_1\tanh\left(\frac{\sqrt{2}}{2}t + \frac{\sqrt{2}}{2}P\right) + U,\\
v &= -G_1^2\tanh^2\left(\frac{\sqrt{2}}{2}t + \frac{\sqrt{2}}{2}P\right)\\
&\quad + \left(2G_1^2 + \sqrt{2}G_2\right)\tanh\left(\frac{\sqrt{2}}{2}t + \frac{\sqrt{2}}{2}P\right) + V,\\
g_1 &= -\frac{2G_1}{1 + \cosh\left(\sqrt{2}t + \sqrt{2}P\right)},\\
g_2 &= \frac{2G_2 + 4\sqrt{2}G_1^2\left(1 + e^{\sqrt{2}(t+P)}\right)^{-1}}{1 + \cosh\left(\sqrt{2}t + \sqrt{2}P\right)}.
\end{aligned} \tag{29}$$

Here U, V, P, G_1, and G_2 are all group invariant functions of $X(= x - \alpha t)$. Substituting the group invariant solutions (29) into the prolonged system leads to the symmetry reduction equations for U, V, P, G_1, and G_2:

$$\begin{aligned}
P' &= -G_1,\\
G_2 &= -G_1' - \sqrt{2}G_1^2,\\
U &= -\frac{G_1' - \alpha G_1 - 1}{G_1},\\
V &= -\frac{1}{2}U^2 + \alpha U + c_0
\end{aligned} \tag{30}$$

with

$$\begin{aligned}
G_1'' &= \frac{3}{2}\frac{G_1'^2}{G_1} - \frac{2G_1'}{G_1} + G_1^3 - \alpha_0 G_1 + \frac{1}{2G_1},\\
\alpha_0 &= 1 + \frac{1}{2}\alpha^2 + c_0.
\end{aligned} \tag{31}$$

Here, and in the latter of the paper, the primes $', ''$ on the functions with only one independent variable denote derivatives. Thus by the transformation

$$\begin{aligned}
G_1 &= \frac{R}{\sqrt[4]{2}R'},\\
R &\equiv R\left(\sqrt[4]{2}X\right),
\end{aligned} \tag{32}$$

(31) becomes

$$R''' = \frac{R''^2}{2R'} - \frac{R^2}{2R'} + \frac{\sqrt{2}}{2}\alpha_0 R'. \tag{33}$$

For different values of α_0, (33) is exactly solvable. For example, if we take $\alpha_0 = 3/2$, this equation has a solution

$$R = c_1 e^{\sqrt{2}X} + c_2 e^X + c_3 e^{-X} + \frac{c_2 c_3}{2c_1}e^{\sqrt{2}X}. \tag{34}$$

Then, by solving (30), one can directly obtain an exact solution of the WBK system (1)-(2):

$$\begin{aligned}
u = &-\frac{2\left(2c_2c_3e^X + c_1^2e^{-X} + c_1c_2e^{\sqrt{2}X} + c_1c_3e^{-\sqrt{2}X}\right)}{2\sqrt{2}c_2c_3e^X - \sqrt{2}c_1^2e^{-X} + 2c_1c_2e^{\sqrt{2}X} - 2c_1c_3e^{-\sqrt{2}X}}\tanh\left(\frac{\sqrt{2}}{2}t + \frac{1}{2}\ln\frac{c_1e^{\sqrt{2}X} + \sqrt{2}c_3e^X}{c_1 - \sqrt{2}c_2e^{(1+\sqrt{2})X}} + c_4\right)\\
&+ \frac{2\sqrt{2}c_2c_3(1+\alpha)e^X - \sqrt{2}c_1^2(\alpha - 1)e^{-X} + 2c_1c_2\left(\alpha + \sqrt{2}\right)e^{\sqrt{2}X} - 2c_1c_3\left(\sqrt{2} - \alpha\right)e^{-\sqrt{2}X}}{2\sqrt{2}c_2c_3e^X - \sqrt{2}c_1^2e^{-X} + 2c_1c_2e^{\sqrt{2}X} - 2c_1c_3e^{-\sqrt{2}X}}, \quad v = u_x - 1.
\end{aligned} \tag{35}$$

(2) One equivalent case of ω_5 is $K_6 + K_8 + K_3 - K_4$ and the group invariant solution in this case is

$$\begin{aligned}
\phi &= \sqrt{2}\tanh\left(\frac{\sqrt{2}}{2}t + \frac{\sqrt{2}}{2}P\right),\\
u &= -\sqrt{2}G_1\tanh\left(\frac{\sqrt{2}}{2}t + \frac{\sqrt{2}}{2}P\right) + U + t + P,\\
v &= -G_1^2\tanh^2\left(\frac{\sqrt{2}}{2}t + \frac{\sqrt{2}}{2}P\right)\\
&\quad + \left(2G_1^2 + \sqrt{2}G_2\right)\tanh\left(\frac{\sqrt{2}}{2}t + \frac{\sqrt{2}}{2}P\right) + V,\\
g_1 &= -\frac{2G_1}{1 + \cosh\left(\sqrt{2}t + \sqrt{2}P\right)},\\
g_2 &= \frac{2G_2 + 4\sqrt{2}G_1^2 + 2G_2e^{\sqrt{2}(t+P)}}{\left(1 + e^{\sqrt{2}(t+P)}\right)\left(1 + \cosh\left(\sqrt{2}t + \sqrt{2}P\right)\right)}.
\end{aligned} \tag{36}$$

Here U, V, P, G_1, and G_2 are all group invariant functions of $Z(= x - t^2/2)$, which should satisfy

$$\begin{aligned} P' &= -G_1, \\ G_2 &= -G_1' - \sqrt{2}G_1^2, \\ U &= \frac{1 - G_1'}{G_1} - P, \\ V &= U_x + G_1^2 - G_1 - 1. \end{aligned} \tag{37}$$

And

$$G_1'' = \frac{3}{2}\frac{G_1'^2}{G_1} - \frac{2G_1'}{G_1} + G_1^3 + (X - c_0)G_1 + \frac{1}{2G_1}. \tag{38}$$

Thus by means of the transformation

$$\begin{aligned} G_1 &= \frac{W}{\sqrt[4]{2}W'}, \\ W &\equiv W(Y), \\ Y &= \sqrt[4]{2}X, \end{aligned} \tag{39}$$

we can transform (38) into

$$W''' = \frac{W''^2}{2W'} - \frac{W^2}{2W'} + \frac{1}{2}\left(\sqrt{2}c_0 - \sqrt[4]{2}Y\right)W'. \tag{40}$$

On differentiation, (40) becomes linear and of the fourth order:

$$W'''' = -\frac{\sqrt[4]{2}}{2}W' + \left(\sqrt{2}c_0 - \sqrt[4]{2}Y\right)W'' - W. \tag{41}$$

Thus, the general solution of (38) is a rational function of the constants of integration [32].

Remark 1. It is known that the solitary wave is a very important phenomenon in the nature and physics. In fact, these solitary waves must interact with other waves or spread on some nonconstant background. Here, the forms of solutions (29) and (36) directly embody this kind of interactions between the soliton and other nonlinear waves.

(3) For ω_{10} and ω_{11}, it is not difficult to obtain the following group invariant solutions:

$$\begin{aligned} u &= \pm\left[\frac{\sqrt{2}}{2t}\tanh\left(\frac{\sqrt{2}}{4}\frac{x + c_1 t + c_2}{t}\right) + \frac{x + c_2}{t}\right], \\ v &= u_x - 1. \end{aligned} \tag{42}$$

(4) Instead of studying ω_{15}, consider an equivalent case: $v_3 - v_4 + v_6$. The corresponding group invariant solution of the WBK system reads

$$\begin{aligned} u &= -t + U(X) - \frac{G_1(X)}{t + 2P(X)}, \\ v &= V(X) - \frac{2 - U(X)G_1(X)}{t + 2P(X)} - \frac{G_1(X)^2}{2(t + 2P(X))^2}, \\ X &= x + \frac{1}{2}t^2, \end{aligned} \tag{43}$$

where

$$\begin{aligned} U(X) &= \frac{\sqrt[3]{4}\left(c_1\mathrm{Ai}(1, X_1) + \mathrm{Bi}(1, X_1)\right)}{c_1\mathrm{Ai}(X_1) + \mathrm{Bi}(X_1)}, \\ V(X) &= X - 1 + \sqrt[3]{2}\left(\frac{c_1\mathrm{Ai}(1, X_1) + \mathrm{Bi}(1, X_1)}{c_1\mathrm{Ai}(X_1) + \mathrm{Bi}(X_1)}\right)^2, \\ G_1(X) &= 2X - 2\sqrt[3]{2}\frac{\left(c_1\mathrm{Ai}(1, X_1) + \mathrm{Bi}(1, X_1)\right)^2 + c_2}{\left(c_1\mathrm{Ai}(X_1) + \mathrm{Bi}(X_1)\right)^2}, \\ P(X) &= -\frac{1}{4}X^2 \\ &\quad + \frac{\sqrt[3]{2}}{2}\int_{X_0}^{X}\frac{\left(c_1\mathrm{Ai}(1, X_1) + \mathrm{Bi}(1, X_1)\right)^2 + c_2}{\left(c_1\mathrm{Ai}(X_1) + \mathrm{Bi}(X_1)\right)^2}dX, \\ X_1 &= \frac{X}{\sqrt[3]{2}}. \end{aligned} \tag{44}$$

Here, $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are the Airy wave functions, which are linearly independent solutions for $w'' - xw = 0$.

(5) For ω_{19}, it is equivalent to $K_3 + \alpha K_5 + K_6$ with α being an arbitrary constant. The group invariant solution is presented as

$$\begin{aligned} u &= U(X) - \frac{G_1(X)}{t + 2P(X)}, \\ v &= V(X) - \frac{G_2(X)}{t + 2P(X)} - \frac{G_1(X)^2}{2(t + 2P(X))^2}, \quad X = x - \alpha t \end{aligned} \tag{45}$$

with

$$\begin{aligned} U(X) &= \alpha + 2c_1\tanh(c_1X + c_2), \\ V(X) &= 2c_1^2\mathrm{sech}^2(c_1X + c_2) - 1, \\ P(X) &= -\frac{X}{4c_1} + \frac{2c_1X - c_3}{4c_1^2\left(1 + e^{2c_1X + 2c_2}\right)} + c_4, \\ G_1(X) &= \frac{2c_1X + \sinh(2c_1X + 2c_2) - c_3}{c_1\left(1 + \cosh(2c_1X + 2c_2)\right)}, \\ G_2(X) &= \frac{4 + 4\cosh(2c_1X + 2c_2) + 2(c_3 - 2c_1X)\sinh(2c_1X + 2c_2)}{\left(1 + \cosh(2c_1X + 2c_2)\right)^2}. \end{aligned} \tag{46}$$

(6) By solving ω_{21}, one can easily give a simple exact solution:

$$\begin{aligned} u &= \frac{2}{x - c_1t + 2c_2}, \\ v &= -\frac{2}{(x - c_1t + 2c_2)^2} - 1. \end{aligned} \tag{47}$$

4. Summary

In the beginning of [31], the authors also discussed the nonlocal symmetry (6) of the WBK system. To realize the localization of the nonlocal symmetry, they referred to six

dependent variables. In fact, we point that five dependent variables are enough in this paper. Based on a Bäcklund transformation, [31] constructed some special exact solutions, which have no direct connection with the given nonlocal symmetry. The purpose of our article is to give a classification of the eight-dimensional Lie algebra (15) and then to find out some inequivalent group invariant solutions with respect to the nonlocal symmetry (6).

In this paper, we firstly localize the nonlocal symmetries of the (1+1)-dimensional WBK system (1)-(2) to Lie point symmetries by prolonging the original system to a larger one. Then, all the Lie point symmetries of the whole prolonged system are presented, which constitute an eight-dimensional Lie algebra (15). The one-dimensional optimal system (28) is constructed to give a classification of the elements in the Lie algebra. The adjoint transformation matrix and all the invariants of the Lie algebra are also displayed. Finally, we demonstrate six kinds of inequivalent group invariant solutions, some of which reflect the interactions between soliton and other nonlinear waves.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

[1] S. Y. Lou and X. B. Hu, "Non-local symmetries via Darboux transformations," *Journal of Physics A: Mathematical and Theoretical*, vol. 30, no. 5, pp. L95–L100, 1997.

[2] X.-R. Hu, S.-Y. Lou, and Y. Chen, "Explicit solutions from eigenfunction symmetry of the korteweg-de vries equation," *Physical Review E*, vol. 85, no. 5, Article ID 056607, 8 pages, 2012.

[3] X.-P. Cheng, S. Y. Lou, C.-L. Chen, and X.-Y. Tang, "Interactions between solitons and other nonlinear Schrödinger waves," *Physical Review E*, vol. 89, Article ID 043202, 2014.

[4] J. Chen, X. Xin, and Y. Chena, "Nonlocal symmetries of the Hirota-Satsuma coupled Korteweg-de Vries and their applications: Exact interaction solutions and integrable hierarchy system," *Journal of Mathematical Physics*, vol. 55, no. 5, Article ID 053508, 2014.

[5] Y. Li, J. Chen, Y. Chen, and S. Lou, "Darboux transformations via lie point symmetries: kdv equation," *Chinese Physics Letters*, vol. 31, no. 1, Article ID 010201, 2014.

[6] S. Y. Lou, X. Hu, and Y. Chen, "Nonlocal symmetries related to Bäcklund transformation and their applications," *Journal of Physics A: Mathematical and Theoretical*, vol. 45, no. 15, Article ID 155209, 2012.

[7] X. Hu and C. Yong, "Nonlocal symmetries and negative hierarchies related to bilinear Bäcklund transformation," *Chinese Physics B*, vol. 24, no. 3, Article ID 030201, 2015.

[8] R. Conte, "Invariant painlevé analysis of partial differential equations," *Physics Letters A*, vol. 140, pp. 383–390, 1989.

[9] A. Pickering, "A new truncation in Painlevé analysis," *Journal of Physics A: Mathematical and General*, vol. 26, pp. 4395–4405, 1993.

[10] S. Y. Lou, "Residual symmetries and Bäcklund transformations," 2013, https://arxiv.org/abs/1308.1140.

[11] L. V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic, New York, NY, USA, 1982.

[12] J. Patera, P. Winternitz, and H. Zassenhaus, "Continuous subgroups of the fundamental groups of physics. I. General method and the Poincaré group," *Journal of Mathematical Physics*, vol. 16, pp. 1597–1615, 1975.

[13] J. Patera, R. T. Sharp, P. Winternitz, and H. Zassenhaus, "Invariants of real low dimension Lie algebras," *Journal of Mathematical Physics*, vol. 17, no. 6, pp. 986–994, 1976.

[14] F. Galas and E. W. Richter, "Exact similarity solutions of ideal MHD equations for plane motions," *Physica D: Nonlinear Phenomena*, vol. 50, no. 2, pp. 297–307, 1991.

[15] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer, New York, NY, USA, 2nd edition, 1993.

[16] N. H. Ibragimov, *Transformation Groups Applied to Mathematical Physics*, Reidel, Dordrecht, Netherlands, 1985.

[17] N. H. Ibragimov, *Lie Group Analysis of Differential Equations*, CRC Press, Boca Raton, Fla, USA, 1994.

[18] G. W. Bluman and S. Kumei, *Symmetries and Differential Equations*, Springer-Verlag, New York, NY, USA, 1989.

[19] G. Bluman, A. Cheviakov, and S. Anco, *Applications of Symmetry Methods to Partial Differential Equations*, Springer-Verlag, New York, NY, USA, 2010.

[20] K.-S. Chou, G.-X. Li, and C. Qu, "A note on optimal systems for the heat equation," *Journal of Mathematical Analysis and Applications*, vol. 261, no. 2, pp. 741–751, 2001.

[21] K.-S. Chou and C. Qu, "Optimal systems and group classification of (1+2)-dimensional heat equation," *Acta Applicandae Mathematicae*, vol. 83, no. 3, pp. 257–287, 2004.

[22] K.-S. Chou and G.-X. Li, "Optimal systems and invariant solutions for the curve shortening problem," *Communications in Analysis and Geometry*, vol. 10, no. 2, pp. 241–274, 2002.

[23] X. Hu, Y. Li, and Y. Chen, "A direct algorithm of one-dimensional optimal system for the group invariant solutions," *Journal of Mathematical Physics*, vol. 56, Article ID 053504, 2015.

[24] G. B. Whitham, "Variational methods and applications to water waves," *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, vol. 299, pp. 6–25, 1967.

[25] L. J. Broer, "Approximate equations for long water waves," *Applied Scientific Research*, vol. 31, no. 5, pp. 377–395, 1975.

[26] D. J. Kaup, "A higher-order water-wave equation and the method for solving it," *Progress of Theoretical and Experimental Physics*, vol. 54, no. 2, pp. 396–408, 1975.

[27] V. B. Matveev and M. I. Yavor, "Solutions presque périodiques eta N-solitons de l'équation hydrodynamique non tinéaire de Kaup," *Annales de l'Institut Henri Poincaré. Section A*, no. 1, pp. 25–41, 2009.

[28] C.-l. Chen and S.-y. Lou, "Soliton excitations and periodic waves without dispersion relation in shallow water system," *Chaos, Solitons and Fractals*, vol. 16, no. 1, pp. 27–35, 2003.

[29] A. Wazwaz, "Multiple soliton solutions for the Whitham-Broer-Kaup model in shallow water small-amplitude regime," *Physica Scripta*, vol. 88, Article ID 035007, 2013.

[30] Z. Zhang, X. Yong, and Y. Chen, "Symmetry analysis for whitham-Broer-Kaup equations," *Journal of Nonlinear Mathematical Physics*, vol. 15, no. 4, pp. 383–397, 2008.

[31] W. Zhou and B. Lu, "Nonlocal symmetry, CTE solvability and interaction solutions of Whitham-Broer Kaup equations," *Communications in Theoretical Physics*, vol. 67, no. 2, pp. 143–146, 2017.

[32] B. Gambier, "Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est a points critiques fixes," *Acta Mathematica*, vol. 33, no. 1, pp. 1–55, 1910.