Research Article

Upper Semicontinuity of Random Attractors for Nonautonomous Stochastic Reversible Selkov System with Multiplicative Noise

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In this paper, the existence of random attractors for nonautonomous stochastic reversible Selkov system with multiplicative noise has been proved through Ornstein-Uhlenbeck transformation. Furthermore, the upper semicontinuity of random attractors is discussed when the intensity of noise approaches zero. The main difficulty is to prove the asymptotic compactness for establishing the existence of tempered pullback random attractor.

1. Introduction

Selkov equations [1] were a system of ODEs originally proposed by Selkov [2, 3] as a simplified model of a biochemical process called glycolysis, through which living cells get energy from breaking down sugar. The existence and robustness of global attractors of the reversible Selkov system have been investigated in the deterministic case (*i.e.*, $\sigma = 0$) by You in [4]. Li proved the existence of random attractor of the stochastic reversible Selkov system on infinite lattice with additive noise in [5]. In this paper, we consider the upper semicontinuity of random attractors for nonautonomous stochastic reversible Selkov system with multiplicative noise

$$du = (d_1 \Delta u + \rho - au + u^2 v - Gu^3 + f_1(t, x)) dt + \sigma u \circ dW(t), \quad (1)$$

$$dv = (d_2 \Delta v + \beta - bv - u^2 v + Gu^3 + f_2(t, x)) dt + \sigma v \circ dW(t), \quad (2)$$

where $\Gamma \subset \mathbb{R}^n$ ($n \leq 3$) is a bounded smooth domain. There are boundary conditions

$$u(t, x) = v(t, x) = 0, \quad x \in \partial\Gamma, \quad (3)$$

and initial conditions

$$\begin{aligned} u(\tau, x) &= u_\tau(x), \\ v(\tau, x) &= v_\tau(x). \end{aligned} \quad (4)$$

All the coefficients $d_1, d_2, \rho, \beta, a, b$, and G are arbitrarily given positive constants; f_i ($i = 1, 2$) denote the time-dependent external forces. $\{W(t)\}_{t \in \mathbb{R}}$ is a one-dimensional two-sided standard Wiener process on a probability space. The terms $\sigma u \circ dW(t)$, $\sigma v \circ dW(t)$ indicate that (1)-(2) are interpreted in the sense of the Stratonovich integration.

In the past decades, great progress has been made in various aspects of random dynamical systems; see [6–22]. The long-time behavior of solutions for nonautonomous three-component reversible Gray-Scott system was discussed in [23] and the existence and upper semicontinuity of random attractor for stochastic three-component reversible Gray-Scott system was proved in [24]. In [25], Tu and You introduced the pullback asymptotic compactness of stochastic Brusselator system with multiplicative noise. Notice that (1)

and (2) are nonautonomous stochastic equations with time-dependent external terms f_1 and f_2 . For such a nonautonomous stochastic system, Wang established an efficacious theory about the existence and upper semicontinuity of the random attractor by introducing two parametric spaces [26–29].

In this paper, we first obtain the existence of random attractors for the stochastic perturbed reversible Selkov system (1)–(4) defined on $\Gamma \subset \mathbf{R}^n (n \leq 3)$. When $\sigma \rightarrow 0$, we will consider the limit behaviors of random attractors and prove the upper semicontinuity of these perturbed random attractors. As usual, we must prove the pullback asymptotic compactness of the solution operators in $L^2(\Gamma)$. The main difficulty for proving such compactness lines is the uniform estimates of the solutions in $[L^6(\Gamma)]^2$ when we need to obtain some estimates in $[H^1(\Gamma)]^2$. We obtain the important $L^6(\Gamma)$ -norm estimates by using the Mean Value Theorem.

The outline of the paper is as follows. In Section 2, some basic concepts related to the nonautonomous random dynamical system and upper semicontinuity of random attractors are introduced. Section 3 is devoted to the pullback asymptotic compactness and the existence of random attractors. In Section 4, we establish the upper semicontinuity of random attractors when the coefficient σ approaches zero.

2. Preliminaries

In this section, we use some concepts of nonautonomous random dynamical system and random attractor. Let $(X, \|\cdot\|_X)$ be a real separable Banach space with Borel σ -algebra $\mathcal{B}(X)$ and $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbf{R}})$ be an ergodic metric dynamical system. Some concepts and definitions of continuous random dynamical system have been seen in [6, 8, 17, 30] for more details. The following theorem is used to obtain the existence of random attractors for the continuous random dynamical system.

Theorem 1 (see [6, 8]). *Let φ be a continuous random dynamical system on X over $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbf{R}})$. If there exists a closed random tempered absorbing set $\{B(\tau, \omega) : \tau \in \mathbf{R}, \omega \in \Omega\}$ of φ and φ is asymptotically compact in X , then $\mathcal{A}(\tau, \omega)$ is a random attractor of φ , where*

$$\mathcal{A}(\tau, \omega) = \bigcap_{r \geq 0} \bigcup_{t \geq r} \overline{\varphi(t, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega))}, \quad (5)$$

$$\tau \in \mathbf{R}, \omega \in \Omega.$$

Moreover, $\mathcal{A}(\tau, \omega)$ is the random attractor of φ .

Now, we define the product Hilbert spaces

$$H = [L^2(\Gamma)]^2, \quad (6)$$

$$E = [H_0^1(\Gamma)]^2.$$

The norm and inner-product in H or $L^2(\Gamma)$ will be denoted by $\|\cdot\|$ and $(\cdot, \cdot)$, respectively. The norm in $L^p(\Gamma)$ or $[L^p(\Gamma)]^2$ will be denoted by $\|\cdot\|_{L^p}$, if $p \neq 2$. The norm in E will be denoted by $\|\cdot\|_E$.

Let $\{W(t)\}_{t \in \mathbf{R}}$ be the standard one-dimensional, two-sided Wiener process in the complete probability space $(\Omega, \mathcal{F}, P)$, where

$$\Omega = \{\omega \in C(\mathbf{R}, \mathbf{R}) : \omega(0) = 0\}, \quad (7)$$

the σ -algebra $\mathcal{F}$ is generated by the compact-open topology on Ω , and P is the corresponding Wiener measure on $\mathcal{F}$; see [6, 8]. The Wiener shift θ_t is defined by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, t \in \mathbf{R}. \quad (8)$$

Consider the Ornstein-Uhlenbeck process

$$\begin{aligned} z(\theta_t \omega) &:= - \int_{-\infty}^0 e^s (\theta_t \omega)(s) ds \\ &= - \int_{-\infty}^0 e^s \omega(t+s) ds + \omega(t), \end{aligned} \quad (9)$$

which solves the Itô equation

$$dz + zdt = dW(t). \quad (10)$$

Proposition 2 (see [6, 8]). *Let the metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbf{R}})$ and the Ornstein-Uhlenbeck process $\{z(\theta_t \omega)\}_{t \in \mathbf{R}}$ be defined as above. There is a θ_t -invariant set $\tilde{\Omega} \in \Omega$ of full P -measure such that for every $\omega \in \tilde{\Omega}$, the following statements hold:*

(1) *The Ornstein-Uhlenbeck process $\{z(\theta_t \omega)\}_{t \in \mathbf{R}}$ has the asymptotically sublinear growth property; i.e.,*

$$\lim_{t \rightarrow \pm\infty} \frac{|z(\theta_t \omega)|}{|t|} = 0. \quad (11)$$

(2) *$z(\theta_t \omega)$ is continuous in t and, for any fixed $t_0 \in \mathbf{R}$,*

$$\lim_{t \rightarrow \pm\infty} \frac{1}{t - t_0} \int_{t_0}^t z(\theta_s \omega) ds = 0. \quad (12)$$

In the sequel, we consider $\omega \in \tilde{\Omega}$ only and will always write Ω for $\tilde{\Omega}$.

Proposition 3 ((upper semicontinuity of random attractors) (see [8, 28])). *Suppose the following conditions are satisfied:*

(1) *Given $\sigma > 0$, we suppose that φ_σ is a random dynamical system over a metric system $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbf{R}})$. For P -a.e. $\tau \in \mathbf{R}$, $\omega \in \Omega$, $t \geq 0$, $\sigma_n \rightarrow 0$, and $x_n, x \in X$ with $x_n \rightarrow x$, there holds*

$$\lim_{n \rightarrow \pm\infty} \varphi_\sigma(t, \tau, \omega, x_n) = \varphi(t, \tau) x. \quad (13)$$

(2) *Assume that φ_σ has a random attractor $\mathcal{A}_\sigma(\tau, \omega) = \{\mathcal{A}_\sigma(\tau, \omega) : \tau \in \mathbf{R}, \omega \in \Omega\} \in \mathcal{D}$ and a random absorbing set $K_\sigma(\tau, \omega) = \{K_\sigma(\tau, \omega) : \tau \in \mathbf{R}, \omega \in \Omega\} \in \mathcal{D}$ such that for some deterministic positive constant c and for P -a.e. $\tau \in \mathbf{R}, \omega \in \Omega$*

$$\limsup_{\sigma \rightarrow 0} \|K_\sigma(\tau, \omega)\|_X \leq c, \quad (14)$$

where $\|K_\sigma(\tau, \omega)\|_X = \sup_{x \in K_\sigma(\tau, \omega)} \|x\|_X$.

(3) There exists a $\sigma_0 > 0$ such that for P-a.e. $\tau \in \mathbf{R}, \omega \in \Omega$

$$\bigcup_{0 < \sigma \leq \sigma_0} \mathcal{A}_\sigma(\tau, \omega) \text{ is precompact in } X. \quad (15)$$

Then, for P-a.e. $\tau \in \mathbf{R}, \omega \in \Omega$,

$$\text{dist}(\mathcal{A}_\sigma(\tau, \omega), \mathcal{A}_0(\tau, \omega)) \longrightarrow 0, \text{ as } \sigma \longrightarrow 0. \quad (16)$$

To investigate the random dynamics of stochastic PDEs, we convert the stochastic evolutionary system to a system of pathwise PDEs with the random parameter. Let

$$\begin{aligned} U(t, \tau, \omega, U_\tau) &= e^{-\sigma z(\theta_t \omega)} u(t, \tau, \omega, U_\tau), \\ V(t, \tau, \omega, V_\tau) &= e^{-\sigma z(\theta_t \omega)} v(t, \tau, \omega, V_\tau), \end{aligned} \quad (17)$$

where $z(\theta_t \omega)$ is the Ornstein-Uhlenbeck process. Then

$$\begin{aligned} dU &= e^{-\sigma z(\theta_t \omega)} du - \sigma e^{-\sigma z(\theta_t \omega)} u \circ dz(\theta_t \omega), \\ dV &= e^{-\sigma z(\theta_t \omega)} dv - \sigma e^{-\sigma z(\theta_t \omega)} v \circ dz(\theta_t \omega). \end{aligned} \quad (18)$$

So we can consider the following equations with random coefficients, but without white noise:

$$\begin{aligned} \frac{dU}{dt} &= d_1 \Delta U + e^{-\sigma z(\theta_t \omega)} \rho - aU + e^{2\sigma z(\theta_t \omega)} U^2 V \\ &\quad - Ge^{2\sigma z(\theta_t \omega)} U^3 + e^{-\sigma z(\theta_t \omega)} f_1(t, x) \\ &\quad + \sigma U z(\theta_t \omega), \end{aligned} \quad (19)$$

$$\begin{aligned} \frac{dV}{dt} &= d_2 \Delta V + e^{-\sigma z(\theta_t \omega)} \beta - bV - e^{2\sigma z(\theta_t \omega)} U^2 V \\ &\quad + Ge^{2\sigma z(\theta_t \omega)} U^3 + e^{-\sigma z(\theta_t \omega)} f_2(t, x) \\ &\quad + \sigma V z(\theta_t \omega), \end{aligned} \quad (20)$$

for $x \in \Gamma$ and $t \geq \tau$, with the homogeneous Dirichlet boundary conditions

$$U(t, x) = V(t, x) = 0, \quad t \geq \tau \in \mathbf{R}, \quad x \in \partial\Gamma, \quad (21)$$

and the initial value conditions

$$\begin{aligned} U(\tau, x) &= U_\tau(x) = e^{-\sigma z(\theta_\tau \omega)} u_\tau(x), \\ V(\tau, x) &= V_\tau(x) = e^{-\sigma z(\theta_\tau \omega)} v_\tau(x). \end{aligned} \quad (22)$$

For every $\omega \in \Omega, \tau \in \mathbf{R}$, the initial-boundary value problem (19)–(22) is formulated into the following reversible Selkov equation:

$$\frac{dg}{dt} = Ag + J(g) + f(t, x), \quad (23)$$

$$g(\tau, x) = g_\tau(x) = (U_\tau(x), V_\tau(x))^\top, \quad (24)$$

where $g(t, \tau, \omega, g_\tau) = (U(t, \tau, \omega, U_\tau), V(t, \tau, \omega, V_\tau))^\top$, and

$$A = \begin{pmatrix} d_1 \Delta & 0 \\ 0 & d_2 \Delta \end{pmatrix}, \quad (25)$$

and

$$J(g, \theta_t \omega) = \begin{pmatrix} e^{-\sigma z(\theta_t \omega)} \rho - aU + e^{2\sigma z(\theta_t \omega)} U^2 V - Ge^{2\sigma z(\theta_t \omega)} U^3 + e^{-\sigma z(\theta_t \omega)} f_1(t, x) + \sigma U z(\theta_t \omega) \\ e^{-\sigma z(\theta_t \omega)} \beta - bV - e^{2\sigma z(\theta_t \omega)} U^2 V + Ge^{2\sigma z(\theta_t \omega)} U^3 + e^{-\sigma z(\theta_t \omega)} f_2(t, x) + \sigma V z(\theta_t \omega) \end{pmatrix}, \quad (26)$$

for any $t \geq \tau$ with initial data

$$\begin{aligned} g_\tau(x) &= (U_\tau(x), V_\tau(x))^\top \\ &= (e^{-\sigma z(\theta_\tau \omega)} u_\tau, e^{-\sigma z(\theta_\tau \omega)} v_\tau)^\top. \end{aligned} \quad (27)$$

By the Galerkin method, we can prove the existence and uniqueness of the weak solution, which is continuously depending on the initial data and

$$g(t, \tau, \omega, g_\tau) \in C([\tau, +\infty); H) \cap L^2([\tau, +\infty); E). \quad (28)$$

This is similar to the autonomous case studied in [4]. Define the continuous random dynamical system φ associated with the problem (23)–(24) as follows:

$$\begin{aligned} \varphi : \mathbf{R}^+ \times \mathbf{R} \times \Omega \times H &\longrightarrow H, \\ \varphi(t, \tau, \omega, g_\tau) &= g(t + \tau, \tau, \theta_{-t} \omega, g_\tau) \\ &= (U(t + \tau, \tau, \theta_{-t} \omega, U_\tau), V(t + \tau, \tau, \theta_{-t} \omega, V_\tau)) \end{aligned} \quad (29)$$

for all $(t, \tau, \omega, g_\tau) \in \mathbf{R}^+ \times \mathbf{R} \times \Omega \times H$. The continuous dynamical system φ and a continuous random dynamical system associated with the solutions of (1)–(4) are equivalent. Thus, we only discuss the dynamical system φ induced by (19)–(22).

Assume that

$$\begin{aligned} X &= \left\{ f \text{ is measurable: } \int_{-\infty}^0 e^{(\gamma/2)s} (\|f(s + \tau, \cdot)\|_{L^p}^p ds \right. \\ &\quad \left. < +\infty, p = 2, 4, 6 \right\}. \end{aligned} \quad (30)$$

Sometimes, we also assume that f_1 and f_2 are tempered in the following sense. For every $c > 0$,

$$\begin{aligned} \lim_{\lambda \rightarrow -\infty} e^{c\lambda} \int_{-\infty}^0 e^{(\gamma/2)s} (\|f_1(s - \lambda, \cdot)\|_{L^p}^p \\ + \|f_2(s - \lambda, \cdot)\|_{L^p}^p) ds = 0, \quad p = 2, 4, 6. \end{aligned} \quad (31)$$

3. Existence of Random Attractors

In this section, we derive uniform estimates of solution for the stochastic reversible Selkov equations; these estimates are necessary for proving the existence of bounded absorbing sets and the asymptotic compactness of the random dynamical system. We will use $\mathcal{D}$ to denote the collection of all tempered random subsets of H from now on.

For brevity, we write $U(t, \tau, \omega, U_\tau), V(t, \tau, \omega, V_\tau)$ as $U(t, \omega), V(t, \omega)$ or simply as U, V ; similarly, we write weak solution $g(t, \tau, \omega, g_\tau)$ as $g(t, \omega)$ or g .

Lemma 4. Suppose that $f_i(t, x) \in X, i = 1, 2$. Then, for every $\tau \in \mathbf{R}, \omega \in \Omega$, and $B = \{B(\tau, \omega) : \tau \in \mathbf{R}, \omega \in \Omega\} \in \mathcal{D}$, there exist a number $T_0(\tau, \omega, B, \sigma) \geq 0$ and a tempered random variable $M_0(\tau, \omega)$

$$\begin{aligned} M_0(\tau, \omega) &= 1 + C_1 \int_{-\infty}^0 e^{-2\sigma z(\theta_s, \omega) - \int_0^s 2\sigma z(\theta_r, \omega) dr + \gamma s} ds \\ &\quad + \frac{2G}{a} \int_{-\infty}^0 e^{-2\sigma z(\theta_s, \omega) - \int_0^s 2\sigma z(\theta_r, \omega) dr + \gamma s} \|f_1(s + \tau)\|^2 ds \\ &\quad + \frac{2}{b} \int_{-\infty}^0 e^{-2\sigma z(\theta_s, \omega) - \int_0^s 2\sigma z(\theta_r, \omega) dr + \gamma s} \|f_2(s + \tau)\|^2 ds, \end{aligned} \quad (32)$$

such that for all $t \geq T_0$, the solution g of (19)–(22) with $g_{\tau-t} \in B(\tau - t, \theta_{-\tau}\omega)$ satisfies

$$\|g(\tau, \tau - t, \theta_{-\tau}\omega, g_{\tau-t})\| \leq M_0(\tau, \omega), \quad \forall t \geq T_0(\tau, \omega, B, \sigma). \quad (33)$$

Proof. Multiplying (19) by GU and (20) by V , we get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} G \|U\|^2 + G d_1 \|\nabla U\|^2 \\ &= e^{-\sigma z(\theta_t, \omega)} \int_{\Gamma} GU \rho \, dx - aG \|U\|^2 \\ &\quad + e^{2\sigma z(\theta_t, \omega)} \int_{\Gamma} GU^3 V \, dx - e^{2\sigma z(\theta_t, \omega)} \int_{\Gamma} G^2 U^4 \, dx \\ &\quad + \int_{\Gamma} e^{-\sigma z(\theta_t, \omega)} f_1(t, x) GU \, dx \\ &\quad + \sigma z(\theta_t, \omega) G \|U\|^2, \end{aligned} \quad (34)$$

and

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|V\|^2 + d_2 \|\nabla V\|^2 \\ &= e^{-\sigma z(\theta_t, \omega)} \int_{\Gamma} V \beta \, dx - b \|V\|^2 \\ &\quad - e^{2\sigma z(\theta_t, \omega)} \int_{\Gamma} U^2 V^2 \, dx + e^{2\sigma z(\theta_t, \omega)} \int_{\Gamma} GU^3 V \, dx \\ &\quad + \int_{\Gamma} e^{-\sigma z(\theta_t, \omega)} f_2(t, x) V \, dx + \sigma z(\theta_t, \omega) \|V\|^2. \end{aligned} \quad (35)$$

Adding up the above two equations, we obtain

$$\begin{aligned} &\frac{d}{dt} (G \|U\|^2 + \|V\|^2) + 2G d_1 \|\nabla U\|^2 + 2d_2 \|\nabla V\|^2 \\ &\quad + 2(aG \|U\|^2 + b \|V\|^2) \\ &\quad - 2\sigma z(\theta_t, \omega) (G \|U\|^2 + \|V\|^2) \\ &= 2e^{-\sigma z(\theta_t, \omega)} \int_{\Gamma} (GU f_1(t, x) + V f_2(t, x)) \, dx \\ &\quad + 2e^{-\sigma z(\theta_t, \omega)} \int_{\Gamma} (GU \rho + V \beta) \, dx \\ &\quad - 2e^{2\sigma z(\theta_t, \omega)} \int_{\Gamma} (GU^2 - UV)^2 \, dx. \end{aligned} \quad (36)$$

By the ϵ -Young inequality, we have

$$\begin{aligned} &e^{-\sigma z(\theta_t, \omega)} \int_{\Gamma} (GU \rho + V \beta) \, dx \\ &\leq \frac{aG}{4} \|U\|^2 + \frac{b}{4} \|V\|^2 + \frac{e^{-2\sigma z(\theta_t, \omega)} \rho^2 |\Gamma| G}{a} \\ &\quad + \frac{e^{-2\sigma z(\theta_t, \omega)} \beta^2 |\Gamma|}{b}, \end{aligned} \quad (37)$$

$$\begin{aligned} &e^{-\sigma z(\theta_t, \omega)} \int_{\Gamma} (GU f_1(t, x) + V f_2(t, x)) \, dx \\ &\leq \frac{aG}{4} \|U\|^2 + \frac{b}{4} \|V\|^2 + \frac{e^{-2\sigma z(\theta_t, \omega)} G}{a} \|f_1\|^2 \\ &\quad + \frac{e^{-2\sigma z(\theta_t, \omega)}}{b} \|f_2\|^2. \end{aligned} \quad (38)$$

It follows that

$$\begin{aligned} &\frac{d}{dt} (G \|U\|^2 + \|V\|^2) + 2G d_1 \|\nabla U\|^2 + 2d_2 \|\nabla V\|^2 \\ &\quad + 2(aG \|U\|^2 + b \|V\|^2) \\ &\quad - 2\sigma z(\theta_t, \omega) (G \|U\|^2 + \|V\|^2) \\ &\leq aG \|U\|^2 + b \|V\|^2 \\ &\quad + e^{-2\sigma z(\theta_t, \omega)} \left(\frac{2\rho^2 |\Gamma| G}{a} + \frac{2\beta^2 |\Gamma|}{b} \right) \\ &\quad + e^{-2\sigma z(\theta_t, \omega)} \left(\frac{2G}{a} \|f_1\|^2 + \frac{2}{b} \|f_2\|^2 \right). \end{aligned} \quad (39)$$

Let $\gamma = \min\{a, b\}$, $C_1 = 2\rho^2 |\Gamma| G/a + 2\beta^2 |\Gamma|/b$. It is easy to see that

$$\begin{aligned} &\frac{d}{dt} (G \|U\|^2 + \|V\|^2) \\ &\quad + (\gamma - 2\sigma z(\theta_t, \omega)) (G \|U\|^2 + \|V\|^2) \end{aligned}$$

$$\begin{aligned}
& + 2Gd_1 \|\nabla U\|^2 + 2d_2 \|\nabla V\|^2 \\
& \leq C_1 e^{-2\sigma z(\theta_t, \omega)} + e^{-2\sigma z(\theta_t, \omega)} \left(\frac{2G}{a} \|f_1\|^2 + \frac{2}{b} \|f_2\|^2 \right). \quad (40)
\end{aligned}$$

By Gronwall's lemma on the interval $[\tau - t, r](r \geq \tau - t)$ for above inequality and substituting ω by $\theta_{-\tau}\omega$, we can deduce

$$\begin{aligned}
& G \|U(r, \tau - t, \theta_{-\tau}\omega, U_{\tau-t})\|^2 + \|V(r, \tau - t, \theta_{-\tau}(\omega), \\
& V_{\tau-t})\|^2 \leq e^{\int_{-\tau}^{r-\tau} 2\sigma z(\theta_s, \omega) ds - \gamma(r+t-\tau)} \left(G \|U_{\tau-t}\|^2 \right. \\
& \left. + \|V_{\tau-t}\|^2 \right) \\
& + C_1 \int_{-\tau}^{r-\tau} e^{-2\sigma z(\theta_s, \omega) + \int_s^{r-\tau} 2\sigma z(\theta_r, \omega) dr + \gamma(s+\tau-r)} ds + \frac{2G}{a} \\
& \cdot \int_{-\tau}^{r-\tau} e^{-2\sigma z(\theta_s, \omega) + \int_s^{r-\tau} 2\sigma z(\theta_r, \omega) dr + \gamma(s+\tau-r)} \|f_1(s \\
& + \tau)\|^2 ds + \frac{2}{b} \\
& \cdot \int_{-\tau}^{r-\tau} e^{-2\sigma z(\theta_s, \omega) + \int_s^{r-\tau} 2\sigma z(\theta_r, \omega) dr + \gamma(s+\tau-r)} \|f_2(s \\
& + \tau)\|^2 ds. \quad (41)
\end{aligned}$$

Note that

$$\begin{aligned}
& \frac{2G}{a} \int_{-\tau}^{r-\tau} e^{-2\sigma z(\theta_s, \omega) + \int_s^{r-\tau} 2\sigma z(\theta_r, \omega) dr + \gamma(s+\tau-r)} \|f_1(s + \tau)\|^2 ds \\
& = \frac{2G}{a} e^{\gamma(\tau-r)} e^{-\int_{r-t}^0 2\sigma z(\theta_r, \omega) dr} \int_{-\tau}^{r-\tau} e^{\gamma s} e^{-\int_0^s 2\sigma z(\theta_r, \omega) dr - 2\sigma z(\theta_s, \omega)} \|f_1(s + \tau)\|^2 ds. \quad (42)
\end{aligned}$$

By (11) and (12), we find that there exists a $T_1 = T_1(\tau, \omega, B, \sigma_0)$ ($0 < \sigma < \sigma_0$) such that for $\forall t \geq T_1 > 0$, we have

$$|z(\theta_{-t}\omega)| \leq \frac{\gamma t}{8\sigma_0}, \quad (43)$$

$$\left| \int_0^{-t} z(\theta_r \omega) dr \right| \leq \frac{\gamma t}{8\sigma_0}. \quad (44)$$

We get for all $t \geq T_1$,

$$\begin{aligned}
& \int_{-t}^{-T_1} e^{\gamma s} e^{-\int_0^s 2\sigma z(\theta_r, \omega) dr - 2\sigma z(\theta_s, \omega)} \|f_1(s + \tau)\|^2 ds \\
& \leq \int_{-t}^{-T_1} e^{(\gamma/2)s} \|f_1(s + \tau)\|^2 ds. \quad (45)
\end{aligned}$$

For the term on the right-hand side of (45), since $f_1, f_2 \in X$, we have

$$\int_{-t}^{-T_1} e^{(\gamma/2)s} (\|f_1(s + \tau)\|^2 + \|f_2(s + \tau)\|^2) ds < +\infty. \quad (46)$$

Thus, we get

$$\begin{aligned}
& \frac{2G}{a} \int_{-t}^{r-\tau} e^{-2\sigma z(\theta_s, \omega) - \int_{r-\tau}^s 2\sigma z(\theta_r, \omega) dr + \gamma(s+\tau-r)} \|f_1(s \\
& + \tau)\|^2 ds \leq \frac{2G}{a} \\
& \cdot \int_{-\infty}^{r-\tau} e^{-2\sigma z(\theta_s, \omega) - \int_{r-\tau}^s 2\sigma z(\theta_r, \omega) dr + \gamma(s+\tau-r)} \|f_1(s \\
& + \tau)\|^2 ds, \quad (47)
\end{aligned}$$

where the integral is convergent due to (46). Similarly, we also obtain

$$\begin{aligned}
& \frac{2}{b} \int_{-t}^{r-\tau} e^{-2\sigma z(\theta_s, \omega) - \int_{r-\tau}^s 2\sigma z(\theta_r, \omega) dr + \gamma(s+\tau-r)} \|f_2(s + \tau)\|^2 ds \\
& \leq \frac{2}{b} \int_{-\infty}^{r-\tau} e^{-2\sigma z(\theta_s, \omega) - \int_{r-\tau}^s 2\sigma z(\theta_r, \omega) dr + \gamma(s+\tau-r)} \|f_2(s \\
& + \tau)\|^2 ds. \quad (48)
\end{aligned}$$

Thus, we yield

$$\begin{aligned}
& G \|U(r, \tau - t, \theta_{-\tau}\omega, U_{\tau-t})\|^2 \\
& + \|V(r, \tau - t, \theta_{-\tau}(\omega), V_{\tau-t})\|^2 \\
& \leq e^{\int_{-t}^{r-\tau} 2\sigma z(\theta_s, \omega) ds - \gamma(r+t-\tau)} \left(G \|U_{\tau-t}\|^2 + \|V_{\tau-t}\|^2 \right) \\
& + C_1 \int_{-\tau}^{r-\tau} e^{-2\sigma z(\theta_s, \omega) + \int_s^{r-\tau} 2\sigma z(\theta_r, \omega) dr + \gamma(s+\tau-r)} ds \\
& + \frac{2G}{a} \int_{-\infty}^{r-\tau} e^{-2\sigma z(\theta_s, \omega) + \int_s^{r-\tau} 2\sigma z(\theta_r, \omega) dr + \gamma(s+\tau-r)} \|f_1(s + \tau)\|^2 ds \\
& + \frac{2}{b} \int_{-\infty}^{r-\tau} e^{-2\sigma z(\theta_s, \omega) + \int_s^{r-\tau} 2\sigma z(\theta_r, \omega) dr + \gamma(s+\tau-r)} \|f_2(s + \tau)\|^2 ds. \quad (49)
\end{aligned}$$

By (49), we get that

$$\begin{aligned}
& G \|U(\tau, \tau - t, \theta_{-\tau}\omega, U_{\tau-t})\|^2 \\
& + \|V(\tau, \tau - t, \theta_{-\tau}\omega, V_{\tau-t})\|^2 \\
& \leq e^{\int_{-t}^0 2\sigma z(\theta_s, \omega) ds - \gamma t} \left(G \|U_{\tau-t}\|^2 + \|V_{\tau-t}\|^2 \right) \\
& + C_1 \int_{-\infty}^0 e^{-2\sigma z(\theta_s, \omega) + \int_s^0 2\sigma z(\theta_r, \omega) dr + \gamma s} ds \\
& + \frac{2G}{a} \int_{-\infty}^0 e^{-2\sigma z(\theta_s, \omega) + \int_s^0 2\sigma z(\theta_r, \omega) dr + \gamma s} \|f_1(s + \tau)\|^2 ds \\
& + \frac{2}{b} \int_{-\infty}^0 e^{-2\sigma z(\theta_s, \omega) + \int_s^0 2\sigma z(\theta_r, \omega) dr + \gamma s} \|f_2(s + \tau)\|^2 ds. \quad (50)
\end{aligned}$$

According to the properties of the Ornstein-Uhlenbeck process,

$$\int_{-\infty}^0 e^{-2\sigma z(\theta_s, \omega) + \int_s^0 2\sigma z(\theta_r, \omega) dr + \gamma s} ds < +\infty. \quad (51)$$

Noticing that $\{B(\tau, \omega) : \tau \in \mathbf{R}, \omega \in \Omega\} \in \mathcal{D}$ is tempered, then for any $(U_{\tau-t}, V_{\tau-t}) \in B(\tau-t, \theta_{-t}\omega)$, we have

$$\lim_{t \rightarrow +\infty} e^{\int_{-t}^0 (2\sigma z(\theta_s \omega) - \gamma) ds} (G \|U_{\tau-t}\|^2 + \|V_{\tau-t}\|^2) = 0. \quad (52)$$

Then, $M_0(\tau, \omega)$ is tempered. By (49), the proof is completed. $\square$

Lemma 5. Suppose that $f_i(t, x) \in X$, $i = 1, 2$. Then, for every $\tau \in \mathbf{R}$, $\omega \in \Omega$, and $B = \{B(\tau, \omega) : \tau \in \mathbf{R}, \omega \in \Omega\} \in \mathcal{D}$, there exist a number $T_2(\tau, \omega, B, \sigma) \geq 0$ and tempered random variables $M_1(\tau, \omega)$, $M_2(\tau, \omega)$

$$\begin{aligned} M_1(\tau, \omega) &= 1 \\ &+ \frac{C_1}{2} \\ &\cdot e^{\int_{-1}^0 |2\sigma z(\theta_s \omega)| ds + \gamma} \int_{-\infty}^0 e^{-2\sigma z(\theta_s \omega) + \int_s^0 2\sigma z(\theta_r \omega) dr + \gamma s} ds \\ &+ \frac{G}{a} e^{\int_{-1}^0 |2\sigma z(\theta_s \omega)| ds + \gamma} \int_{-\infty}^0 e^{-2\sigma z(\theta_s \omega) + \int_s^0 2\sigma z(\theta_r \omega) dr + \gamma s} \|f_1(s + \tau)\|^2 ds \\ &+ \frac{1}{b} e^{\int_{-1}^0 |2\sigma z(\theta_s \omega)| ds + \gamma} \int_{-\infty}^0 e^{-2\sigma z(\theta_s \omega) + \int_s^0 2\sigma z(\theta_r \omega) dr + \gamma s} \|f_2(s + \tau)\|^2 ds, \end{aligned} \quad (53)$$

$$\begin{aligned} M_2(\tau, \omega) &= 1 \\ &+ \frac{C_1}{2} \\ &\cdot e^{\int_{-2}^{-1} |2\sigma z(\theta_s \omega)| ds + (\gamma+1)} \int_{-\infty}^{-1} e^{-2\sigma z(\theta_s \omega) + \int_s^{-1} 2\sigma z(\theta_r \omega) dr + \gamma(s+1)} ds \\ &+ \frac{G}{a} e^{\int_{-2}^{-1} |2\sigma z(\theta_s \omega)| ds + (\gamma+1)} \int_{-\infty}^{-1} e^{-2\sigma z(\theta_s \omega) + \int_s^{-1} 2\sigma z(\theta_r \omega) dr + \gamma(s+1)} \|f_1(s + \tau)\|^2 ds \\ &+ \frac{1}{b} e^{\int_{-2}^{-1} |2\sigma z(\theta_s \omega)| ds + (\gamma+1)} \int_{-\infty}^{-1} e^{-2\sigma z(\theta_s \omega) + \int_s^{-1} 2\sigma z(\theta_r \omega) dr + \gamma(s+1)} \|f_2(s + \tau)\|^2 ds. \end{aligned}$$

such that for all $t \geq T_2$, the solution g of (19)–(22) with $g_{\tau-t} \in B(\tau-t, \theta_{-t}\omega)$ satisfies

$$\begin{aligned} &\int_{\tau-1}^{\tau} \|\nabla U(s, \tau-t, \theta_{-t}\omega, U_{\tau-t})\|^2 ds \\ &+ \int_{\tau-1}^{\tau} \|\nabla V(s, \tau-t, \theta_{-t}\omega, V_{\tau-t})\|^2 ds \leq M_1(\tau, \omega), \\ &\int_{\tau-2}^{\tau-1} \|\nabla U(s, \tau-t, \theta_{-t}\omega, U_{\tau-t})\|^2 ds \\ &+ \int_{\tau-2}^{\tau-1} \|\nabla V(s, \tau-t, \theta_{-t}\omega, V_{\tau-t})\|^2 ds \leq M_2(\tau, \omega), \end{aligned} \quad (54)$$

Proof. Substituting r for $\tau-1$ in (41), for $t \geq 1$, we yield

$$\begin{aligned} &G \|U(\tau-1, \tau-t, \theta_{-t}\omega, U_{\tau-t})\|^2 \\ &+ \|V(\tau-1, \tau-t, \theta_{-t}\omega, V_{\tau-t})\|^2 \\ &\leq e^{\int_{-t}^{-1} 2\sigma z(\theta_s \omega) ds - \gamma(t-1)} (G \|U_{\tau-t}\|^2 + \|V_{\tau-t}\|^2) \\ &+ C_1 \int_{-t}^{-1} e^{-2\sigma z(\theta_s \omega) + \int_s^{-1} 2\sigma z(\theta_r \omega) dr + \gamma(1+s)} ds + \frac{2G}{a} \\ &\cdot \int_{-t}^{-1} e^{-2\sigma z(\theta_s \omega) + \int_s^{-1} 2\sigma z(\theta_r \omega) dr + \gamma(1+s)} \|f_1(s + \tau)\|^2 ds \\ &+ \frac{2}{b} \\ &\cdot \int_{-t}^{-1} e^{-2\sigma z(\theta_s \omega) + \int_s^{-1} 2\sigma z(\theta_r \omega) dr + \gamma(1+s)} \|f_2(s + \tau)\|^2 ds. \end{aligned} \quad (55)$$

Applying Gronwall's lemma to (40) over $[\tau-1, \tau]$ and by (55), we obtain

$$\begin{aligned} &d_1 G \int_{\tau-1}^{\tau} e^{\int_s^{\tau} 2\sigma z(\theta_{s-\tau}\omega) ds - \gamma(\tau-s)} \|\nabla U(s, \tau \\ &- t, \theta_{-t}\omega, U_{\tau-t})\|^2 ds \\ &+ d_2 \int_{\tau-1}^{\tau} e^{\int_s^{\tau} 2\sigma z(\theta_{s-\tau}\omega) ds - \gamma(\tau-s)} \|\nabla V(s, \tau \\ &- t, \theta_{-t}\omega, V_{\tau-t})\|^2 ds \leq \frac{1}{2} \\ &\cdot e^{\int_{-t}^0 2\sigma z(\theta_s \omega) ds - \gamma t} (G \|U_{\tau-t}(\theta_{-t}\omega)\|^2 \\ &+ \|V_{\tau-t}(\theta_{-t}\omega)\|^2) + \frac{C_1}{2} \\ &\cdot \int_{-\infty}^0 e^{-2\sigma z(\theta_s \omega) + \int_s^0 2\sigma z(\theta_r \omega) dr + \gamma s} ds + \frac{G}{a} \\ &\cdot \int_{-\infty}^0 e^{-2\sigma z(\theta_s \omega) + \int_s^0 2\sigma z(\theta_r \omega) dr + \gamma s} \|f_1(s + \tau)\|^2 ds \\ &+ \frac{1}{b} \int_{-\infty}^0 e^{-2\sigma z(\theta_s \omega) + \int_s^0 2\sigma z(\theta_r \omega) dr + \gamma s} \|f_2(s + \tau)\|^2 ds. \end{aligned} \quad (56)$$

It is easy to show that

$$\begin{aligned} &\int_{\tau-1}^{\tau} e^{\int_s^{\tau} 2\sigma z(\theta_{s-\tau}\omega) ds - \gamma(\tau-s)} \|\nabla U(s, \tau \\ &- t, \theta_{-t}\omega, U_{\tau-t})\|^2 ds \\ &\geq e^{-\int_{-1}^0 |2\sigma z(\theta_s \omega)| ds - \gamma} \int_{\tau-1}^{\tau} \|\nabla U(s, \tau \\ &- t, \theta_{-t}\omega, U_{\tau-t})\|^2 ds, \end{aligned} \quad (57)$$

and

$$\begin{aligned}
& \int_{\tau-1}^{\tau} e^{\int_s^{\tau} 2\sigma z(\theta_s, \omega) ds - \gamma(\tau-s)} \|\nabla V(s, \tau \\
& - t, \theta_{-\tau}\omega, V_{\tau-t})\|^2 ds \\
& \geq e^{-\int_{\tau-1}^0 |2\sigma z(\theta_s, \omega)| ds - \gamma} \int_{\tau-1}^{\tau} \|\nabla V(s, \tau \\
& - t, \theta_{-\tau}\omega, V_{\tau-t})\|^2 ds.
\end{aligned} \tag{58}$$

Now let $d = \min\{d_1, d_2, d_1 G\}$. According to (56), (57), and (58), there exists a $T_3(\tau, \omega, B, \sigma)$ such that for all $t \geq T_3(\tau, \omega, B, \sigma)$, we can get

$$\begin{aligned}
& \int_{\tau-1}^{\tau} \|\nabla U(s, \tau - t, \theta_{-\tau}\omega, U_{\tau-t})\|^2 ds \\
& + \int_{\tau-1}^{\tau} \|\nabla V(s, \tau \\
& - t, \theta_{-\tau}\omega, V_{\tau-t})\|^2 ds \\
& \leq \frac{1}{d} e^{\int_{\tau-1}^0 |2\sigma z(\theta_s, \omega)| ds + \gamma} \left[\int_{\tau-1}^{\tau} e^{\int_s^{\tau} 2\sigma z(\theta_s, \omega) ds - \gamma(\tau-s)} \|\nabla U(s, \tau \\
& - t, \theta_{-\tau}\omega, U_{\tau-t})\|^2 ds + \int_{\tau-1}^{\tau} e^{\int_s^{\tau} 2\sigma z(\theta_s, \omega) ds - \gamma(\tau-s)} \|\nabla V(s, \tau \\
& - t, \theta_{-\tau}\omega, V_{\tau-t})\|^2 ds \right] \leq 1 \\
& + \frac{C_1}{2} \\
& \cdot e^{\int_{\tau-1}^0 |2\sigma z(\theta_s, \omega)| ds + \gamma} \int_{-\infty}^0 e^{-2\sigma z(\theta_s, \omega) + \int_s^0 2\sigma z(\theta_s, \omega) dr + \gamma s} ds \\
& + \frac{G}{a} e^{\int_{\tau-1}^0 |2\sigma z(\theta_s, \omega)| ds + \gamma} \int_{-\infty}^0 e^{-2\sigma z(\theta_s, \omega) + \int_s^0 2\sigma z(\theta_s, \omega) dr + \gamma s} \|f_1(s \\
& + \tau)\|^2 ds \\
& + \frac{1}{b} \\
& \cdot e^{\int_{\tau-1}^0 |2\sigma z(\theta_s, \omega)| ds + \gamma} \int_{-\infty}^0 e^{-2\sigma z(\theta_s, \omega) + \int_s^0 2\sigma z(\theta_s, \omega) dr + \gamma s} \|f_2(s + \tau)\|^2 ds \doteq M_1(\tau, \omega).
\end{aligned} \tag{59}$$

Similarly, substituting r for $\tau - 2$ in (41), applying Gronwall's lemma to (40) over $[\tau - 2, \tau - 1]$, and combining the estimate (49), there exists a $T_4(\tau, \omega, B, \sigma)$ such that for all $t \geq T_4(\tau, \omega, B, \sigma)$, we get

$$\begin{aligned}
& \int_{\tau-2}^{\tau-1} \|\nabla U(s, \tau - t, \theta_{-\tau}\omega, U_{\tau-t})\|^2 ds \\
& + \int_{\tau-2}^{\tau-1} \|\nabla V(s, \\
& \tau - t, \theta_{-\tau}\omega, V_{\tau-t})\|^2 ds \leq 1 \\
& + \frac{C_1}{2} \\
& \cdot e^{\int_{\tau-2}^{-1} |2\sigma z(\theta_s, \omega)| ds + (\gamma+1)} \int_{-\infty}^{-1} e^{-2\sigma z(\theta_s, \omega) + \int_s^{-1} 2\sigma z(\theta_s, \omega) dr + \gamma(s+1)} ds \\
& + \frac{G}{a} e^{\int_{\tau-2}^{-1} |2\sigma z(\theta_s, \omega)| ds + (\gamma+1)} \int_{-\infty}^{-1} e^{-2\sigma z(\theta_s, \omega) + \int_s^{-1} 2\sigma z(\theta_s, \omega) dr + \gamma(s+1)} \|f_1(s \\
& + \tau)\|^2 ds
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{b} e^{\int_{\tau-2}^{-1} |2\sigma z(\theta_s, \omega)| ds + (\gamma+1)} \int_{-\infty}^{-1} e^{-2\sigma z(\theta_s, \omega) + \int_s^{-1} 2\sigma z(\theta_s, \omega) dr + \gamma(s+1)} \|f_2(s \\
& + \tau)\|^2 ds \doteq M_2(\tau, \omega).
\end{aligned} \tag{60}$$

It is easy to check that $M_1(\tau, \omega)$ and $M_2(\tau, \omega)$ are tempered. Let $T_2 = \max\{T_3, T_4\}$. This completes the proof. $\square$

Lemma 6. Let $f_i(t, x) \in X, i = 1, 2, q \in [\tau - 1, \tau]$. If there exists a $t_0 \in (\tau - t, q)$ such that $g(t_0, \tau - t, \theta_{-\tau}\omega, g_{\tau-t}) \in E$ with $\|g(t_0, \tau - t, \theta_{-\tau}\omega, g_{\tau-t})\|_E \leq R(\tau, \omega)$, where $R(\tau, \omega) > 0$ is any given random variable. Then for every $\tau \in \mathbf{R}, \omega \in \Omega$, and $B(\tau, \omega) \in \mathcal{D}$, there exist a number $T_5(\tau, \omega, B, \sigma) \geq 0$ and two tempered random variables $M_3(\tau, \omega)$ and $M_4(\tau, \omega)$

$$\begin{aligned}
M_3(\tau, \omega) &= 1 + C_2 \int_{-\infty}^{q-\tau} e^{-4\sigma z(\theta_s, \omega) + \int_s^{q-\tau} 4\sigma z(\theta_s, \omega) ds + \gamma(s-q+\tau)} ds \\
&+ \frac{8}{a^3} \int_{-\infty}^{q-\tau} e^{-4\sigma z(\theta_s, \omega) + \int_s^{q-\tau} 4\sigma z(\theta_s, \omega) ds + \gamma(s-q+\tau)} \|f_1(s + \tau)\|_{L^4}^4 ds \\
&+ \frac{8}{b^3 G^3} \int_{-\infty}^{q-\tau} e^{-4\sigma z(\theta_s, \omega) + \int_s^{q-\tau} 4\sigma z(\theta_s, \omega) ds + \gamma(s-q+\tau)} \|f_2(s + \tau)\|_{L^4}^4 ds,
\end{aligned} \tag{61}$$

and

$$\begin{aligned}
M_4(\tau, \omega) &= 1 \\
&+ C_3 \int_{-\infty}^{q-\tau} e^{-6\sigma z(\theta_s, \omega) + \int_s^{q-\tau} 6\sigma z(\theta_s, \omega) ds + \gamma(s-q+\tau)} ds + \frac{32}{a^5} \\
&\cdot \int_{-\infty}^{q-\tau} e^{-6\sigma z(\theta_s, \omega) + \int_s^{q-\tau} 6\sigma z(\theta_s, \omega) ds + \gamma(s-q+\tau)} \\
&\cdot \|f_1(s + \tau)\|_{L^6}^6 ds + \frac{32}{b^5 G^5} \\
&\cdot \int_{-\infty}^{q-\tau} e^{-6\sigma z(\theta_s, \omega) + \int_s^{q-\tau} 6\sigma z(\theta_s, \omega) ds + \gamma(s-q+\tau)} \\
&\cdot \|f_2(s + \tau)\|_{L^6}^6 ds,
\end{aligned} \tag{62}$$

such that for all $t \geq T_5$, the solution g of (19)–(22) with $g_{\tau-t} \in B(\tau - t, \theta_{-\tau}\omega)$ satisfies

$$\begin{aligned}
& \|g(q, \tau - t, \theta_{-\tau}\omega, g_{\tau-t})\|_{L^4}^4 \leq M_3(\tau, \omega), \\
& \|g(q, \tau - t, \theta_{-\tau}\omega, g_{\tau-t})\|_{L^6}^6 \leq M_4(\tau, \omega).
\end{aligned} \tag{63}$$

Proof. Here we invoked Sobolev's embedding theorem to assert that $H_0^1(\Gamma) \hookrightarrow L^6(\Gamma) \hookrightarrow L^4(\Gamma)$ is a continuous embedding for $n \leq 3$. For $g(t_0, \tau - t, \theta_{-\tau}\omega, g_{\tau-t}) \in E$ with $\|g(t_0, \tau - t, \theta_{-\tau}\omega, g_{\tau-t})\|_E \leq R(\tau, \omega)$, it follows that

$$\begin{aligned}
& \|g(t_0, \tau - t, \theta_{-\tau}\omega, g_{\tau-t})\|_{L^6} \leq \eta R(\tau, \omega), \\
& \|g(t_0, \tau - t, \theta_{-\tau}\omega, g_{\tau-t})\|_{L^4} \leq \lambda R(\tau, \omega),
\end{aligned} \tag{64}$$

where η and λ are positive constants.

Let $\tilde{U} = U$, $\tilde{V} = V/G$. Then, the transformed system (19)-(20) becomes a rescaled system

$$\begin{aligned} \frac{d\tilde{U}}{dt} &= d_1 \Delta \tilde{U} + e^{-\sigma z(\theta_s \omega)} \rho - a \tilde{U} + e^{2\sigma z(\theta_s \omega)} G \tilde{U}^2 \tilde{V} \\ &\quad - G e^{2\sigma z(\theta_s \omega)} \tilde{U}^3 + e^{-\sigma z(\theta_s \omega)} f_1(t, x) \\ &\quad + \sigma \tilde{U} z(\theta_s \omega), \end{aligned} \quad (65)$$

$$\begin{aligned} \frac{d\tilde{V}}{dt} &= d_2 \Delta \tilde{V} + \frac{e^{-\sigma z(\theta_s \omega)} \beta}{G} - b \tilde{V} - e^{2\sigma z(\theta_s \omega)} \tilde{U}^2 \tilde{V} \\ &\quad + e^{2\sigma z(\theta_s \omega)} \tilde{U}^3 + \frac{e^{-\sigma z(\theta_s \omega)} f_2(t, x)}{G} \\ &\quad + \sigma \tilde{V} z(\theta_s \omega). \end{aligned} \quad (66)$$

Multiplying (65) by $\tilde{U}^3$, we obtain

$$\begin{aligned} &\frac{1}{4} \frac{d}{dt} \|\tilde{U}\|_{L^4}^4 + 3d_1 \|\tilde{U} \nabla \tilde{U}\|^2 \\ &= \int_{\Gamma} e^{-\sigma z(\theta_s \omega)} (\rho \tilde{U}^3 + f_1 \tilde{U}^3) dx - a \|\tilde{U}^4\|_{L^4} \\ &\quad + \int_{\Gamma} e^{2\sigma z(\theta_s \omega)} (G \tilde{U}^5 \tilde{V} - G \tilde{U}^6) dx \\ &\quad + \sigma z(\theta_s \omega) \|\tilde{U}\|_{L^4}^4. \end{aligned} \quad (67)$$

Multiplying (66) by $G \tilde{V}^3$, we get

$$\begin{aligned} &\frac{1}{4} \frac{d}{dt} G \|\tilde{V}\|_{L^4}^4 + 3Gd_2 \|\tilde{V} \nabla \tilde{V}\|^2 \\ &= \int_{\Gamma} e^{-\sigma z(\theta_s \omega)} (\beta \tilde{V}^3 + f_2 \tilde{V}^3) dx - bG \|\tilde{V}^4\|_{L^4} \\ &\quad + \int_{\Gamma} e^{2\sigma z(\theta_s \omega)} (-G \tilde{U}^2 \tilde{V}^4 + G \tilde{U}^3 \tilde{V}^3) dx \\ &\quad + \sigma G z(\theta_s \omega) \|\tilde{V}\|_{L^4}^4. \end{aligned} \quad (68)$$

Summing up the resulting equalities, we have

$$\begin{aligned} &\frac{1}{4} \frac{d}{dt} (\|\tilde{U}\|_{L^4}^4 + G \|\tilde{V}\|_{L^4}^4) \\ &\quad + 3d (\|\tilde{U} \nabla \tilde{U}\|^2 + G \|\tilde{V} \nabla \tilde{V}\|^2) \\ &\leq e^{-\sigma z(\theta_s \omega)} \int_{\Gamma} (\rho \tilde{U}^3 + \beta \tilde{V}^3 + f_1 \tilde{U}^3 + f_2 \tilde{V}^3) dx \end{aligned}$$

$$\begin{aligned} &-a \|\tilde{U}^4\|_{L^4} - bG \|\tilde{V}^4\|_{L^4} \\ &\quad + e^{2\sigma z(\theta_s \omega)} G \int_{\Gamma} (-\tilde{U}^6 + \tilde{U}^5 \tilde{V} + \tilde{U}^3 \tilde{V}^3 - \tilde{U}^2 \tilde{V}^4) dx \\ &\quad + \sigma z(\theta_s \omega) (\|\tilde{U}\|_{L^4}^4 + G \|\tilde{V}\|_{L^4}^4) \\ &= e^{-\sigma z(\theta_s \omega)} \int_{\Gamma} (\rho \tilde{U}^3 + \beta \tilde{V}^3 + f_1 \tilde{U}^3 + f_2 \tilde{V}^3) dx \\ &\quad -a \|\tilde{U}^4\|_{L^4} - bG \|\tilde{V}^4\|_{L^4} \\ &\quad + e^{2\sigma z(\theta_s \omega)} G \int_{\Gamma} \tilde{U}^2 (-\tilde{U}^4 + \tilde{U}^3 \tilde{V} + \tilde{U} \tilde{V}^3 - \tilde{V}^4) dx \\ &\quad + \sigma z(\theta_s \omega) (\|\tilde{U}\|_{L^4}^4 + G \|\tilde{V}\|_{L^4}^4). \end{aligned} \quad (69)$$

Using the Young inequality, we have

$$\begin{aligned} &-\tilde{U}^4 + \tilde{U}^3 \tilde{V} + \tilde{U} \tilde{V}^3 - \tilde{V}^4 \\ &\leq -\tilde{U}^4 + \left(\frac{3}{4} \tilde{U}^4 + \frac{1}{4} \tilde{V}^4\right) + \left(\frac{1}{4} \tilde{U}^4 + \frac{3}{4} \tilde{V}^4\right) - \tilde{V}^4 \\ &\leq 0, \end{aligned} \quad (70)$$

and

$$\begin{aligned} &e^{-\sigma z(\theta_s \omega)} \int_{\Gamma} (\rho \tilde{U}^3 + \beta \tilde{V}^3 + f_1 \tilde{U}^3 + f_2 \tilde{V}^3) dx \\ &\leq e^{-\sigma z(\theta_s \omega)} \left(\rho |\Gamma|^{1/4} \|\tilde{U}\|_{L^4}^3 + \beta |\Gamma|^{1/4} \|\tilde{V}\|_{L^4}^3 \right) \\ &\quad + e^{-\sigma z(\theta_s \omega)} \left(\|f_1\|_{L^4} \|\tilde{U}\|_{L^4}^3 + \|f_2\|_{L^4} \|\tilde{V}\|_{L^4}^3 \right) \\ &\leq \frac{2}{a^3} e^{-4\sigma z(\theta_s \omega)} \rho^4 |\Gamma| + \frac{3a}{8} \|\tilde{U}\|_{L^4}^4 + \frac{3bG}{8} \|\tilde{V}\|_{L^4}^4 \\ &\quad + \frac{2}{b^3 G^3} e^{-4\sigma z(\theta_s \omega)} \beta^4 |\Gamma| + \frac{2}{a^3} e^{-4\sigma z(\theta_s \omega)} \|f_1\|_{L^4}^4 \\ &\quad + \frac{3a}{8} \|\tilde{U}\|_{L^4}^4 + \frac{2}{b^3 G^3} e^{-4\sigma z(\theta_s \omega)} \|f_2\|_{L^4}^4 \\ &\quad + \frac{3bG}{8} \|\tilde{V}\|_{L^4}^4 \\ &= \frac{3}{4} \left(a \|\tilde{U}\|_{L^4}^4 + bG \|\tilde{V}\|_{L^4}^4 \right) \\ &\quad + e^{-4\sigma z(\theta_s \omega)} \left(\frac{2\rho^4 |\Gamma|}{a^3} + \frac{2\beta^4 |\Gamma|}{b^3 G^3} \right) \\ &\quad + e^{-4\sigma z(\theta_s \omega)} \left(\frac{2\|f_1\|_{L^4}^4}{a^3} + \frac{2\|f_2\|_{L^4}^4}{b^3 G^3} \right). \end{aligned} \quad (71)$$

Recalling that $\gamma = \min\{a, b\}$, it follows that

$$\begin{aligned} & \frac{d}{dt} \left(\|\tilde{U}\|_{L^4}^4 + G \|\tilde{V}\|_{L^4}^4 \right) \\ & + (\gamma - 4\sigma z) \left(\|\tilde{U}\|_{L^4}^4 + G \|\tilde{V}\|_{L^4}^4 \right) \\ & \leq e^{-4\sigma z(\theta_s \omega)} C_2 \\ & + e^{-4\sigma z(\theta_s \omega)} \left(\frac{8 \|f_1\|_{L^4}^4}{a^3} + \frac{8 \|f_2\|_{L^4}^4}{b^3 G^3} \right), \end{aligned} \quad (72)$$

where $C_2 = 8\rho^4 |\Gamma|/a^3 + 8\beta^4 |\Gamma|/b^3 G^3$. By Gronwall's lemma on the interval $[t_0, q]$, $q \in [\tau - 1, \tau]$, for the above inequality, and replacing ω by $\theta_{-\tau}\omega$, we get

$$\begin{aligned} & \|\tilde{U}(q, \tau - t, \theta_{-\tau}\omega, \tilde{U}_{\tau-t})\|_{L^4}^4 \\ & + G \|\tilde{V}(q, \tau - t, \theta_{-\tau}\omega, \tilde{V}_{\tau-t})\|_{L^4}^4 \\ & \leq e^{\int_{t_0-\tau}^{q-\tau} 4\sigma z(\theta_s \omega) ds - \gamma(q-t_0)} \left(\|\tilde{U}(t_0, \tau - t, \theta_{-\tau}\omega, \tilde{U}_{\tau-t})\|_{L^4}^4 \right. \\ & \quad \left. + G \|\tilde{V}(t_0, \tau - t, \theta_{-\tau}\omega, \tilde{V}_{\tau-t})\|_{L^4}^4 \right) \\ & + C_2 \int_{-\infty}^{q-\tau} e^{-4\sigma z(\theta_s \omega) + \int_s^{q-\tau} 4\sigma z(\theta_s \omega) ds + \gamma(s-q+\tau)} ds \\ & + \frac{8}{a^3} \int_{-\infty}^{q-\tau} e^{-4\sigma z(\theta_s \omega) + \int_s^{q-\tau} 4\sigma z(\theta_s \omega) ds + \gamma(s-q+\tau)} \|f_1(s) + \tau\|_{L^4}^4 ds \\ & + \frac{8}{b^3 G^3} \int_{-\infty}^{q-\tau} e^{-4\sigma z(\theta_s \omega) + \int_s^{q-\tau} 4\sigma z(\theta_s \omega) ds + \gamma(s-q+\tau)} \|f_2(s) + \tau\|_{L^4}^4 ds. \end{aligned} \quad (73)$$

Recalling that $\tilde{U} = U$, $\tilde{V} = V/G$, we have

$$\begin{aligned} & \|U(q, \tau - t, \theta_{-\tau}\omega, U_{\tau-t})\|_{L^4}^4 \\ & + G^{-3} \|V(q, \tau - t, \theta_{-\tau}\omega, V_{\tau-t})\|_{L^4}^4 \leq 1 \\ & + C_2 \int_{-\infty}^{q-\tau} e^{-4\sigma z(\theta_s \omega) + \int_s^{q-\tau} 4\sigma z(\theta_s \omega) ds + \gamma(s-q+\tau)} ds \\ & + \frac{8}{a^3} \int_{-\infty}^{q-\tau} e^{-4\sigma z(\theta_s \omega) + \int_s^{q-\tau} 4\sigma z(\theta_s \omega) ds + \gamma(s-q+\tau)} \|f_1(s) + \tau\|_{L^4}^4 ds \end{aligned}$$

$$\begin{aligned} & + \frac{8}{b^3 G^3} \int_{-\infty}^{q-\tau} e^{-4\sigma z(\theta_s \omega) + \int_s^{q-\tau} 4\sigma z(\theta_s \omega) ds + \gamma(s-q+\tau)} \|f_2(s) + \tau\|_{L^4}^4 ds \doteq M_3(\tau, \omega). \end{aligned} \quad (74)$$

Similarly, multiplying (65) by $\tilde{U}^5$ and (66) by $G\tilde{V}^5$, we get

$$\begin{aligned} & \frac{d}{dt} \left(\|\tilde{U}\|_{L^6}^6 + G \|\tilde{V}\|_{L^6}^6 \right) \\ & + (\gamma - 6\sigma z(\theta_s \omega)) \left(\|\tilde{U}\|_{L^6}^6 + G \|\tilde{V}\|_{L^6}^6 \right) \\ & \leq e^{-6\sigma z(\theta_s \omega)} \left(\frac{32\rho^6 |\Gamma|}{a^5} + \frac{32\beta^6 |\Gamma|}{b^5 G^5} \right) \\ & + e^{-6\sigma z(\theta_s \omega)} \left(\frac{32 \|f_1\|_{L^6}^6}{a^5} + \frac{32 \|f_2\|_{L^6}^6}{b^5 G^5} \right). \end{aligned} \quad (75)$$

Put $C_3 = 32\rho^6 |\Gamma|/a^5 + 32\beta^6 |\Gamma|/b^5 G^5$. Applying Gronwall's inequality over $[t_0, q]$ and replacing ω by $\theta_{-\tau}\omega$, we have

$$\begin{aligned} & \|\tilde{U}(q, \tau - t, \theta_{-\tau}\omega, \tilde{U}_{\tau-t})\|_{L^6}^6 + G \|\tilde{V}(q, \tau - t, \theta_{-\tau}\omega, \tilde{V}_{\tau-t})\|_{L^6}^6 \\ & \leq e^{\int_{t_0}^q 6\sigma z(\theta_{s-\tau}\omega) ds - \gamma(q-t_0)} \left(\|\tilde{U}(t_0, \tau - t, \theta_{-\tau}\omega, \tilde{U}_{\tau-t})\|_{L^6}^6 \right. \\ & \quad \left. + G \|\tilde{V}(t_0, \tau - t, \theta_{-\tau}\omega, \tilde{V}_{\tau-t})\|_{L^6}^6 \right) \\ & + C_3 \int_{t_0}^q e^{-6\sigma z(\theta_{s-\tau}\omega) + \int_s^q 6\sigma z(\theta_{s-\tau}\omega) ds - \gamma(q-s)} ds + \frac{32}{a^5} \\ & \cdot \int_{t_0}^q e^{-6\sigma z(\theta_{s-\tau}\omega) + \int_s^q 6\sigma z(\theta_{s-\tau}\omega) ds - \gamma(q-s)} \|f_1(s)\|_{L^6}^6 ds \\ & + \frac{32}{b^5 G^5} \\ & \cdot \int_{t_0}^q e^{-6\sigma z(\theta_{s-\tau}\omega) + \int_s^q 6\sigma z(\theta_{s-\tau}\omega) ds - \gamma(q-s)} \|f_2(s)\|_{L^6}^6 ds. \end{aligned} \quad (76)$$

Recalling that $\tilde{U} = U$, $\tilde{V} = V/G$, we get

$$\begin{aligned} & \|U(q, \tau - t, \theta_{-\tau}\omega, U_{\tau-t})\|_{L^6}^6 \\ & + G^{-5} \|V(q, \tau - t, \theta_{-\tau}\omega, V_{\tau-t})\|_{L^6}^6 \leq 1 \\ & + C_3 \int_{-\infty}^{q-\tau} e^{-6\sigma z(\theta_s \omega) + \int_s^{q-\tau} 6\sigma z(\theta_s \omega) ds + \gamma(s-q+\tau)} ds \\ & + \frac{32}{a^5} \int_{-\infty}^{q-\tau} e^{-6\sigma z(\theta_s \omega) + \int_s^{q-\tau} 6\sigma z(\theta_s \omega) ds + \gamma(s-q+\tau)} \|f_1(s) + \tau\|_{L^6}^6 ds \end{aligned}$$

$$\begin{aligned}
& + \frac{32}{b^5 G^5} \int_{-\infty}^{q-\tau} e^{-6\sigma z(\theta_s, \omega) + \int_s^{q-\tau} 6\sigma z(\theta_s, \omega) ds + \gamma(s-q+\tau)} \|f_2(s + \tau)\|_{L^6}^6 ds \doteq M_4(\tau, \omega).
\end{aligned} \tag{77}$$

The proof is completed. $\square$

Lemma 7. Suppose that $f_i(t, x) \in X$, $i = 1, 2$. Then, for every $\tau \in \mathbf{R}$, $\omega \in \Omega$, and $B = \{B(\tau, \omega) : \tau \in \mathbf{R}, \omega \in \Omega\} \in \mathcal{D}$, there exist a number $T_6(\tau, \omega, B, \sigma) \geq 0$ and a tempered random variable $M_5(\tau, \omega)$

$$\begin{aligned}
M_5(\tau, \omega) &= \left(1 + 2\sigma \max_{-1 \leq s \leq 0} |z(\theta_s, \omega)| - 2\gamma\right) M_1 \\
&+ M_4 \int_{-1}^0 \frac{4e^{4\sigma z(\theta_s, \omega)}}{3d} (2 + 3G^2) ds \\
&+ \int_{-1}^0 e^{-2\sigma z(\theta_s, \omega)} |\Gamma| \frac{\rho^2 + \beta^2}{d} ds \\
&+ \int_{-1}^0 e^{-2\sigma z(\theta_s, \omega)} \frac{\|f_1(s + \tau)\|^2 + \|f_2(s + \tau)\|^2}{d} ds.
\end{aligned} \tag{78}$$

such that for all $t \geq T_6$, the solution g of (19)–(22) with $g_{\tau-t} \in B(\tau - t, \theta_{-\tau}\omega)$ satisfies

$$\begin{aligned}
& \|\nabla U(\tau, \tau - t, \theta_{-\tau}\omega, U_{\tau-t}(\theta_{-\tau}\omega))\|^2 \\
& + \|\nabla V(\tau, \tau - t, \theta_{-\tau}\omega, V_{\tau-t}(\omega))\|^2 \leq M_5(\tau, \omega),
\end{aligned} \tag{79}$$

where $g_{\tau-t} = (U_{\tau-t}, V_{\tau-t}) \in B(\tau - t, \theta_{-\tau}\omega)$.

Proof. Multiplying (19) by $-\Delta U$ and (20) by $-\Delta V$, we have

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\nabla U\|^2 &= -d_1 \|\Delta U\|^2 + (\rho e^{-\sigma z(\theta_t, \omega)}, -\Delta U) \\
&- a \|\nabla U\|^2 - (e^{2\sigma z(\theta_t, \omega)} U^2 V, \Delta U) \\
&+ (G e^{2\sigma z(\theta_t, \omega)} U^3, \Delta U) \\
&+ (e^{-\sigma z(\theta_t, \omega)} f_1, -\Delta U) \\
&+ \sigma z(\theta_t, \omega) \|\nabla U\|^2,
\end{aligned} \tag{80}$$

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\nabla V\|^2 &= -d_2 \|\Delta V\|^2 + (\beta e^{-\sigma z(\theta_t, \omega)}, -\Delta V) \\
&- b \|\nabla V\|^2 + (e^{2\sigma z(\theta_t, \omega)} U^2 V, \Delta V) \\
&- (G e^{2\sigma z(\theta_t, \omega)} U^3, \Delta V) \\
&+ (e^{-\sigma z(\theta_t, \omega)} f_2, -\Delta V) \\
&+ \sigma z(\theta_t, \omega) \|\nabla V\|^2.
\end{aligned} \tag{81}$$

Then summing up the resulting equalities, we have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} (\|\nabla U\|^2 + \|\nabla V\|^2) + d_1 \|\Delta U\|^2 + d_2 \|\Delta V\|^2 \\
& + a \|\nabla U\|^2 + b \|\nabla V\|^2 \\
& - \sigma z(\theta_t, \omega) (\|\nabla U\|^2 + \|\nabla V\|^2) \\
& = e^{-\sigma z(\theta_t, \omega)} \int_{\Gamma} (-\rho \Delta U - \beta \Delta V) dx \\
& + e^{-\sigma z(\theta_t, \omega)} \int_{\Gamma} (-f_1 \Delta U - f_2 \Delta V) \\
& + e^{2\sigma z(\theta_t, \omega)} \int_{\Gamma} (-U^2 V \Delta U + G U^3 \Delta U) dx \\
& + e^{2\sigma z(\theta_t, \omega)} \int_{\Gamma} (U^2 V \Delta V - G U^3 \Delta V) dx.
\end{aligned} \tag{82}$$

Due to Hölder's inequality and ϵ -Young inequality, we obtain

$$\begin{aligned}
& e^{-\sigma z(\theta_t, \omega)} \int_{\Gamma} (-\rho \Delta U - \beta \Delta V) dx \\
& \leq e^{-2\sigma z(\theta_t, \omega)} \left(\frac{\rho^2 |\Omega|}{d_1} + \frac{\beta^2 |\Omega|}{d_2} \right) + \frac{d_1}{4} \|\Delta U\|^2 \\
& + \frac{d_2}{4} \|\Delta V\|^2. \\
& e^{-\sigma z(\theta_t, \omega)} \int_{\Gamma} (-f_1 \Delta U - f_2 \Delta V) \\
& \leq e^{-2\sigma z(\theta_t, \omega)} \left(\frac{\|f_1\|^2}{d_1} + \frac{\|f_2\|^2}{d_2} \right) + \frac{d_1}{4} \|\Delta U\|^2 \\
& + \frac{d_2}{4} \|\Delta V\|^2.
\end{aligned} \tag{83}$$

For the third term on the right-hand side of (82), we have

$$\begin{aligned}
& e^{2\sigma z(\theta_t, \omega)} \int_{\Gamma} (-U^2 V \Delta U + G U^3 \Delta U) dx \leq \frac{e^{4\sigma z(\theta_t, \omega)}}{d_1} \\
& \cdot \int_{\Gamma} U^4 V^2 dx + \frac{e^{4\sigma z(\theta_t, \omega)}}{d_1} \int_{\Gamma} G^2 U^6 dx + \frac{d_1}{4} \|\Delta U\|^2 \\
& + \frac{d_1}{4} \|\Delta U\|^2 \leq \frac{d_1}{2} \|\Delta U\|^2 \\
& + \frac{e^{4\sigma z(\theta_t, \omega)}}{3d_1} \left(\int_{\Gamma} 3G^2 U^6 dx \right. \\
& + \left. \int_{\Gamma} (U^6 + U^6 + V^6) dx \right) \leq \frac{d_1}{2} \|\Delta U\|^2 \\
& + \frac{e^{4\sigma z(\theta_t, \omega)}}{3d_1} (2 + 3G^2) (\|U\|_{L^6}^6 + \|V\|_{L^6}^6).
\end{aligned} \tag{84}$$

For the last term in (82), we get

$$\begin{aligned}
& e^{2\sigma z(\theta_t, \omega)} \int_{\Gamma} (U^2 V \Delta V - G U^3 \Delta V) dx \leq \frac{e^{4\sigma z(\theta_t, \omega)}}{d_2} \\
& \cdot \int_{\Gamma} U^4 V^2 dx + \frac{e^{4\sigma z(\theta_t, \omega)}}{d_2} \int_{\Gamma} G^2 U^6 dx + \frac{d_2}{4} \|\Delta V\|^2 \\
& + \frac{d_2}{4} \|\Delta V\|^2 \leq \frac{d_2}{2} \|\Delta V\|^2 \\
& + \frac{e^{4\sigma z(\theta_t, \omega)}}{3d_2} \left(\int_{\Gamma} 3G^2 U^6 dx \right. \\
& \left. + \int_{\Gamma} (U^6 + U^6 + V^6) dx \right) \leq \frac{d_2}{2} \|\Delta V\|^2 \\
& + \frac{e^{4\sigma z(\theta_t, \omega)}}{3d_2} (2 + 3G^2) (\|U\|_{L^6}^6 + \|V\|_{L^6}^6).
\end{aligned} \tag{85}$$

Recalling that $d = \min\{d_1, d_2\}$ and $\gamma = \min\{a, b\}$, it follows that

$$\begin{aligned}
& \frac{d}{dt} (\|\nabla U\|^2 + \|\nabla V\|^2) + (2\gamma - 2\sigma z(\theta_t, \omega)) \\
& \cdot (\|\nabla U\|^2 + \|\nabla V\|^2) \leq \frac{2e^{4\sigma z(\theta_t, \omega)}}{3} \left(\frac{1}{d_1} + \frac{1}{d_2} \right) \\
& \cdot (2 + 3G^2) (\|U\|_{L^6}^6 + \|V\|_{L^6}^6) + 2e^{-2\sigma z(\theta_t, \omega)} |\Gamma| \\
& \cdot \left(\frac{\rho^2}{d_1} + \frac{\beta^2}{d_2} \right) + 2e^{-2\sigma z(\theta_t, \omega)} \left(\frac{\|f_1\|^2}{d_1} + \frac{\|f_2\|^2}{d_2} \right) \\
& \leq \frac{4e^{4\sigma z(\theta_t, \omega)}}{3d} (2 + 3G^2) (\|U\|_{L^6}^6 + \|V\|_{L^6}^6) \\
& + 2e^{-2\sigma z(\theta_t, \omega)} |\Gamma| \frac{\rho^2 + \beta^2}{d} + 2e^{-2\sigma z(\theta_t, \omega)} \\
& \cdot \frac{\|f_1\|^2 + \|f_2\|^2}{d}.
\end{aligned} \tag{86}$$

For $t \geq T_5(\tau, \omega, B) \geq 1$ and $s \in [\tau - 1, \tau]$, we integrate (86) over $[s, \tau]$ to get

$$\begin{aligned}
& \|\nabla U(\tau, \tau - t, \theta_{-\tau}\omega, U_{\tau-t})\|^2 + \|\nabla V(\tau, \tau \\
& - t, \theta_{-\tau}\omega, V_{\tau-t})\|^2 \leq \|\nabla U(s, \tau - t, \theta_{-\tau}\omega, U_{\tau-t})\|^2 \\
& + \|\nabla V(s, \tau - t, \theta_{-\tau}\omega, V_{\tau-t})\|^2 + \int_{\tau-1}^{\tau} 2(\sigma z(\theta_{s-\tau}\omega) \\
& - \gamma) (\|\nabla U(s, \tau - t, \theta_{-\tau}\omega, U_{\tau-t})\|^2 \\
& + \|\nabla V(s, \tau - t, \theta_{-\tau}\omega, V_{\tau-t})\|^2) ds \\
& + \int_{\tau-1}^{\tau} \frac{4e^{4\sigma z(\theta_{s-\tau}\omega)}}{3d} (2 + 3G^2) (\|U\|_{L^6}^6 + \|V\|_{L^6}^6) ds
\end{aligned}$$

$$\begin{aligned}
& + \int_{\tau-1}^{\tau} 2e^{-2\sigma z(\theta_{s-\tau}\omega)} |\Gamma| \frac{\rho^2 + \beta^2}{d} ds \\
& + \int_{\tau-1}^{\tau} 2e^{-2\sigma z(\theta_{s-\tau}\omega)} \frac{\|f_1\|^2 + \|f_2\|^2}{d} ds.
\end{aligned} \tag{87}$$

Now integrating the above with respect to s over $(\tau - 1, \tau)$, we find that for $t \geq T_6(\tau, \omega, B, \sigma)$, it follows that

$$\begin{aligned}
& \|\nabla U(\tau, \tau - t, \theta_{-\tau}\omega, U_{\tau-t})\|^2 + \|\nabla V(\tau, \tau \\
& - t, \theta_{-\tau}\omega, V_{\tau-t})\|^2 \leq \left(1 + 2\sigma \max_{-1 \leq s \leq 0} |z(\theta_s\omega)| - 2\gamma \right) \\
& \cdot \int_{\tau-1}^{\tau} (\|\nabla U(s, \tau - t, \theta_{-\tau}\omega, U_{\tau-t})\|^2 \\
& + \|\nabla V(s, \tau - t, \theta_{-\tau}\omega, V_{\tau-t})\|^2) ds \\
& + \int_{\tau-1}^{\tau} \frac{4e^{4\sigma z(\theta_{s-\tau}\omega)}}{3d} (2 + 3G^2) (\|U(s)\|_{L^6}^6 \\
& + \|V(s)\|_{L^6}^6) ds + \int_{\tau-1}^{\tau} 2e^{-2\sigma z(\theta_{s-\tau}\omega)} |\Gamma| \\
& \cdot \frac{\rho^2 + \beta^2}{d} ds + \int_{\tau-1}^{\tau} 2e^{-2\sigma z(\theta_{s-\tau}\omega)} \\
& \cdot \frac{\|f_1(s)\|^2 + \|f_2(s)\|^2}{d} ds.
\end{aligned} \tag{88}$$

By the Mean Value Theorem in (60), there is a time $t_0 \in [\tau - 2, \tau - 1]$ such that

$$\begin{aligned}
& \|\nabla U(t_0, \tau - t, \theta_{-\tau}\omega, U_{\tau-t})\|^2 \\
& + \|\nabla V(t_0, \tau - t, \theta_{-\tau}\omega, V_{\tau-t})\|^2 \leq M_2(\tau, \omega).
\end{aligned} \tag{89}$$

Then by Lemma 6, for $q \in [\tau - 1, \tau]$, there is a random variable $M_4(\tau, \omega) > 0$ such that

$$\begin{aligned}
& \|U(q, \tau - t, \theta_{-\tau}\omega, U_{\tau-t})\|_{L^6}^6 \\
& + \|V(q, \tau - t, \theta_{-\tau}\omega, V_{\tau-t})\|_{L^6}^6 \leq M_4(\tau, \omega).
\end{aligned} \tag{90}$$

By Lemma 5, it follows from (77) that, for all $t \geq T_5(\tau, \omega, B, \sigma)$,

$$\begin{aligned}
& \|\nabla U(\tau, \tau - t, \theta_{-\tau}\omega, U_{\tau-t})\|^2 \\
& + \|\nabla V(\tau, \tau - t, \theta_{-\tau}\omega, V_{\tau-t})\|^2
\end{aligned}$$

$$\begin{aligned}
&\leq \left(1 + 2\sigma \max_{-1 \leq s \leq 0} |z(\theta_s \omega)| - 2\gamma\right) M_1 \\
&+ M_4 \int_{-1}^0 \frac{4e^{4\sigma z(\theta_s \omega)}}{3d} (2 + 3G^2) ds \\
&+ \int_{-1}^0 e^{-2\sigma z(\theta_s \omega)} |\Gamma| \frac{\rho^2 + \beta^2}{d} ds \\
&+ \int_{-1}^0 e^{-2\sigma z(\theta_s \omega)} \frac{\|f_1(s + \tau)\|^2 + \|f_2(s + \tau)\|^2}{d} ds \\
&\doteq M_5(\tau, \omega).
\end{aligned} \tag{91}$$

It is easy to see that $M_5(\tau, \omega)$ is tempered. This completes the proof. $\square$

Lemma 8. Suppose that $f_i(t, x) \in X(i = 1, 2)$. Then for each $\tau \in \mathbf{R}$ and $\omega \in \Omega$, the continuous cocycle φ associated with problem (19)–(22) has a closed measurable $\mathcal{D}$ -pullback absorbing set $B_\sigma(\tau, \omega) \in \mathcal{D}$, which is given by

$$B_\sigma(\tau, \omega) = \{(U, V) \in H : \|(U, V)\|^2 \leq R_\sigma(\tau, \omega)\}, \tag{92}$$

where $R_\sigma(\tau, \omega)$ is defined as in (32).

Theorem 9. Suppose that $f_i(t, x) \in X(i = 1, 2)$. Then the continuous cocycle φ associated with problem (19)–(22) has a unique $\mathcal{D}$ -pullback attractor $\mathcal{A}_\sigma = \{\mathcal{A}_\sigma(\tau, \omega) : \tau \in \mathbf{R}, \omega \in \Omega\}$ in H .

Proof. By Lemma 8, the continuous cocycle φ has a $\mathcal{D}$ -pullback absorbing set $B_\tau(\sigma, \omega)$. By the compact embedding $E \hookrightarrow H$, the continuous cocycle φ is pullback asymptotically compact in H . According to Theorem 1, there exists a unique random attractor $\mathcal{A}_\sigma$ in H for the continuous cocycle φ . $\square$

4. Upper Semicontinuity of Random Attractors

In this section, we prove the upper semicontinuity of random pullback attractors for the nonautonomous stochastic reversible Selkov system when the intensity of noise $\sigma \rightarrow 0$. Given $\sigma \in [0, 1]$, we write the cocycle of problem (19)–(22) as φ_σ to indicate its dependence on σ .

The deterministic system corresponding to (19)–(22) is given by

$$\frac{du}{dt} = d_1 \Delta u + \rho - au + u^2 v - Gu^3 + f_1(t, x), \tag{93}$$

$$\frac{dv}{dt} = d_2 \Delta v + \beta - bv - u^2 v + Gu^3 + f_2(t, x), \tag{94}$$

with the homogeneous Dirichlet boundary conditions

$$u(t, x) = v(t, x) = 0, \quad t > \tau, \quad x \in \partial\Gamma, \tag{95}$$

and the initial conditions

$$\begin{aligned}
u(\tau, x) &= u_\tau(x), \\
v(\tau, x) &= v_\tau(x), \\
x &\in \Gamma.
\end{aligned} \tag{96}$$

Similarly, we can prove that problem (93)–(96) generates a continuous cocycle φ_0 in H . Moreover, φ_0 has a unique pullback attractor $\mathcal{A}_0 = \{\mathcal{A}_0(\tau), \tau \in \mathbf{R}\}$ in H and has a pullback absorbing set $B_0(\tau) = \{B_0(\tau), \tau \in \mathbf{R}\}$, where $B_0(\tau)$ is given by

$$B_0(\tau) = \{(u, v) \in H : \|(u, v)\|^2 \leq R_0(\tau)\} \tag{97}$$

and

$$\begin{aligned}
R_0(\tau) &= 1 + C_1 \int_{-\infty}^0 e^{\gamma s} ds \\
&+ \frac{G}{a} \int_{-\infty}^0 e^{\gamma s} \|f_1(s + \tau)\|^2 ds \\
&+ \frac{1}{b} \int_{-\infty}^0 e^{\gamma s} \|f_2(s + \tau)\|^2 ds.
\end{aligned} \tag{98}$$

The next lemma shows the convergence of solutions of (19)–(22) as $\sigma \rightarrow 0$.

Lemma 10. Suppose that $f_i(t, x) \in X(i = 1, 2)$. Let $g_\sigma = (U, V)$ and $g = (u, v)$ be the solution of (19)–(22) and (93)–(96) with initial conditions $g_{\tau, \sigma}$ and g_τ , respectively. If $\lim_{\sigma \rightarrow 0} g_{\sigma, \tau} = g_\tau$, then, for $\omega \in \Omega$, $t > \tau$ and $\varepsilon \in [0, 1]$,

$$\lim_{\sigma \rightarrow 0} g_\sigma(t, \tau, \omega, g_{\tau, \sigma}) = g(t, \tau, g_\tau). \tag{99}$$

Proof. Let U and V be the solution of (19)–(22), and $\tilde{u} = U - u$, $\tilde{v} = V - v$, and $k = (\tilde{u}, \tilde{v}) = (U - u, V - v) = g_\sigma - g$. Then from (19)–(20) and (93)–(94), we get

$$\begin{aligned}
\frac{d\tilde{u}}{dt} &= d_1 \Delta \tilde{u} + (e^{-\sigma z(\theta_t \omega)} - 1) \rho - a\tilde{u} + e^{2\sigma z(\theta_t \omega)} U^2 V \\
&- u^2 v - Ge^{2\sigma z(\theta_t \omega)} U^3 + Gu^3 \\
&+ e^{-\sigma z(\theta_t \omega)} f_1(t, x) - f_1(t, x) + \sigma Uz(\theta_t \omega), \\
\frac{d\tilde{v}}{dt} &= d_2 \Delta \tilde{v} + (e^{-\sigma z(\theta_t \omega)} - 1) \beta - b\tilde{v} - e^{2\sigma z(\theta_t \omega)} U^2 V \\
&+ u^2 v + Ge^{2\sigma z(\theta_t \omega)} U^3 - Gu^3 \\
&+ e^{-\sigma z(\theta_t \omega)} f_2(t, x) - f_2(t, x) + \sigma Vz(\theta_t \omega),
\end{aligned} \tag{100}$$

which implies

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\tilde{u}\|^2 + d_1 \|\nabla \tilde{u}\|^2 + a \|\tilde{u}\|^2 &= \int_\Gamma (e^{-\sigma z(\theta_t \omega)} - 1) \\
&\cdot \rho \tilde{u} dx + \int_\Gamma (e^{-\sigma z(\theta_t \omega)} - 1) f_1 \tilde{u} dx \\
&+ \int_\Gamma (e^{2\sigma z(\theta_t \omega)} U^2 V - u^2 v - Ge^{2\sigma z(\theta_t \omega)} U^3 + Gu^3) \\
&\cdot \tilde{u} dx + \int_\Gamma \sigma z(\theta_t \omega) U \tilde{u} dx,
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\tilde{v}\|^2 + d_2 \|\nabla \tilde{v}\|^2 + b \|\tilde{v}\|^2 = \int_{\Gamma} \left(e^{-\sigma z(\theta_t, \omega)} - 1 \right) \\
& \quad \cdot \beta \tilde{v} \, dx + \int_{\Gamma} \left(e^{-\sigma z(\theta_t, \omega)} - 1 \right) f_2 \tilde{v} \, dx \\
& \quad - \int_{\Gamma} \left(e^{2\sigma z(\theta_t, \omega)} U^2 V - u^2 v - G e^{2\sigma z(\theta_t, \omega)} U^3 + G u^3 \right) \\
& \quad \cdot \tilde{v} \, dx + \int_{\Gamma} \sigma z(\theta_t, \omega) V \tilde{v} \, dx.
\end{aligned} \tag{101}$$

Summing up the two equalities together, it yields that

$$\begin{aligned}
& \frac{d}{dt} \left(\|\tilde{u}\|^2 + \|\tilde{v}\|^2 \right) + 2d \left(\|\nabla \tilde{u}\|^2 + \|\nabla \tilde{v}\|^2 \right) + 2a \|\tilde{u}\|^2 \\
& + 2b \|\tilde{v}\|^2 = 2 \int_{\Gamma} \left(e^{-\sigma z(\theta_t, \omega)} - 1 \right) (\rho \tilde{u} + \beta \tilde{v}) \, dx \\
& + 2 \int_{\Gamma} \left(e^{-\sigma z(\theta_t, \omega)} - 1 \right) (f_1 \tilde{u} + f_2 \tilde{v}) \, dx \\
& + 2 \int_{\Gamma} \sigma z(\theta_t, \omega) (U \tilde{u} + V \tilde{v}) \, dx \\
& + 2 \int_{\Gamma} \left(e^{2\sigma z(\theta_t, \omega)} U^2 V - u^2 v - G e^{2\sigma z(\theta_t, \omega)} U^3 + G u^3 \right) \\
& \cdot (\tilde{u} - \tilde{v}) \, dx.
\end{aligned} \tag{102}$$

Note that

$$\begin{aligned}
& \int_{\Gamma} \left(e^{2\sigma z(\theta_t, \omega)} U^2 V - u^2 v - G e^{2\sigma z(\theta_t, \omega)} U^3 + G u^3 \right) \\
& \cdot (\tilde{u} - \tilde{v}) \, dx = \int_{\Gamma} e^{2\sigma z(\theta_t, \omega)} (U^2 V - G U^3) \\
& \cdot (\tilde{u} - \tilde{v}) \, dx + \int_{\Gamma} (G u^3 - u^2 v) (\tilde{u} - \tilde{v}) \, dx \\
& = \int_{\Gamma} \left(e^{2\sigma z(\theta_t, \omega)} - 1 \right) (U^2 V - G U^3) (\tilde{u} - \tilde{v}) \, dx \\
& + \int_{\Gamma} (U^2 V - G U^3 + G u^3 - u^2 v) (\tilde{u} - \tilde{v}) \, dx \\
& = \int_{\Gamma} \left(e^{2\sigma z(\theta_t, \omega)} - 1 \right) (U^2 V - G U^3) (\tilde{u} - \tilde{v}) \, dx \\
& + \int_{\Gamma} [(U^2 V - u^2 v) - (G U^3 - G u^3)] \\
& \cdot (\tilde{u} - \tilde{v}) \, dx.
\end{aligned} \tag{103}$$

Given that $\tau \in \mathbf{R}$, $\omega \in \Omega$, $T > 0$, and $\varepsilon \in [0, 1]$, since ω is continuous on $\mathbf{R}$, we obtain that there exists a $\sigma_1 = \sigma_1(\omega, T, \varepsilon) > 0$ such that for all $\sigma \in (0, \sigma_1)$ and $t \in [\tau, \tau + T]$,

$$\begin{aligned}
& \left| e^{-\sigma z(\theta_t, \omega)} - 1 \right| + \left| e^{\sigma z(\theta_t, \omega)} - 1 \right| + \left| e^{2\sigma z(\theta_t, \omega)} - 1 \right| \\
& + \left| e^{-2\sigma z(\theta_t, \omega)} - 1 \right| < \varepsilon.
\end{aligned} \tag{104}$$

Thus, we obtain

$$\begin{aligned}
& \frac{d}{dt} \left(\|\tilde{u}\|^2 + \|\tilde{v}\|^2 \right) + 2d \left(\|\nabla \tilde{u}\|^2 + \|\nabla \tilde{v}\|^2 \right) + 2a \|\tilde{u}\|^2 \\
& + 2b \|\tilde{v}\|^2 \leq a \|\tilde{u}\|^2 + b \|\tilde{v}\|^2 + \frac{\varepsilon^2 \rho^2 |\Gamma|}{a} + \frac{\varepsilon^2 \beta^2 |\Gamma|}{b} \\
& + a \|\tilde{u}\|^2 + b \|\tilde{v}\|^2 + \frac{\varepsilon^2 \|f_1\|^2}{a} + \frac{\varepsilon^2 \|f_2\|^2}{b} \\
& + 2\sigma z(\theta_t, \omega) \int_{\Gamma} (|U \tilde{u}| + |V \tilde{v}|) \, dx \\
& + 2\varepsilon \int_{\Gamma} (|U^2 V| + |G U^3|) (|\tilde{u}| + |\tilde{v}|) \, dx \\
& + 2 \int_{\Gamma} (|U^2 V - u^2 v| + |G U^3 - G u^3|) \\
& \cdot (|\tilde{u}| + |\tilde{v}|) \, dx.
\end{aligned} \tag{105}$$

Lemma 6 is true even if $q \in [\tau, t]$. By Hölder's inequality and Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}
& \int_{\Gamma} (|U^2 V - u^2 v| + |G U^3 - G u^3|) (|\tilde{u}| + |\tilde{v}|) \, dx \\
& \leq d (\|\nabla \tilde{u}\| + \|\nabla \tilde{v}\|) + c'_1 (\|g\| + \|g\|_{L^4}^2 + \|g\|_{L^6}^3) \\
& + \|g_{\sigma}\| + \|g_{\sigma}\|_{L^4}^2 + \|g_{\sigma}\|_{L^6}^3 (\|\tilde{u}\|^2 + \|\tilde{v}\|^2),
\end{aligned} \tag{106}$$

where c'_1 is a constant depending on d, G and the Gagliardo-Nirenberg inequality constant. It follows from (105) and (106) that there exists a $\sigma_2 = \sigma_2(\omega, T, \varepsilon) > 0$ such that for all $\sigma \in (0, \sigma_2)$ and $t \in [\tau, \tau + T]$,

$$\begin{aligned}
& \frac{d}{dt} \left(\|\tilde{u}\|^2 + \|\tilde{v}\|^2 \right) \leq \frac{\varepsilon^2 \rho^2 |\Gamma|}{a} + \frac{\varepsilon^2 \beta^2 |\Gamma|}{b} + \frac{\varepsilon^2 \|f_1\|^2}{a} \\
& + \frac{\varepsilon^2 \|f_2\|^2}{b} + 2c'_2 (\|g\| + \|g\|_{L^4}^2 + \|g\|_{L^6}^3 + \|g_{\sigma}\| \\
& + \|g_{\sigma}\|_{L^4}^2 + \|g_{\sigma}\|_{L^6}^3) (\|\tilde{u}\|^2 + \|\tilde{v}\|^2) \leq c_4 \varepsilon^2 \\
& + \frac{\varepsilon^2 \|f_1\|^2}{a} + \frac{\varepsilon^2 \|f_2\|^2}{b} + 2c'_3,
\end{aligned} \tag{107}$$

where $c_4 = 2\rho^2 |\Gamma|/a + 2\beta^2 |\Gamma|/b$, $c'_3 = 2c'_2 (\|g\| + \|g\|_{L^4}^2 + \|g\|_{L^6}^3 + \|g_{\sigma}\| + \|g_{\sigma}\|_{L^4}^2 + \|g_{\sigma}\|_{L^6}^3)$. For all $\sigma \in (0, \sigma_2)$ and $t \in [\tau, \tau + T]$, due to Gronwall's lemma, we get

$$\begin{aligned}
& \|k(t)\|^2 \\
& \leq e^{c'_3(t-\tau)} \left(\|\tilde{u}(\tau)\|^2 + \|\tilde{v}(\tau)\|^2 \right) \\
& + \varepsilon^2 \int_{\tau}^t e^{-c'_3(t-s)} \left(c_4 + \frac{\|f_1(s)\|^2}{a} + \frac{\|f_2(s)\|^2}{b} \right) ds.
\end{aligned} \tag{108}$$

From (108), recalling that $k = (\tilde{u}, \tilde{v}) = (U - u, V - v) = g_\sigma - g$, we find that

$$\begin{aligned} & \|g_\sigma(t, \tau, \omega, g_{\tau, \sigma}) - g(t, \tau, g_\tau)\|^2 \\ & \leq e^{c'_3(t-\tau)} (\|g_{\tau, \sigma} - g_\tau\|^2) \\ & \quad + \varepsilon^2 \int_\tau^t e^{-c'_3(t-s)} \left(c_4 + \frac{\|f_1(r)\|^2}{a} + \frac{\|f_2(r)\|^2}{b} \right) dr, \end{aligned} \quad (109)$$

which finishes the proof. $\square$

Theorem 11. *Provided that $f_i(t, x) \in X(i = 1, 2)$. Then for every $\tau \in \mathbf{R}$ and $\omega \in \Omega$,*

$$\lim_{\sigma \rightarrow 0} d_H(\mathcal{A}_\sigma(\tau, \omega), \mathcal{A}_0(\tau)) = 0. \quad (110)$$

Proof. To achieve the result, it suffices to verify conditions (1), (2), and (3) in Proposition 3. For condition (1), by taking a sequence $\sigma_n \rightarrow 0$ and $g_{\tau, n} \rightarrow g_\tau$, from Lemma 10 we have

$$\varphi_{\sigma_n}(t, \tau, \omega, g_{\tau, n}) \rightarrow \varphi(t, \tau, g_\tau) \quad \text{in } H. \quad (111)$$

By Lemma 8 and Theorem 9, φ_σ has a tempered pullback absorbing set B_σ and a unique $\mathcal{D}$ -pullback random attractor $\mathcal{A}_\sigma \in \mathcal{D}$. Here,

$$B_\sigma(\tau, \omega) = \{U, V \in H : \|(U, V)\|^2 \leq R_\sigma(\tau, \omega)\}, \quad (112)$$

where $R_\sigma(\tau, \omega)$ is defined as in (32). It follows from (97) and (112) that for all $\tau \in \mathbf{R}$ and $\omega \in \Omega$, we have

$$\limsup_{\sigma \rightarrow 0} \|B_\sigma(\tau, \omega)\| \leq \|B_0(\tau)\|, \quad (113)$$

which deduces condition (2) immediately.

Now consider condition (3). By Lemma 7, we know that $\{K_\sigma(\tau, \omega)\} \in \mathcal{D}$ is also a closed and tempered absorbing set for φ_σ in H , where

$$K_\sigma(\tau, \omega) = \{(U, V) \in H : \|(U, V)\|_E^2 \leq M_5(\tau, \omega)\}. \quad (114)$$

Let

$$\begin{aligned} & \widetilde{M}_5(\tau, \omega) \\ & = \left(1 + 2 \max_{-1 \leq s \leq 0} |z(\theta_s, \omega)| - 2\gamma\right) M_1 \\ & \quad + M_4 \int_{-1}^0 \frac{4e^{4|z(\theta_s, \omega)|}}{3d} (2 + 3G^2) ds \\ & \quad + \int_{-1}^0 e^{2|z(\theta_s, \omega)|} |\Gamma| \frac{\rho^2 + \beta^2}{d} ds \\ & \quad + \int_{-1}^0 e^{2|z(\theta_s, \omega)|} \frac{\|f_1(s + \tau)\|^2 + \|f_2(s + \tau)\|^2}{d} ds, \end{aligned} \quad (115)$$

and

$$K(\tau, \omega) = \{(U, V) \in H : \|(U, V)\|_E^2 \leq \widetilde{M}_5(\tau, \omega)\}. \quad (116)$$

We find that for all $\tau \in \mathbf{R}$ and $\omega \in \Omega$,

$$\bigcup_{0 < \sigma \leq 1} \mathcal{A}_\sigma(\tau, \omega) \subseteq \bigcup_{0 < \sigma \leq 1} K_\sigma(\tau, \omega) \subseteq K(\tau, \omega). \quad (117)$$

By the compactness of embedding $H \hookrightarrow E$, we know that $K(\tau, \omega)$ is compact in H . By (117), it follows that $\bigcup_{0 < \sigma \leq 1} \mathcal{A}_\sigma(\tau, \omega)$ is precompact in H . Hence, condition (3) is clear and the proof is finished. $\square$

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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