Research Article

Derivations and Deformations of δ -Jordan Lie Supertriple Systems

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Let T be a δ -Jordan Lie supertriple system. We first introduce the notions of generalized derivations and representations of T and present some properties. Also, we study the low-dimensional cohomology and the coboundary operator of T , and then we investigate the deformations and Nijenhuis operators of T by choosing some suitable cohomologies.

1. Introduction

Lie triple systems arose initially in Cartan's study of Riemannian geometry. Jacobson [1] first introduced Lie triple systems and Jordan triple systems in connection with problems from Jordan theory and quantum mechanics, viewing Lie triple systems as subspaces of Lie algebras that are closed relative to the ternary product. Lister [2] investigated notions of the radical, semisimplicity and solvability as defined for Lie triple systems and determined all simple Lie triple systems over an algebraically closed field. Later, the representation theory, the central extension, the deformation theory, bilinear forms, and the generalized derivation of Lie triple systems and Jordan triple systems have been developed; see [3–11].

In [12], Okubo and Kamiya introduced the notion of δ -Jordan Lie triple system, where $\delta = \pm 1$, which is a generalization of both Lie triple systems ($\delta = 1$) and Jordan Lie triple systems ($\delta = -1$). Later, Kamiya and Okubo [13] studied a construction of simple Jordan superalgebras from certain triple systems. Recently, Ma and Chen [14] discussed the cohomology theory, the deformations, Nijenhuis operators, abelian extensions, and T^* -extensions of δ -Jordan Lie triple system.

As a natural generalization of Lie triple systems, Okubo [15] introduced the notion of Lie supertriple systems in the study of Yang-Baxter equations. Lie supertriple systems have many applications in high energy physics, and many important results on Lie supertriple systems have been obtained; see [13, 15–17]. In [12], Okubo and Kamiya introduced the notion of δ -Jordan Lie supertriple system (they still call it Jordan Lie triple system); they presented some nontrivial examples and discussed their quasiclassical property. In the present paper, we hope to study generalized derivations, cohomology theories, and deformations of δ -Jordan Lie supertriple systems.

This paper is organized as follows. In Section 2, we recall the definition of δ -Jordan Lie supertriple systems and construct a kind of δ -Jordan Lie supertriple systems. Also, we study generalized derivation algebra of a δ -Jordan Lie supertriple system. In Section 3, we introduce notions of the representation and low-dimensional cohomology of a δ -Jordan Lie supertriple system. In Section 4, we consider the theory of deformations of a δ -Jordan Lie supertriple system by choosing a suitable cohomology. In Section 5, we study Nijenhuis operators for a δ -Jordan Lie supertriple system to describe trivial deformations.

2. Generalized Derivations of δ -Jordan Lie Supertriple Systems

In this section, we start by recalling the definition of δ -Jordan Lie supertriple systems and then we study its generalized derivations.

Definition 1 ([12]). A δ -Jordan Lie supertriple system is a Z_2 -graded vector space T together with a triple linear product $[\cdot, \cdot, \cdot] : V \otimes V \otimes V \longrightarrow V$ satisfying

$$(1) \quad |[a, b, c]| = (|a| + |b| + |c|) \pmod{2}; \quad (1)$$

$$(2) \quad [b, a, c] = -\delta(-1)^{|a||b|} [a, b, c]; \quad (2)$$

$$(3) \quad (-1)^{|a||c|} [a, b, c] + (-1)^{|b||a|} [b, c, a] + (-1)^{|c||b|} [c, a, b] = 0; \quad (3)$$

$$(4) \quad [a, b, [c, d, e]] = [[a, b, c], d, e] + (-1)^{|c|(|a|+|b|)} [c, [a, b, d], e] + \delta(-1)^{(|a|+|b|)(|c|+|d|)} [c, d, [a, b, e]], \quad (4)$$

for all $a, b, c, d, e \in T$, where $\delta = \pm 1$ and $|a|$ denotes the degree of the element $a \in T$.

Remark 2. Clearly, T_0^- is an ordinary δ -Jordan Lie triple system in [14]. The case of $\delta = 1$ defines, especially, a Lie supertriple system while the other case of $\delta = -1$ may be termed an anti-Lie supertriple system as in [18].

Example 3 ([12]). Let $(T, [\cdot, \cdot])$ be a δ -Jordan Lie superalgebra. Then $(T, [\cdot, \cdot, \cdot])$ becomes a δ -Jordan Lie supertriple system, where $[a, b, c] = [[a, b], c]$, for all $a, b, c \in T$.

Example 4. Let T be a δ -Jordan Lie supertriple system and t an indeterminate. Set $T' = \{\sum_{i \geq 0} x \otimes t^i \mid x \in T\}$; then T' is a δ -Jordan Lie supertriple system with a triple linear product $[\cdot, \cdot, \cdot]'$ defined by

$$[a \otimes t^i, b \otimes t^j, c \otimes t^k]' = [a, b, c] \otimes t^{i+j+k}, \quad (5)$$

for all $a \otimes t^i, b \otimes t^j, c \otimes t^k \in T'$, where $|a \otimes t^i| = |a|$.

Definition 5. Let T be a δ -Jordan Lie supertriple system and k a nonnegative integer. A homogeneous linear map $D : T \longrightarrow T$ is said to be a homogeneous k -derivation of T if it satisfies

$$\delta^k [D(a), b, c] + \delta^k (-1)^{|D||a|} [a, D(b), c] + \delta^k (-1)^{|D|(|a|+|b|)} [a, b, D(c)] = D([a, b, c]), \quad (6)$$

for all $a, b, c \in T$, where $|D|$ denotes the degree of D .

We denote by $Der(T) = \bigoplus_{k \geq 0} Der_k(T)$, where $Der_k(T)$ is the set of all homogeneous k -derivations of T . Obviously, $Der(T)$ is a subalgebra of $End(T)$ and has a normal Lie superalgebra structure via the bracket product

$$[D, D'] = DD' - (-1)^{|D||D'|} D'D. \quad (7)$$

Definition 6. Let T be a δ -Jordan Lie supertriple system and k a nonnegative integer. $D \in End_{\bar{s}}(T)$ is said to be a homogeneous generalized k -derivation of T , if there exist three endomorphisms $D', D'', D''' \in End_{\bar{s}}(T)$ such that

$$\delta^k [D(a), b, c] + \delta^k (-1)^{|a|} [a, D'(b), c] + \delta^k (-1)^{|a|+|b|} [a, b, D''(c)] = D'''([a, b, c]), \quad (8)$$

for all $a, b, c \in T$.

Definition 7. Let T be a δ -Jordan Lie supertriple system and k a nonnegative integer. $D \in End_{\bar{s}}(T)$ is said to be a homogeneous k -quasiderivation of T , if there exists an endomorphism $D' \in End_{\bar{s}}(T)$ such that

$$\delta^k [D(a), b, c] + \delta^k (-1)^{|a|} [a, D(b), c] + \delta^k (-1)^{|a|+|b|} [a, b, D(c)] = D'([a, b, c]), \quad (9)$$

for all $a, b, c \in T$.

Let $GDer(T)$ and $QDer(T)$ be the sets of homogeneous generalized k -derivations and of homogeneous k -quasiderivations, respectively. That is,

$$GDer(T) = \bigoplus_{k \geq 0} GDer_k(T), \quad (10)$$

$$QDer(T) = \bigoplus_{k \geq 0} QDer_k(T).$$

Definition 8. Let T be a δ -Jordan Lie supertriple system and k a nonnegative integer. The k -centroid of T is the space of linear transformations on T given by

$$C^k(T) = \{D \in End(T) \mid \delta^k [D(a), b, c] = \delta^k (-1)^{|D||a|} [a, D(b), c] + \delta^k (-1)^{|D|(|a|+|b|)} [a, b, D(c)] = D([a, b, c])\}. \quad (11)$$

We denote $C(T) = \bigoplus_{k \geq 0} C^k(T)$ and call it the centroid of T .

Definition 9. Let T be a δ -Jordan Lie supertriple system. The quas centroid of T is the space of linear transformations on T given by

$$QC(T) = \{D \in End(T) \mid D([a, b, c]) = [D(a), b, c], \forall a, b, c \in T\}. \quad (12)$$

Remark 10. Let T be a δ -Jordan Lie supertriple system. Then $QC(T) \subseteq C(T)$.

For any $D \in QC(T)$ and $a, b, c \in T$, we have

$$D([a, b, c]) = [D(a), b, c] = (-1)^{|D||a|} [a, D(b), c] = (-1)^{|D|(|a|+|b|)} [a, b, D(c)]. \quad (13)$$

In fact, by the definition of the δ -Jordan Lie supertriple system, we have

$$\begin{aligned}
 D([a, b, c]) &= -\delta(-1)^{|a||b|} D([b, a, c]) \\
 &= -\delta(-1)^{|a||b|} [D(b), a, c] \\
 &= \delta^2(-1)^{|a||b|} (-1)^{|a|(|b|+|D|)} [a, D(b), c] \\
 &= (-1)^{|D||a|} [a, D(b), c].
 \end{aligned} \tag{14}$$

Similarly, we have

$$\begin{aligned}
 &(-1)^{|a|(|D|+|c|)} [a, b, D(c)] \\
 &= -(-1)^{|a||b|} [b, D(c), a] \\
 &\quad - (-1)^{|b|(|D|+|c|)} [D(c), a, b] \\
 &= -(-1)^{|b|(|a|+|D|)} D([b, c, a]) \\
 &\quad - (-1)^{|b|(|D|+|c|)} D([c, a, b]) \\
 &= -(-1)^{|b||D|} D((-1)^{|b||a|} [b, c, a] + (-1)^{|c||b|} [c, a, b]) \\
 &= (-1)^{|b||D|} D((-1)^{|a||c|} [a, b, c]).
 \end{aligned} \tag{15}$$

It follows that $[a, b, D(c)] = (-1)^{|D|(|a|+|b|)} [a, b, D(c)]$.

Definition 11. Let T be a δ -Jordan Lie supertriple system. $D \in \text{End}(T)$ is said to be a central derivation of T if

$$D([a, b, c]) = [D(a), b, c] = 0, \tag{16}$$

for all $a, b, c \in T$. Denote the set of all central derivations by $\text{ZDer}(T)$.

Remark 12. Let T be a δ -Jordan Lie supertriple system. Then

$$\begin{aligned}
 \text{ZDer}(T) &\subseteq \text{Der}(T) \subseteq \text{QDer}(T) \subseteq \text{GDer}(T) \\
 &\subseteq \text{End}(T).
 \end{aligned} \tag{17}$$

Definition 13. Let T be a δ -Jordan Lie supertriple system. If $Z(T) = \{a \in T \mid [a, b, c] = 0, \forall b, c \in T\}$, then $Z(T)$ is called the center of T .

Proposition 14. Let T be a δ -Jordan Lie supertriple system; then the following statements hold:

- (1) $\text{GDer}(T)$, $\text{QDer}(T)$, and $C(T)$ are subalgebras of $\text{End}(T)$.
- (2) $\text{ZDer}(T)$ is an ideal of $\text{Der}(T)$.

Proof. (1) We only prove that $\text{GDer}(T)$ is a subalgebra of $\text{End}(T)$, and similarly for cases of $\text{QDer}(T)$ and $C(T)$. For any $D_1 \in \text{GDer}_k(T)$, $D_2 \in \text{GDer}_l(T)$ and $a, b, c \in T$, we have

$$\begin{aligned}
 [D_1 D_2(a), b, c] &= \delta^k D_1''' [D_2(a), b, c] \\
 &\quad - (-1)^{|D_1|(|D_2|+|a|)} [D_2(a), D_1'(b), c] \\
 &\quad - (-1)^{|D_1|(|D_2|+|a|+|b|)} [D_2(a), b, D_1''(c)] \\
 &= \delta^k D_1''' \{ \delta^l D_2''' ([a, b, c]) \\
 &\quad - (-1)^{|D_2||a|} [a, D_2'(b), c] \\
 &\quad - (-1)^{|D_2|(|a|+|b|)} [a, b, D_2''(c)] \} - (-1)^{|D_1|(|D_2|+|a|)} \\
 &\quad \cdot [D_2(a), D_1'(b), c] - (-1)^{|D_1|(|D_2|+|a|+|b|)} \\
 &\quad \cdot [D_2(a), b, D_1''(c)] = \delta^{k+l} D_1''' D_2''' ([a, b, c]) \\
 &\quad - \delta^l (-1)^{|D_2||a|} D_1''' [a, D_2'(b), c] - \delta^l (-1)^{|D_2|(|a|+|b|)} \\
 &\quad \cdot D_1''' [a, b, D_2''(c)] - (-1)^{|D_1|(|D_2|+|a|)} \\
 &\quad \cdot [D_2(a), D_1'(b), c] - (-1)^{|D_1|(|D_2|+|a|+|b|)} \\
 &\quad \cdot [D_2(a), b, D_1''(c)] = \delta^{k+l} D_1''' D_2''' ([a, b, c]) \\
 &\quad - \delta^{k+l} (-1)^{|D_2||a|} [D_1(a), D_2'(b), c] \\
 &\quad - \delta^{k+l} (-1)^{(|D_1|+|D_2|)|a|} [a, D_1' D_2'(b), c] \\
 &\quad - \delta^{k+l} (-1)^{|D_2||a|+|D_1|(|a|+|b|+|D_2|)} [a, D_2'(b), D_1''(c)] \\
 &\quad - \delta^{k+l} (-1)^{|D_2|(|a|+|b|)} [D_1(a), b, D_2''(c)] \\
 &\quad - \delta^{k+l} (-1)^{|D_2|(|a|+|b|)+|D_1||a|} [a, D_1'(b), D_2''(c)] \\
 &\quad - \delta^{k+l} (-1)^{(|D_1|+|D_2|)(|a|+|b|)} [a, b, D_1'' D_2''(c)] \\
 &\quad - (-1)^{|D_1|(|D_2|+|a|)} [D_2(a), D_1'(b), c] \\
 &\quad - (-1)^{|D_1|(|D_2|+|a|+|b|)} [D_2(a), b, D_1''(c)].
 \end{aligned} \tag{18}$$

Similarly, we have

$$\begin{aligned}
 [D_2 D_1(a), b, c] &= \delta^{k+l} D_2''' D_1''' ([a, b, c]) \\
 &\quad - \delta^{k+l} (-1)^{|D_1||a|} [D_2(a), D_1'(b), c] \\
 &\quad - \delta^{k+l} (-1)^{(|D_1|+|D_2|)|a|} [a, D_2' D_1'(b), c] \\
 &\quad - \delta^{k+l} (-1)^{|D_1||a|+|D_2|(|a|+|b|+|D_1|)} [a, D_1'(b), D_2''(c)] \\
 &\quad - \delta^{k+l} (-1)^{|D_1|(|a|+|b|)} [D_2(a), b, D_1''(c)] \\
 &\quad - \delta^{k+l} (-1)^{|D_1|(|a|+|b|)+|D_2||a|} [a, D_2'(b), D_1''(c)] \\
 &\quad - \delta^{k+l} (-1)^{(|D_1|+|D_2|)(|a|+|b|)} [a, b, D_2'' D_1''(c)]
 \end{aligned}$$

$$\begin{aligned}
& -(-1)^{|D_2|(|D_1|+|a|)} [D_1(a), D_2'(b), c] \\
& -(-1)^{|D_2|(|D_1|+|a|+|b|)} [D_1(a), b, D_2''(c)].
\end{aligned} \tag{19}$$

It follows that

$$\begin{aligned}
& [[D_1, D_2](a), b, c] = [D_1 D_2(a), b, c] - (-1)^{k+l} \\
& \cdot [D_2 D_1(a), b, c] \\
& = \delta^{k+l} (D_1''' D_2''' - (-1)^{|D_1|+|D_2|} D_2''' D_1''') [a, b, c] \\
& - (-1)^{|a|(|D_1|+|D_2|)} \\
& \cdot [a, (D_1' D_2' - (-1)^{|D_1|+|D_2|} D_2' D_1')(b), c] \\
& - (-1)^{(|a|+|b|)(|D_1|+|D_2|)} \\
& \cdot [a, b, (D_1'' D_2'' - (-1)^{|D_1|+|D_2|} D_2'' D_1'')(c)] \\
& = \delta^{k+l} [D_1''', D_2''', a, b, c] - (-1)^{|a|(|D_1|+|D_2|)} \\
& \cdot [a, [D_1', D_2'](b), c] - (-1)^{(|a|+|b|)(|D_1|+|D_2|)} \\
& \cdot [a, b, [D_1'', D_2''](c)].
\end{aligned} \tag{20}$$

Obviously, $[D_1', D_2'], [D_1'', D_2'']$ and $[D_1''', D_2''']$ are contained in $End(T)$; thus $[D_1, D_2] \in GDer_{k+l}(T) \subseteq GDer(T)$; that is, $GDer(T)$ is a subalgebra of $End(T)$.

(2) For any $D_1 \in ZDer(T)$, $D_2 \in Der_k(T)$, and $a, b, c \in T$, we have

$$\begin{aligned}
& [D_1, D_2]([a, b, c]) = D_1 D_2([a, b, c]) \\
& - (-1)^{|D_1||D_2|} D_2 D_1([a, b, c]) \\
& = 0.
\end{aligned} \tag{21}$$

Also, we have

$$\begin{aligned}
& [[D_1, D_2](a), b, c] = [D_1 D_2(a), b, c] - (-1)^{|D_1||D_2|} \\
& \cdot [D_2 D_1(a), b, c] = 0 - (-1)^{|D_1||D_2|} \\
& \cdot [D_2 D_1(a), b, c] = -(-1)^{|D_1||D_2|} \\
& \cdot (\delta^k D_2([D_1(a), b, c]) \\
& - (-1)^{|D_2|(|D_1|+|a|)} [D_1(a), D_2(b), c]) \\
& + (-1)^{|D_1||D_2|} (-1)^{|D_2|(|D_1|+|a|+|b|)} [D_1(a), b, D_2(c)] \\
& = 0.
\end{aligned} \tag{22}$$

It follows that $[D_1, D_2] \in ZDer(T)$. That is, $ZDer(T)$ is an ideal of $Der(T)$. $\square$

Proposition 15. Let T be a δ -Jordan Lie supertriple system; then the following statements hold:

- (1) $[Der(T), C(T)] \subseteq C(T)$.
- (2) $[QDer(T), QC(T)] \subseteq QC(T)$.
- (3) $[QC(T), QC(T)] \subseteq QDer(T)$.
- (4) $C(T) \subseteq QDer(T)$.

Proof. (1) For any $D_1 \in Der_k(T)$, $D_2 \in C_l(T)$, and $a, b, c \in T$, we have

$$\begin{aligned}
& [D_1, D_2]([a, b, c]) \\
& = D_1 D_2([a, b, c]) - (-1)^{|D_1||D_2|} D_2 D_1([a, b, c]) \\
& = \delta^l D_1([D_2(a), b, c]) \\
& - \delta^k (-1)^{|D_1||D_2|} D_2([D_1(a), b, c]) \\
& - \delta^k (-1)^{|D_1|(|D_2|+|a|)} D_2([a, D_1(b), c]) \\
& - \delta^k (-1)^{|D_1|(|D_2|+|a|+|b|)} D_2([a, b, D_1(c)]) \\
& = \delta^{k+l} [D_1 D_2(a), b, c] \\
& + \delta^{k+l} (-1)^{|D_1|(|D_2|+|a|)} [D_2(a), D_1(b), c] \\
& + \delta^{k+l} (-1)^{|D_1|(|D_2|+|a|+|b|)} [D_2(a), b, D_1(c)] \\
& - \delta^{k+l} (-1)^{|D_1||D_2|} [D_2 D_1(a), b, c] \\
& - \delta^{k+l} (-1)^{|D_1|(|D_2|+|a|)} [D_2(a), D_1(b), c] \\
& - \delta^{k+l} (-1)^{|D_1|(|D_2|+|a|+|b|)} [D_2(a), b, D_1(c)] \\
& = \delta^{k+l} [[D_1, D_2](a), b, c].
\end{aligned} \tag{23}$$

Similarly, one can check that

$$\begin{aligned}
& [[D_1, D_2](a), b, c] = (-1)^{|a|(k+l)} [a, [D_1, D_2](b), c] \\
& = (-1)^{(|a|+|b|)(k+l)} [a, b, [D_1, D_2](c)].
\end{aligned} \tag{24}$$

It follows that $[D_1, D_2] \in C(T)$; thus $[Der(T), C(T)] \subseteq C(T)$.

(2) Similar to the proof of (1).

(3) For any $D_1, D_2 \in QC(T)$ and $a, b, c \in T$, we have

$$\begin{aligned}
& [[D_1, D_2](a), b, c] \\
& + (-1)^{|a|(|D_1|+|D_2|)} [a, [D_1, D_2](b), c] \\
& + (-1)^{(|a|+|b|)(|D_1|+|D_2|)} [a, b, [D_1, D_2](c)] \\
& = [D_1 D_2(a), b, c] \\
& + (-1)^{|a|(|D_1|+|D_2|)} [a, D_1 D_2(b), c] \\
& + (-1)^{(|a|+|b|)(|D_1|+|D_2|)} [a, b, D_1 D_2(c)] \\
& - (-1)^{|D_1||D_2|} [D_2 D_1(a), b, c] \\
& - (-1)^{|a|(|D_1|+|D_2|)+|D_1||D_2|} [a, D_2 D_1(b), c] \\
& - (-1)^{(|a|+|b|)(|D_1|+|D_2|)+|D_1||D_2|} [a, b, D_2 D_1(c)].
\end{aligned} \tag{25}$$

Since $D_1, D_2 \in QC(T)$, we have

$$\begin{aligned} [D_1 D_2(a), b, c] &= (-1)^{|D_1|(|a|+|D_2|)} [D_2(a), D_1(b), c] \\ &= (-1)^{|D_1|(|a|+|D_2|)+|a||D_2|} [a, D_2 D_1(b), c]. \end{aligned} \quad (26)$$

Similarly, one may check that

$$\begin{aligned} [a, D_1 D_2(b), c] &= (-1)^{|b|(|D_1|+|D_2|)+|D_1||D_2|} [a, b, D_2 D_1(c)], \\ [a, b, D_1 D_2(c)] &= (-1)^{(|a|+|b|)(|D_1|+|D_2|)+|D_1||D_2|} [D_2 D_1(a), b, c]. \end{aligned} \quad (27)$$

It follows that

$$\begin{aligned} [[D_1, D_2](a), b, c] &+ (-1)^{|a|(|D_1|+|D_2|)} [a, [D_1, D_2](b), c] \\ &+ (-1)^{(|a|+|b|)(|D_1|+|D_2|)} [a, b, [D_1, D_2](c)] = 0. \end{aligned} \quad (28)$$

Therefore, $[D_1, D_2] \subseteq QDer(T)$ and $[QC(T), QC(T)] \subseteq QDer(T)$.

(4) For any $D \in C_k(T)$ and $a, b, c \in T$, we have

$$\begin{aligned} D([a, b, c]) &= \delta^k [D(a), b, c] \\ &= \delta^k (-1)^{|D||a|} [a, D(b), c] \\ &= \delta^k (-1)^{|D|(|a|+|b|)} [a, b, D(c)]. \end{aligned} \quad (29)$$

Thus

$$\begin{aligned} \delta^k [D(a), b, c] &+ \delta^k (-1)^{|D||a|} [a, D(b), c] \\ &+ \delta^k (-1)^{|D|(|a|+|b|)} [a, b, D(c)] = 3D([a, b, c]), \end{aligned} \quad (30)$$

that is, $D \in QDer_k(T)$ and $QC(T) \subseteq QDer(T)$. $\square$

Theorem 16. Let T be a δ -Jordan Lie supertriple system; then $[C(T), QC(T)] \subseteq End(T, Z(T))$. Moreover, if $Z(T) = \{0\}$, then $[C(T), QC(T)] = \{0\}$.

Proof. For any $D_1 \in C_k(T)$, $D_2 \in QC_l(T)$, and $a, b, c \in T$, we have

$$\begin{aligned} [[D_1, D_2](a), b, c] &= [D_1 D_2(a), b, c] - (-1)^{|D_1||D_2|} [D_2 D_1(a), b, c] \\ &= \delta^k D_1([D_2(a), b, c]) \\ &\quad - (-1)^{|D_1||D_2|} (-1)^{(|D_1|+|a|)|D_2|} [D_1(a), D_2(b), c] \\ &= \delta^k (-1)^{|a||D_2|} D_1([a, D_2(b), c]) \\ &\quad - (-1)^{|a||D_2|} D_1([a, D_2(b), c]) = 0. \end{aligned} \quad (31)$$

So $[D_1, D_2](a) \subseteq Z(T)$ and therefore $[C(T), QC(T)] \subseteq End(T, Z(T))$. Moreover, if $Z(T) = \{0\}$, then it is easy to see that $[C(T), QC(T)] = \{0\}$. $\square$

Theorem 17. Let T be a δ -Jordan Lie supertriple system and $Char\ k \neq 2$; then $ZDert = C(T) \cap Der(T)$.

Proof. For any $D \in C_k(T) \cap Der_k(T)$ and $a, b, c \in T$, we have

$$\begin{aligned} D([a, b, c]) &= \delta^k [D(a), b, c] \\ &\quad + \delta^k (-1)^{|D||a|} [a, D(b), c] \\ &\quad + \delta^k (-1)^{|D|(|a|+|b|)} [a, b, D(c)], \\ D([a, b, c]) &= \delta^k [D(a), b, c] \\ &= \delta^k (-1)^{|D||a|} [a, D(b), c] \\ &= \delta^k (-1)^{|D|(|a|+|b|)} [a, b, D(c)]. \end{aligned} \quad (32)$$

It follows that $2D([a, b, c]) = 0$. Thus $D([a, b, c]) = 0$ since $Char\ k \neq 2$; that is, $D \in ZDert$. So $C(T) \cap Der(T) \subseteq ZDert$.

On the other hand, for any $D \in ZDert$ and $a, b, c \in T$, we have $D([a, b, c]) = 0$. Clearly, (6) and (11) hold; that is, $D \in C_k(T) \cap Der_k(T)$, and, therefore, $ZDert \subseteq C(T) \cap Der(T)$. And this completes the proof. $\square$

3. The Cohomology of δ -Jordan Lie Supertriple Systems

In this section, we introduce the notion of the representation of δ -Jordan Lie supertriple systems and present its low-dimensional cohomologies.

Definition 18. Let T be a δ -Jordan Lie supertriple system and V a Z_2 -graded vector space. Suppose that there exists a bilinear mapping $\theta : T \otimes T \rightarrow End(V)$ satisfying the following axioms:

$$\begin{aligned} &(-1)^{(|a|+|b|)(|c|+|d|)} \theta(c, d) \theta(a, b) \\ &- \delta (-1)^{(|a||b|+|d|(|c|+|a|))} \theta(b, d) \theta(a, c) \\ &- \theta(a, [b, c, d]) \end{aligned} \quad (33)$$

$$+ (-1)^{|a|(|b|+|c|)} D(b, c) \theta(a, d) = 0,$$

$$\begin{aligned} &\delta (-1)^{(|a|+|b|)(|c|+|d|)} \theta(c, d) D(a, b) - \delta D(a, b) \theta(c, d) \\ &+ \theta([a, b, c], d) \end{aligned} \quad (34)$$

$$+ \delta (-1)^{|c|(|a|+|b|)} \theta(c, [a, b, d]) = 0,$$

$$\begin{aligned} &D([a, b, c], d) + (-1)^{|c|(|a|+|b|)} D(c, [a, b, d]) \\ &- \delta D(a, b) D(c, d) \end{aligned} \quad (35)$$

$$+ (-1)^{(|a|+|b|)(|c|+|d|)} D(c, d) D(a, b),$$

for $a, b, c, d \in T$, where $D(a, b) = (-1)^{|a||b|} \theta(b, a) - \delta \theta(a, b)$; then (V, θ) is called the representation of T ; V is called a T -module.

Example 19. Let T be a δ -Jordan Lie supertriple system. Define $\theta : T \otimes T \longrightarrow \text{End}(T)$ by $\theta(a, b)(x) = (-1)^{|x|(|a|+|b|)}[x, a, b]$. It is not hard to check that $D(a, b)(x) = \delta[a, b, x]$ and T itself is a T -module. In this case, (T, θ) is said to be the adjoint representation of T .

Proposition 20. *Let T be a δ -Jordan Lie supertriple system and (V, θ) the representation. Then $T \oplus V$ has a structure of a δ -Jordan Lie supertriple system.*

Proof. Define a triple linear product $[\cdot, \cdot, \cdot] : (T \oplus V) \otimes (T \oplus V) \otimes (T \oplus V) \longrightarrow (T \oplus V)$ by

$$\begin{aligned} & [(a, u), (b, v), (c, w)] \\ &= ([a, b, c], (-1)^{|a|(|b|+|c|)} \theta(b, c)(u) \\ &\quad - \delta(-1)^{|b||c|} \theta(a, c)(v) + \delta D(a, b)(w)), \end{aligned} \quad (36)$$

for all $(a, u), (b, v), (c, w) \in T \oplus V$, where $|a, u| = |a|$.

Now we check that the operation $[\cdot, \cdot, \cdot]$ defined above satisfies axioms in Definition 1. It is easy to see that (1) holds since T is a δ -Jordan Lie supertriple system.

For (2), we take any $(a, u), (b, v), (c, w) \in T \oplus V$ and compute

$$\begin{aligned} & -\delta(-1)^{|a||b|} [(b, v), (a, u), (c, w)] = -\delta(-1)^{|a||b|} \\ & \cdot ([b, a, c], (-1)^{|b|(|a|+|c|)} \theta(a, c)(v) \\ & - \delta(-1)^{|a||c|} \theta(b, c)(u) + \delta D(b, a)(w)) \\ &= (-\delta(-1)^{|a||b|} [b, a, c], -\delta(-1)^{|b||c|} \theta(a, c)(v) \\ & - (-1)^{|a|(|b|+|c|)} \theta(b, c)(u) \\ & - \delta(-1)^{|a||b|} \delta D(b, a)(w)) \\ &= ([a, b, c], (-1)^{|a|(|b|+|c|)} \theta(b, c)(u) \\ & - \delta(-1)^{|b||c|} \theta(a, c)(v) + \delta D(a, b)(w)) \\ &= [(a, u), (b, v), (c, w)]. \end{aligned} \quad (37)$$

The last equality holds since $-(-1)^{|a||b|} [b, a, w] = [a, b, w] = \delta D(a, b)(w)$.

For (3), we have

$$\begin{aligned} & (-1)^{|a||c|} [(a, u), (b, v), (c, w)] + (-1)^{|b||a|} \\ & \cdot [(b, v), (c, w), (a, u)] + (-1)^{|c||b|} \\ & \cdot [(c, w), (a, u), (b, v)] = (-1)^{|a||c|} [a, b, c] \\ & + (-1)^{|b||a|} [b, c, a] + (-1)^{|c||b|} [c, a, b], \Omega = (0, \Omega), \end{aligned} \quad (38)$$

where

$$\begin{aligned} \Omega &= (-1)^{|a||b|} \theta(b, c)(u) - \delta(-1)^{(|a|+|b|)|c|} \theta(a, c)(v) \\ &\quad + \delta(-1)^{|a||c|} D(a, b)(w) + (-1)^{|b||c|} \theta(c, a)(v) \\ &\quad - \delta(-1)^{(|b|+|c|)|a|} \theta(b, a)(w) \\ &\quad + \delta(-1)^{|b||a|} D(b, c)(u) + (-1)^{|c||b|} \theta(a, b)(w) \\ &\quad - \delta(-1)^{(|c|+|a|)|b|} \theta(c, b)(u) \\ &\quad + \delta(-1)^{|c||b|} D(c, a)(v) \\ &= (-1)^{|a||b|} \theta(b, c)(u) + (-1)^{|b||c|} \theta(c, a)(v) \\ &\quad + (-1)^{|c||b|} \theta(a, b)(w) - (-1)^{|a||c|} \theta(a, b)(w) \\ &\quad + (-1)^{|b||a|} \theta(b, c)(u) - (-1)^{|b||c|} \theta(c, a)(v) = 0. \end{aligned} \quad (39)$$

The second equality holds since $D(a, b) = (-1)^{|a||b|} \theta(b, a) - \delta \theta(a, b)$. Then we have

$$\begin{aligned} & (-1)^{|a||c|} [(a, u), (b, v), (c, w)] \\ & + (-1)^{|b||a|} [(b, v), (c, w), (a, u)] \\ & + (-1)^{|c||b|} [(c, w), (a, u), (b, v)] = (0, 0), \end{aligned} \quad (40)$$

as desired.

For (4), we take any $(a, u), (b, v), (c, w), (d, m), (e, n) \in T \oplus V$. First, we calculate the following expression:

$$\begin{aligned} & [[(a, u), (b, v), (c, w)], (d, m), (e, n)] \\ &= [[([a, b, c], (-1)^{|a|(|b|+|c|)} \theta(b, c)(u) \\ &\quad - \delta(-1)^{|b||c|} \theta(a, c)(v) + \delta D(a, b)(w)), (d, m), \\ &\quad (e, n)] = ([[a, b, c], d, e], \Omega_1), \end{aligned} \quad (41)$$

where

$$\begin{aligned} \Omega_1 &= (-1)^{(|a|+|b|+|c|)(|d|+|e|)} (-1)^{|a|(|b|+|c|)} \theta(d, e) \theta(b, c)(u) \\ &\quad - \delta(-1)^{(|a|+|b|+|c|)(|d|+|e|)} (-1)^{|b||c|} \theta(d, e) \theta(a, c)(v) \\ &\quad + \delta(-1)^{(|a|+|b|+|c|)(|d|+|e|)} \theta(d, e) D(a, b)(w) \\ &\quad - \delta(-1)^{|d||e|} \theta([a, b, c], e)(m) \\ &\quad + \delta D([a, b, c], d)(n). \end{aligned} \quad (42)$$

Second, we compute the expression $(-1)^{|c|(|a|+|b|)} [(c, w), [(a, u), (b, v), (d, m)], (e, n)]$:

$$\begin{aligned}
& (-1)^{|c|(|a|+|b|)} [(c, w), [(a, u), (b, v), (d, m)], (e, n)] \\
& = (-1)^{|c|(|a|+|b|)} [(c, w), \\
& \quad ([a, b, d], (-1)^{|a|(|b|+|d|)} \theta(b, d)(u) \\
& \quad - \delta(-1)^{|b||d|} \theta(a, d)(v) + \delta D(a, b)(m)], (e, n)] \\
& = ((-1)^{|c|(|a|+|b|)} [c, [a, b, d], e], \Omega_2),
\end{aligned} \tag{43}$$

where

$$\begin{aligned}
\Omega_2 = & -\delta(-1)^{(|a|+|b|+|d|)|e|} (-1)^{|c|(|a|+|b|)+|a|(|b|+|d|)} \theta(c, e) \\
& \cdot \theta(b, d)(u) + (-1)^{(|a|+|b|+|d|)|e|} \\
& \cdot (-1)^{|c|(|a|+|b|)+|b||d|} \theta(c, e) \theta(a, d)(v) \\
& + (-1)^{(|a|+|b|+|d|+|e|)|c|} (-1)^{|c|(|a|+|b|)} \theta([a, b, d], e) \\
& \cdot (w) - (-1)^{(|a|+|b|+|d|)|e|} (-1)^{|c|(|a|+|b|)} \theta(c, e) \\
& \cdot D(a, b)(m) + \delta(-1)^{|c|(|a|+|b|)} D(c, [a, b, d]) \\
& \cdot (n).
\end{aligned} \tag{44}$$

Third, we compute the expression $\delta(-1)^{(|a|+|b|)(|c|+|d|)} [(c, w), (d, m), [(a, u), (b, v), (e, n)]]$:

$$\begin{aligned}
& \delta(-1)^{(|a|+|b|)(|c|+|d|)} [(c, w), (d, m), \\
& \quad [(a, u), (b, v), (e, n)]] = \delta(-1)^{(|a|+|b|)(|c|+|d|)} [(c, w), \\
& \quad (d, m), ([a, b, e], (-1)^{|a|(|b|+|e|)} \theta(b, e)(u) \\
& \quad - \delta(-1)^{|b||e|} \theta(a, e)(v) + \delta D(a, b)(n))] \\
& = (\delta(-1)^{(|a|+|b|)(|c|+|d|)} [c, d, [a, b, e]], \Omega_3),
\end{aligned} \tag{45}$$

where

$$\begin{aligned}
\Omega_3 = & (-1)^{(|a|+|b|)(|c|+|d|)} (-1)^{|a|(|b|+|e|)} D(c, d) \theta(b, e)(u) \\
& - \delta(-1)^{(|a|+|b|)(|c|+|d|)} (-1)^{|b||e|} D(c, d) \theta(a, e)(v) \\
& + \delta(-1)^{(|a|+|b|)(|c|+|d|)} (-1)^{(|a|+|b|+|d|+|e|)|c|} \\
& \cdot \theta(d, [a, b, e])(w) - (-1)^{(|a|+|b|)(|c|+|d|)} \\
& \cdot (-1)^{(|a|+|b|+|e|)|d|} \theta(c, [a, b, e])(m) \\
& + \delta(-1)^{(|a|+|b|)(|c|+|d|)} D(c, d) D(a, b)(n).
\end{aligned} \tag{46}$$

Fourth, we compute the expression $[(a, u), (b, v), [(c, w), (d, m), (e, n)]]$:

$$\begin{aligned}
& [(a, u), (b, v), [(c, w), (d, m), (e, n)]] = [(a, u), (b, v), \\
& \quad ([c, d, e], (-1)^{|c|(|d|+|e|)} \theta(d, e)(w) \\
& \quad - \delta(-1)^{|d||e|} \theta(c, e)(m) + \delta D(c, d)(n))] \\
& = ([a, b, [c, d, e]], \Omega_4),
\end{aligned} \tag{47}$$

where

$$\begin{aligned}
\Omega_4 = & (-1)^{|a|(|b|+|c|+|d|+|e|)} \theta(b, [c, d, e])(u) \\
& - \delta(-1)^{|b|(|c|+|d|+|e|)} \theta(a, [c, d, e])(v) \\
& + \delta(-1)^{|c|(|d|+|e|)} D(a, b) \theta(d, e)(w) \\
& - (-1)^{|d||e|} D(a, b) \theta(c, e)(m) \\
& + D(a, b) D(c, d)(n).
\end{aligned} \tag{48}$$

Finally, by (33), (34), and (35), we have

$$\begin{aligned}
& [[(a, u), (b, v), (c, w)], (d, m), (e, n)] + (-1)^{|c|(|a|+|b|)} \\
& \cdot [(c, w), [(a, u), (b, v), (d, m)], (e, n)] \\
& + \delta(-1)^{(|a|+|b|)(|c|+|d|)} \\
& \cdot [(c, w), (d, m), [(a, u), (b, v), (e, n)]] \\
& = ([a, b, c], d, e) + (-1)^{|c|(|a|+|b|)} [c, [a, b, d], e] \\
& + \delta(-1)^{(|a|+|b|)(|c|+|d|)} [c, d, [a, b, e]], \Omega_1 + \Omega_2 \\
& + \Omega_3 = ([a, b, [c, d, e]], \Omega_4) \\
& = [(a, u), (b, v), [(c, w), (d, m), (e, n)]],
\end{aligned} \tag{49}$$

as desired, and this finishes the proof. $\square$

Corollary 21. Any δ -Jordan Lie supertriple system can be considered as a subspace of a δ -Jordan Lie superalgebra.

Proof. It is straightforward from Example 19 and Proposition 20. $\square$

Definition 22. Let T be a δ -Jordan Lie supertriple system and V a T -module by a bilinear map θ . If an n -linear map $f : T \times T \times \cdots \times T \rightarrow T$ satisfies the following axioms

$$\begin{aligned}
(1) \quad & f(x_1, x_2, \dots, x, y, \dots, x_n) = \\
& -\delta(-1)^{|x||y|} f(x_1, x_2, \dots, x, y, \dots, x_n); \\
(2) \quad & (-1)^{|x||z|} f(x_1, x_2, \dots, x_{n-3}, x, y, z) + \\
& (-1)^{|y||x|} f(x_1, x_2, \dots, x_{n-3}, y, z, x) + \\
& (-1)^{|z||y|} f(x_1, x_2, \dots, x_{n-3}, z, x, y) = 0,
\end{aligned}$$

then f is called an n -cochain on T . Denote by $C_\delta^n(T, V)$ the set of all n -cochains, for $n \geq 1$.

Definition 23. Let T be a δ -Jordan Lie supertriple system and V a T -module by a bilinear map θ . For $n = 1, 2$, the coboundary operator $d_n : C_\delta^n(T, V) \rightarrow C_\delta^{n+2}(T, V)$ is defined as follows:

$$\begin{aligned}
& (i) \text{ If } f \in C_\delta^1(T, V), \text{ then} \\
& d^1 f(x_1, x_2, x_3) \\
& = (-1)^{(|f|+|x_1|)(|x_2|+|x_3|)} \theta(x_2, x_3) f(x_1) \\
& - f([x_1, x_2, x_3])
\end{aligned}$$

$$\begin{aligned}
& + \delta(-1)^{|f|(|x_1|+|x_2|)} D(x_1, x_2) f(x_3) \\
& - \delta(-1)^{|x_2||x_3|+|f|(|x_1|+|x_3|)} \theta(x_1, x_3) f(x_2).
\end{aligned} \tag{50}$$

(ii) If $f \in C_\delta^2(T, V)$, then

$$\begin{aligned}
& d^2 f(y, x_1, x_2, x_3) \\
& = (-1)^{(|f|+|y|+|x_1|)(|x_2|+|x_3|)} \theta(x_2, x_3) f(y, x_1) \\
& \quad - f(y, [x_1, x_2, x_3]) \\
& \quad - \delta(-1)^{|x_2||x_3|+|f|(|x_1|+|x_3|)} \theta(x_1, x_3) f(y, x_2) \\
& \quad + \delta(-1)^{(|f|+|y|)(|x_1|+|x_2|)} D(x_1, x_2) f(y, x_3).
\end{aligned} \tag{51}$$

(iii) If $f \in C_\delta^3(T, V)$, then

$$\begin{aligned}
& d^3 f(x_1, x_2, x_3, x_4, x_5) = (-1)^{(|f|+|x_1|+|x_2|+|x_3|)(|x_4|+|x_5|)} \\
& \quad \cdot \theta(x_4, x_5) f(x_1, x_2, x_3) \\
& \quad - \delta(-1)^{(|f|+|x_1|+|x_2|)(|x_3|+|x_5|)+|x_4||x_5|} \theta(x_3, x_5) \\
& \quad \cdot f(x_1, x_2, x_4) - \delta(-1)^{|f|(|x_1|+|x_2|)} D(x_1, x_2) \\
& \quad \cdot f(x_3, x_4, x_5) + (-1)^{(|f|+|x_1|+|x_2|)(|x_3|+|x_4|)} \\
& \quad \cdot D(x_3, x_4) f(x_1, x_2, x_5) + f([x_1, x_2, x_3], x_4, x_5) \\
& \quad - f(x_1, x_2, [x_3, x_4, x_5]) + (-1)^{|x_3|(|x_1|+|x_2|)} \\
& \quad \cdot f(x_3, [x_1, x_2, x_4], x_5) + \delta(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)} \\
& \quad \cdot f(x_3, x_4, [x_1, x_2, x_5]).
\end{aligned} \tag{52}$$

(iv) If $f \in C_\delta^4(T, V)$, then

$$\begin{aligned}
& d^4 f(y, x_1, x_2, x_3, x_4, x_5) \\
& = (-1)^{(|f|+|y|+|x_1|+|x_2|+|x_3|)(|x_4|+|x_5|)} \theta(x_4, x_5) \\
& \quad \cdot f(y, x_1, x_2, x_3) \\
& \quad - \delta(-1)^{(|f|+|y|+|x_1|+|x_2|)(|x_3|+|x_5|)+|x_4||x_5|} \theta(x_3, x_5) \\
& \quad \cdot f(y, x_1, x_2, x_4) - \delta(-1)^{(|f|+|y|)(|x_1|+|x_2|)} \\
& \quad \cdot D(x_1, x_2) f(y, x_3, x_4, x_5) \\
& \quad + (-1)^{(|f|+|y|+|x_1|+|x_2|)(|x_3|+|x_4|)} D(x_3, x_4) \\
& \quad \cdot f(y, x_1, x_2, x_5) + f(y, [x_1, x_2, x_3], x_4, x_5) \\
& \quad - f(y, x_1, x_2, [x_3, x_4, x_5]) + (-1)^{|x_3|(|x_1|+|x_2|)} \\
& \quad \cdot f(y, x_3, [x_1, x_2, x_4], x_5) \\
& \quad + \delta(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)} f(y, x_3, x_4, [x_1, x_2, x_5]).
\end{aligned} \tag{53}$$

Theorem 24. Let T be a δ -Jordan Lie supertriple system and V a T -module by a bilinear map θ . The coboundary operator d^n defined above satisfies $d^{n+2}d^n = 0$, $n = 1, 2$.

Proof. From the definition of the coboundary operator, it follows immediately that $d^3d^1 = 0$ implies $d^4d^2 = 0$. Then, we only need to prove $d^3d^1 = 0$:

$$d^3(d^1 f)(x_1, x_2, x_3, x_4, x_5) \tag{54}$$

$$= (-1)^{(|f|+|x_1|+|x_2|+|x_3|)(|x_4|+|x_5|)} \theta(x_4, x_5) (d^1 f) \tag{55}$$

$$\begin{aligned}
& \cdot (x_1, x_2, x_3) \\
& - \delta(-1)^{(|f|+|x_1|+|x_2|)(|x_3|+|x_5|)+|x_4||x_5|} \theta(x_3, x_5) (d^1 f) \\
& \cdot (x_1, x_2, x_4)
\end{aligned} \tag{56}$$

$$- \delta(-1)^{|f|(|x_1|+|x_2|)} D(x_1, x_2) (d^1 f)(x_3, x_4, x_5) \tag{57}$$

$$+ (-1)^{(|f|+|x_1|+|x_2|)(|x_3|+|x_4|)} D(x_3, x_4) (d^1 f)(x_1, x_2, x_5) \tag{58}$$

$$\begin{aligned}
& + (d^1 f)([x_1, x_2, x_3], x_4, x_5) - (d^1 f) \\
& \cdot (x_1, x_2, [x_3, x_4, x_5])
\end{aligned} \tag{59}$$

$$+ (-1)^{|x_3|(|x_1|+|x_2|)} (d^1 f)(x_3, [x_1, x_2, x_4], x_5) \tag{60}$$

$$+ \delta(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)} (d^1 f)(x_3, x_4, [x_1, x_2, x_5]). \tag{61}$$

By Definition 23, we have

$$\begin{aligned}
(55) & = (-1)^{(|f|+|x_1|+|x_2|+|x_3|)(|x_4|+|x_5|)} \theta(x_4, x_5) (d^1 f) \\
& \cdot (x_1, x_2, x_3) = (-1)^{(|f|+|x_1|+|x_2|+|x_3|)(|x_4|+|x_5|)} \\
& \cdot \theta(x_4, x_5) \left\{ -f([x_1, x_2, x_3]) + (-1)^{(|f|+|x_1|)(|x_2|+|x_3|)} \right. \\
& \cdot \theta(x_2, x_3) f(x_1) + \delta(-1)^{|f|(|x_1|+|x_2|)} D(x_1, x_2) \\
& \cdot f(x_3) - \delta(-1)^{|x_2||x_3|+|f|(|x_1|+|x_3|)} \theta(x_1, x_3) \\
& \cdot f(x_2) \left. \right\},
\end{aligned} \tag{62}$$

$$\begin{aligned}
(56) & = -\delta(-1)^{(|f|+|x_1|+|x_2|)(|x_3|+|x_5|)+|x_4||x_5|} \theta(x_3, x_5) \\
& \cdot (d^1 f)(x_1, x_2, x_4) \\
& = -\delta(-1)^{(|f|+|x_1|+|x_2|)(|x_3|+|x_5|)+|x_4||x_5|} \theta(x_3, x_5) \\
& \cdot \left\{ -f([x_1, x_2, x_4]) + (-1)^{(|f|+|x_1|)(|x_2|+|x_4|)} \theta(x_2, x_4) \right. \\
& \cdot f(x_1) + \delta(-1)^{|f|(|x_1|+|x_2|)} D(x_1, x_2) f(x_4) \\
& \left. - \delta(-1)^{|x_2||x_4|+|f|(|x_1|+|x_4|)} \theta(x_1, x_4) f(x_2) \right\},
\end{aligned} \tag{63}$$

$$\begin{aligned}
(57) &= -\delta(-1)^{|f|(|x_1|+|x_2|)} D(x_1, x_2) (d^1 f)(x_3, x_4, x_5) \\
&= -\delta(-1)^{|f|(|x_1|+|x_2|)} D(x_1, x_2) \{ -f([x_3, x_4, x_5]) \\
&+ (-1)^{(|f|+|x_3|)(|x_4|+|x_5|)} \theta(x_4, x_5) f(x_3) \\
&+ \delta(-1)^{|f|(|x_3|+|x_4|)} D(x_3, x_4) f(x_5) \\
&- \delta(-1)^{|x_4||x_5|+|f|(|x_3|+|x_5|)} \theta(x_3, x_5) f(x_4) \},
\end{aligned} \tag{64}$$

$$\begin{aligned}
(58) &= (-1)^{(|f|+|x_1|+|x_2|)(|x_3|+|x_4|)} D(x_3, x_4) (d^1 f) \\
&\cdot (x_1, x_2, x_5) = (-1)^{(|f|+|x_1|+|x_2|)(|x_3|+|x_4|)} D(x_3, x_4) \\
&\cdot \{ -f([x_1, x_2, x_5]) + (-1)^{(|f|+|x_1|)(|x_2|+|x_5|)} \theta(x_2, x_5) \\
&\cdot f(x_1) + \delta(-1)^{|f|(|x_1|+|x_2|)} D(x_1, x_2) f(x_5) \\
&- \delta(-1)^{|x_2||x_5|+|f|(|x_1|+|x_5|)} \theta(x_1, x_5) f(x_2) \},
\end{aligned} \tag{65}$$

$$\begin{aligned}
(59) &= (d^1 f)([x_1, x_2, x_3], x_4, x_5) - (d^1 f) \\
&\cdot (x_1, x_2, [x_3, x_4, x_5]) \\
&= (-1)^{(|f|+|x_1|+|x_2|+|x_3|)(|x_4|+|x_5|)} \theta(x_4, x_5) \\
&\cdot f([x_1, x_2, x_3]) - f([x_1, x_2, x_3], x_4, x_5) \\
&+ \delta(-1)^{|f|(|x_1|+|x_2|+|x_3|+|x_4|)} D([x_1, x_2, x_3], x_4) \\
&\cdot f(x_5) - \delta(-1)^{|x_4||x_5|+|f|(|x_1|+|x_2|+|x_3|+|x_5|)} \\
&\cdot \theta([x_1, x_2, x_3], x_5) f(x_4) \\
&- (-1)^{(|f|+|x_1|)(|x_2|+|x_3|+|x_4|+|x_5|)} \theta(x_2, [x_3, x_4, x_5]) \\
&\cdot f(x_1) + f([x_1, x_2, [x_3, x_4, x_5]]) \\
&- \delta(-1)^{|f|(|x_1|+|x_2|)} D(x_1, x_2) f([x_3, x_4, x_5]) \\
&+ \delta(-1)^{|f|(|x_1|+|x_3|+|x_4|+|x_5|)+|x_2|(|x_3|+|x_4|+|x_5|)} \\
&\cdot \theta(x_1, [x_3, x_4, x_5]) f(x_2),
\end{aligned} \tag{66}$$

$$\begin{aligned}
(60) &= (-1)^{|x_3|(|x_1|+|x_2|)} (d^1 f)(x_3, [x_1, x_2, x_4], x_5) \\
&= (-1)^{|x_3|(|x_1|+|x_2|)+(|f|+|x_3|)(|x_1|+|x_2|+|x_4|+|x_5|)} \\
&\cdot \theta([x_1, x_2, x_4], x_5) f(x_3) - (-1)^{|x_3|(|x_1|+|x_2|)} \\
&\cdot f([x_3, [x_1, x_2, x_4], x_5]) \\
&+ \delta(-1)^{|x_3|(|x_1|+|x_2|)+|f|(|x_1|+|x_2|+|x_3|+|x_4|)} \\
&\cdot D(x_3, [x_1, x_2, x_4]) f(x_5) \\
&- \delta(-1)^{(|f|+|x_1|+|x_2|)(|x_3|+|x_5|)+|x_4||x_5|} \theta(x_3, x_5) \\
&\cdot f([x_1, x_2, x_4]),
\end{aligned} \tag{67}$$

$$\begin{aligned}
(61) &= \delta(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)} (d^1 f) \\
&\cdot (x_3, x_4, [x_1, x_2, x_5]) = \delta(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)} \\
&\cdot \{ (-1)^{(|f|+|x_3|)(|x_1|+|x_2|+|x_4|+|x_5|)} \theta(x_4, [x_1, x_2, x_5]) \\
&\cdot f(x_3) - f([x_3, x_4, [x_1, x_2, x_5]]) \\
&+ \delta(-1)^{|f|(|x_3|+|x_4|)} D(x_3, x_4) f([x_1, x_2, x_5]) \\
&- \delta(-1)^{|x_4|(|x_1|+|x_2|+|x_5|)+|f|(|x_1|+|x_2|+|x_3|+|x_5|)} \\
&\cdot \theta(x_3, [x_1, x_2, x_5]) f(x_4) \}.
\end{aligned} \tag{68}$$

By (62)–(68), we have

$$\begin{aligned}
&d^3 (d^1 f)(x_1, x_2, x_3, x_4, x_5) \\
&= -f([x_1, x_2, x_3], x_4, x_5) \\
&- (-1)^{|x_3|(|x_1|+|x_2|)} f(x_3, [x_1, x_2, x_4], x_5) \\
&- \delta(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)} f(x_3, x_4, [x_1, x_2, x_5]) \\
&+ f(x_1, x_2, [x_3, x_4, x_5]) \\
&+ (-1)^{(|f|+|x_1|)(|x_2|+|x_3|+|x_4|+|x_5|)} \Lambda_1 f(x_1) \\
&- \delta(-1)^{(|f|+|x_2|)(|x_3|+|x_4|+|x_5|)+|f||x_1|} \Lambda_2 f(x_2) \\
&+ (-1)^{|f|(|x_1|+|x_2|+|x_4|+|x_5|)+|x_3|(|x_4|+|x_5|)} \Lambda_3 f(x_3) \\
&- (-1)^{|f|(|x_1|+|x_2|+|x_3|+|x_5|)+|x_4||x_5|} \Lambda_4 f(x_4) \\
&+ \delta(-1)^{|f|(|x_1|+|x_2|+|x_3|+|x_4|)} \Lambda_5 f(x_5) \\
&= (-1)^{(|f|+|x_1|)(|x_2|+|x_3|+|x_4|+|x_5|)} \Lambda_1 f(x_1) \\
&- \delta(-1)^{(|f|+|x_2|)(|x_3|+|x_4|+|x_5|)+|f||x_1|} \Lambda_2 f(x_2) \\
&+ (-1)^{|f|(|x_1|+|x_2|+|x_4|+|x_5|)+|x_3|(|x_4|+|x_5|)} \Lambda_3 f(x_3) \\
&- (-1)^{|f|(|x_1|+|x_2|+|x_3|+|x_5|)+|x_4||x_5|} \Lambda_4 f(x_4) \\
&+ \delta(-1)^{|f|(|x_1|+|x_2|+|x_3|+|x_4|)} \Lambda_5 f(x_5),
\end{aligned} \tag{69}$$

where

$$\begin{aligned}
\Lambda_1 &= (-1)^{(|x_2|+|x_3|)(|x_4|+|x_5|)} \theta(x_4, x_5) \theta(x_2, x_3) \\
&- \delta(-1)^{|x_2||x_3|+(|x_2|+|x_4|)|x_5|} \theta(x_3, x_5) \theta(x_2, x_4) \\
&- \theta(x_2, [x_3, x_4, x_5]) \\
&+ \delta(-1)^{|x_2|(|x_3|+|x_4|)} D(x_3, x_4) \theta(x_2, x_5) = 0,
\end{aligned}$$

by Eq. (33)

$$\begin{aligned}
\Lambda_2 &= (-1)^{(|x_1|+|x_3|)(|x_4|+|x_5|)} \theta(x_4, x_5) \theta(x_1, x_3) \\
&\quad - \delta(-1)^{|x_1||x_3|+|x_5|(|x_1|+|x_4|)} \theta(x_3, x_5) \theta(x_1, x_4) \\
&\quad - \theta(x_1, [x_3, x_4, x_5]) \\
&\quad + \delta(-1)^{|x_1|(|x_3|+|x_4|)} D(x_3, x_4) \theta(x_1, x_5) = 0,
\end{aligned}$$

by Eq. (33)

$$\begin{aligned}
\Lambda_3 &= (-1)^{(|x_1|+|x_2|)(|x_4|+|x_5|)} \theta(x_4, x_5) D(x_1, x_2) \\
&\quad - \delta D(x_1, x_2) \theta(x_4, x_5) + \theta([x_1, x_2, x_4], x_5) \\
&\quad + \delta(-1)^{|x_4|(|x_1|+|x_2|)} \theta(x_4, [x_1, x_2, x_5]) = 0,
\end{aligned}$$

by Eq. (34)

$$\begin{aligned}
\Lambda_4 &= (-1)^{(|x_1|+|x_2|)(|x_3|+|x_5|)} \theta(x_3, x_5) D(x_1, x_2) \\
&\quad - \delta D(x_1, x_2) \theta(x_3, x_5) + \theta([x_1, x_2, x_3], x_5) \\
&\quad + \delta(-1)^{|x_3|(|x_1|+|x_2|)} \theta(x_3, [x_1, x_2, x_5]) = 0,
\end{aligned}$$

by Eq. (34)

$$\begin{aligned}
\Lambda_5 &= D([x_1, x_2, x_3], x_4) \\
&\quad + (-1)^{|x_3|(|x_1|+|x_2|)} D(x_3, [x_1, x_2, x_4]) \\
&\quad - \delta D(x_1, x_2) D(x_3, x_4) \\
&\quad + (-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)} D(x_3, x_4) D(x_1, x_2) \\
&= 0. \quad \text{by Eq. (35)}
\end{aligned}$$

It follows that $d^3(d^1 f)(x_1, x_2, x_3, x_4, x_5) = 0$, as desired. And this finishes the proof. $\square$

For $n = 1, 2$, the map $f \in C_\delta^n(T, V)$ is called an n -cocycle if $d^n f = 0$. We denote by $Z_\delta^n(T, V)$ the subspace spanned by n -cocycles and $B_\delta^n(T, V) = d^{n-2} C_\delta^{n-2}(T, V)$. By Theorem 24, $B_\delta^n(T, V)$ is a subspace of $Z_\delta^n(T, V)$. Therefore, we can define a cohomology space $H_\delta^n(T, V)$ of the δ -Jordan Lie supertriple system T as the factor space $Z_\delta^n(T, V)/B_\delta^n(T, V)$.

4. 1-Parameter Formal Deformations of δ -Jordan Lie Supertriple Systems

Let T be a δ -Jordan Lie supertriple system and $k[[t]]$ the power series ring in one variable t with coefficients in k . Assume that $T[[t]]$ is the set of formal series whose coefficients are elements of the vector space T .

Definition 25. Let T be a δ -Jordan Lie supertriple system. A 1-parameter formal deformation of T is a formal power series $f_t : T[[t]] \times T[[t]] \times T[[t]] \rightarrow T[[t]]$ given by

$$\begin{aligned}
f_t &= \sum_{i \geq 0} f_i(x_1, x_2, x_3) t^i \\
&= f_0(x_1, x_2, x_3) + f_1(x_1, x_2, x_3) t \\
&\quad + f_2(x_1, x_2, x_3) t^2 + \cdots,
\end{aligned} \tag{71}$$

where each f_i is a k -trilinear map $f_i : T \times T \times T \rightarrow T$ (extended to be $k[[t]]$ -trilinear) and $f_0(x_1, x_2, x_3) = [x_1, x_2, x_3]$, satisfying the following axioms:

$$|f_t(x_1, x_2, x_3)| = |x_1| + |x_2| + |x_3|; \tag{72}$$

$$f_t(x_2, x_1, x_3) = -\delta(-1)^{|x_1||x_2|} f_t(x_1, x_2, x_3); \tag{73}$$

$$\begin{aligned}
&(-1)^{|x_1||x_3|} f_t(x_1, x_2, x_3) + (-1)^{|x_2||x_1|} f_t(x_2, x_3, x_1) \\
&\quad + (-1)^{|x_3||x_2|} f_t(x_3, x_1, x_2) = 0;
\end{aligned} \tag{74}$$

$$\begin{aligned}
&f_t(x_1, x_2, f_t(x_3, x_4, x_5)) = (-1)^{|x_3|(|x_1|+|x_2|)} \\
&\quad \cdot f_t(x_3, f_t(x_1, x_2, x_4), x_5) \\
&\quad + f_t(f_t(x_1, x_2, x_3), x_4, x_5) \\
&\quad + \delta(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)} f_t(x_3, x_4, f_t(x_1, x_2, x_5)).
\end{aligned} \tag{75}$$

Remark 26. Equations (72)–(75) are equivalent to ($n = 0, 1, 2, \dots$)

$$|f_i(x_2, x_1, x_3)| = |x_1| + |x_2| + |x_3|; \tag{76}$$

$$f_i(x_2, x_1, x_3) = -\delta(-1)^{|x_1||x_2|} f_i(x_1, x_2, x_3); \tag{77}$$

$$\begin{aligned}
&(-1)^{|x_1||x_3|} f_i(x_1, x_2, x_3) + (-1)^{|x_2||x_1|} f_i(x_2, x_3, x_1) \\
&\quad + (-1)^{|x_3||x_2|} f_i(x_3, x_1, x_2) = 0;
\end{aligned} \tag{78}$$

$$\begin{aligned}
&\sum_{i+j=n} f_i(x_1, x_2, f_j(x_3, x_4, x_5)) = \sum_{i+j=n} \left((-1)^{|x_3|(|x_1|+|x_2|)} \right. \\
&\quad \cdot f_i(x_3, f_j(x_1, x_2, x_4), x_5) \\
&\quad + f_i(f_j(x_1, x_2, x_3), x_4, x_5) \\
&\quad + \delta(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)} \\
&\quad \cdot f_i(x_3, x_4, f_j(x_1, x_2, x_5)) \Big).
\end{aligned} \tag{79}$$

Furthermore, we can rewrite the deformation Equation (79) by the equality $\sum_{i+j=n} f_i f_j = 0$, where

$$\begin{aligned}
&f_i f_j(x_1, x_2, x_3, x_4, x_5) = -f_i(x_1, x_2, f_j(x_3, x_4, x_5)) \\
&\quad + (-1)^{|x_3|(|x_1|+|x_2|)} f_i(x_3, f_j(x_1, x_2, x_4), x_5) \\
&\quad + f_i(f_j(x_1, x_2, x_3), x_4, x_5) \\
&\quad + \delta(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)} f_i(x_3, x_4, f_j(x_1, x_2, x_5)).
\end{aligned} \tag{80}$$

When $n = 1$, (79) is equivalent to $f_0 f_1 + f_1 f_0 = 0$. When $n \geq 2$, (79) is equivalent to $-(f_0 f_n + f_n f_0) = f_1 f_{n-1} + f_2 f_{n-2} + \cdots + f_{n-1} f_1$.

By Example 19, (T, θ) is the adjoint representation of T itself, where $\theta(a, b)(x) = (-1)^{|x|(|a|+|b|)} [x, a, b]$ and $D(a, b)(x) = \delta[a, b, x]$. It is easy to see that $f_i \in C_\delta^3(T, V)$ and

therefore $f_i f_j \in C_\delta^5(T, V)$. Since $f_0(x_1, x_2, x_3) = [x_1, x_2, x_3]$, we have

$$\begin{aligned}
 f_0 f_1(x_1, x_2, x_3, x_4, x_5) &= -f_0(x_1, x_2, f_1(x_3, x_4, x_5)) \\
 &+ (-1)^{|x_3|(|x_1|+|x_2|)} f_0(x_3, f_1(x_1, x_2, x_4), x_5) \\
 &+ f_0(f_1(x_1, x_2, x_3), x_4, x_5) \\
 &+ \delta (-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)} f_0(x_3, x_4, f_1(x_1, x_2, x_5)) \\
 &= -[x_1, x_2, f_1(x_3, x_4, x_5)] + (-1)^{|x_3|(|x_1|+|x_2|)} \\
 &\cdot [x_3, f_1(x_1, x_2, x_4), x_5] + [f_1(x_1, x_2, x_3), x_4, x_5] \\
 &+ \delta (-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)} [x_3, x_4, f_1(x_1, x_2, x_5)] \quad (81) \\
 &= -\delta D(x_1, x_2) f(x_3, x_4, x_5) \\
 &- \delta (-1)^{(|x_1|+|x_2|)(|x_3|+|x_5|)+|x_4||x_5|} D(x_3, x_4) \\
 &\cdot f(x_1, x_2, x_5) + (-1)^{(|x_1|+|x_2|+|x_3|)(|x_4|+|x_5|)} \\
 &\cdot \theta(x_4, x_5) f(x_1, x_2, x_3) + (-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)} \\
 &\cdot D(x_3, x_4) f(x_1, x_2, x_5)
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 f_1 f_0(x_1, x_2, x_3, x_4, x_5) &= -f_1(x_1, x_2, f_0(x_3, x_4, x_5)) \\
 &+ (-1)^{|x_3|(|x_1|+|x_2|)} f_1(x_3, f_0(x_1, x_2, x_4), x_5) \\
 &+ f_1(f_0(x_1, x_2, x_3), x_4, x_5) \\
 &+ \delta (-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)} f_1(x_3, x_4, f_0(x_1, x_2, x_5)) \quad (82) \\
 &= -f_1(x_1, x_2, [x_3, x_4, x_5]) \\
 &+ (-1)^{|x_3|(|x_1|+|x_2|)} f_1(x_3, [x_1, x_2, x_4], x_5) \\
 &+ f_1([x_1, x_2, x_3], x_4, x_5) \\
 &+ \delta (-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)} f_1(x_3, x_4, [x_1, x_2, x_5]).
 \end{aligned}$$

It follows that

$$\begin{aligned}
 (f_0 f_1 + f_1 f_0)(x_1, x_2, x_3, x_4, x_5) \\
 = d^3 f_1(x_1, x_2, x_3, x_4, x_5). \quad (83)
 \end{aligned}$$

Therefore, we deduce $d^3 f_1 = 0$ since $f_0 f_1 + f_1 f_0 = 0$. Also we can obtain $-d^3 f_n = f_1 f_{n-1} + f_2 f_{n-2} + \dots + f_{n-1} f_1$. And f_1 is called the infinitesimal deformation of f_t .

Definition 27. Let T be a δ -Jordan Lie supertriple system. Two 1-parameter formal deformations f_t and f'_t of T are said to

be equivalent, denoted by $f_t \sim f'_t$, if there exists a formal isomorphism of $k[[t]]$ -modules

$$\begin{aligned}
 \phi_t(x) &= \sum_{i \geq 0} \phi_i(x) t^i : (T[[t]], f_t, \delta) \\
 &\longrightarrow (T[[t]], f'_t, \delta), \quad (84)
 \end{aligned}$$

where $\phi_0 = id_T$, $\phi_i : T \rightarrow T$ is a k -linear map (extended to be $k[[t]]$ -linear) such that

$$\begin{aligned}
 \phi_t f_t(x_1, x_2, x_3) &= f'_t(\phi_t(x_1), \phi_t(x_2), \phi_t(x_3)), \\
 \forall x_1, x_2, x_3 \in T. \quad (85)
 \end{aligned}$$

In particular, if $f_1 = f_2 = \dots = 0$, then $f_1 = f_0$ is called the null deformation. If $f_t \sim f_0$, then f_t is called the trivial deformation. If every 1-parameter formal deformation f_t is trivial, then T is called an analytically rigid δ -Jordan Lie supertriple system.

Theorem 28. Let $f_t = \sum_{i \geq 0} f_i(x_1, x_2, x_3) t^i$ and $f'_t = \sum_{i \geq 0} f'_i(x_1, x_2, x_3) t^i$ be two equivalent 1-parameter formal deformations of T . Then the infinitesimal deformations f_1 and f'_1 belong to the same cohomology class in $H_\delta^3(T, T)$.

Proof. By the assumption that f_1 and f'_1 are equivalent, there exists a formal isomorphism $\phi_t(x) = \sum_{i \geq 0} \phi_i(x) t^i$ of $k[[t]]$ -modules satisfying

$$\begin{aligned}
 \sum_{i \geq 0} \phi_i \left(\sum_{j \geq 0} f_j(x_1, x_2, x_3) t^j \right) &= \sum_{i \geq 0} f'_i \left(\sum_{k \geq 0} \phi_k(x_1) t^k, \right. \\
 &\left. \sum_{l \geq 0} \phi_l(x_2) t^l, \sum_{m \geq 0} \phi_m(x_3) t^m \right) t^i, \quad (86)
 \end{aligned}$$

for any $x_1, x_2, x_3 \in T$. Compared with the coefficients of t^1 for two sides of the above equation, we have

$$\begin{aligned}
 f_1(x_1, x_2, x_3) + \phi_1([x_1, x_2, x_3]) \\
 = f'_1(x_1, x_2, x_3) + [\phi_1(x_1), x_2, x_3] \\
 + [x_1, \phi_1(x_2), x_3] + [x_1, x_2, \phi_1(x_3)]. \quad (87)
 \end{aligned}$$

It follows that

$$\begin{aligned}
 f_1(x_1, x_2, x_3) - f'_1(x_1, x_2, x_3) \\
 = [\phi_1(x_1), x_2, x_3] + [x_1, \phi_1(x_2), x_3] \\
 + [x_1, x_2, \phi_1(x_3)] - \phi_1([x_1, x_2, x_3]) \\
 = (-1)^{|x_1|(|x_2|+|x_3|)} \theta(x_2, x_3) \phi_1(x_1) \\
 - \phi_1([x_1, x_2, x_3]) + \delta D(x_1, x_2) \phi_1(x_3) \\
 - \delta (-1)^{|x_2||x_3|} \theta(x_1, x_3) \phi_1(x_2) \\
 = d^1 \phi_1(x_1, x_2, x_3). \quad (88)
 \end{aligned}$$

So $f_1 - f'_1 = d^1 \phi_1 \in B_\delta^3(T, T)$, as desired. The proof is completed. $\square$

Theorem 29. Let T be a δ -Jordan Lie supertriple system with $H_\delta^3(T, T) = 0$; then T is analytically rigid.

Proof. Let $f_t = \sum_{i \geq 0} f_i t^i$ be a 1-parameter formal deformation of T . Then $d^3 f_n = f_1 f_{n-1} + f_2 f_{n-2} + \cdots + f_{n-1} f_1 = 0$. By the assumption $H_\delta^3(T, T) = 0$, we have $f_n \in Z_\delta^3(T, T) = B_\delta^3(T, T)$; that is, there exists $g_n \in C_\delta^1(T, T)$ such that $f_n = d^1 g^n$.

Set $\phi_t = id_T - g_n t^n$; then

$$\begin{aligned} & \phi_t (id_T + g_n t^n + g_n^2 t^{2n} + g_n^3 t^{3n} + \cdots) \\ &= (id_T - g_n t^n) (id_T + g_n t^n + g_n^2 t^{2n} + g_n^3 t^{3n} + \cdots) \\ &= (id_T + g_n t^n + g_n^2 t^{2n} + g_n^3 t^{3n} + \cdots) \\ & \quad - (g_n t^n + g_n^2 t^{2n} + g_n^3 t^{3n} + \cdots) = id_T. \end{aligned} \quad (89)$$

Similarly, one may check that $(id_T + g_n t^n + g_n^2 t^{2n} + g_n^3 t^{3n} + \cdots) \phi_t = id_T$. So $\phi_t : (T[[t]], f_t, \delta) \rightarrow (T[[t]], f'_t, \delta)$ is a linear isomorphism. Thus we can define another 1-parameter formal deformation by ϕ_t^{-1} in the form of

$$f'_t(x_1, x_2, x_3) = \phi_t^{-1} f_t(\phi_t(x_1), \phi_t(x_2), \phi_t(x_3)). \quad (90)$$

Obviously, $f_t \sim f'_t$. Set $f'_t = \sum_{i \geq 0} f'_i t^i$; then we have

$$\begin{aligned} & \sum_{i \geq 0} f'_i(x_1, x_2, x_3) t^i - \sum_{i \geq 0} g_n f_i(x_1, x_2, x_3) t^{i+n} = f_0(x_1, \\ & x_2, x_3) - \{f_0(g_n(x_1), x_2, x_3) + f_0(x_1, g_n(x_2), x_3) \\ & + f_0(x_1, x_2, g_n(x_3))\} t^n \\ & + \{f_0(g_n(x_1), g_n(x_2), x_3) \\ & + f_0(x_1, g_n(x_2), g_n(x_3)) \\ & + f_0(x_1, g_n(x_2), g_n(x_3))\} t^{2n} + f_0(g_n(x_1), \\ & g_n(x_2), g_n(x_3)) t^{3n} + \sum_{i \geq n} f_i(x_1, x_2, x_3) t^i \\ & - \sum_{i \geq n} \{f_i(g_n(x_1), x_2, x_3) + f_i(x_1, g_n(x_2), x_3) \\ & + f_i(x_1, x_2, g_n(x_3))\} t^{i+n} \\ & + \sum_{i \geq n} \{f_i(g_n(x_1), g_n(x_2), x_3) \\ & + f_i(x_1, g_n(x_2), g_n(x_3)) \\ & + f_i(g_n(x_1), x_2, g_n(x_3))\} t^{i+2n} \\ & - \sum_{i \geq n} f_i(g_n(x_1), g_n(x_2), g_n(x_3)) t^{i+3n}. \end{aligned} \quad (91)$$

By the above equation, it follows that

$$\begin{aligned} & f'_0(x_1, x_2, x_3) = f_0(x_1, x_2, x_3) = [x_1, x_2, x_3], \\ & f'_1(x_1, x_2, x_3) = f'_1(x_1, x_2, x_3) = \cdots \\ &= f'_{n-1}(x_1, x_2, x_3) = 0, \\ & f'_n(x_1, x_2, x_3) - g_n([x_1, x_2, x_3]) \\ &= f_n(x_1, x_2, x_3) - [g_n(x_1), x_2, x_3] \\ & \quad - [x_1, g_n(x_2), x_3] - [x_1, x_2, g_n(x_3)] \\ &= f_n(x_1, x_2, x_3) \\ & \quad - (-1)^{|x_1|(|x_2|+|x_3|)} \theta(x_2, x_3) g_n(x_1) \\ & \quad + \delta(-1)^{|x_2||x_3|} \theta(x_1, x_3) g_n(x_2) \\ & \quad - \delta D(x_1, x_2) g_n(x_3). \end{aligned} \quad (92)$$

Therefore, we deduce

$$\begin{aligned} & f'_n(x_1, x_2, x_3) = f_n(x_1, x_2, x_3) - d^1 g_n(x_1, x_2, x_3) \\ &= 0. \end{aligned} \quad (93)$$

It follows that $f'_t = f_0 + \sum_{i \geq n+1} f'_i t^i$. By induction, we have $f_t \sim f_0$; that is, T is analytically rigid. The proof is finished. $\square$

5. Nijenhuis Operators of δ -Jordan Lie Supertriple Systems

In this section, we introduce the notion of Nijenhuis operators for δ -Jordan Lie supertriple systems. Also, we give trivial deformations of this kind of operators.

Let T be a δ -Jordan Lie supertriple system and let $\psi : T \times T \times T \rightarrow T$ be an even trilinear map. Consider a λ -parametrized family of linear operations:

$$[x_1, x_2, x_3]_\lambda = [x_1, x_2, x_3] + \lambda \psi(x_1, x_2, x_3), \quad (94)$$

for any $x_1, x_2, x_3 \in T$, where λ is a formal variable.

If $[\cdot, \cdot, \cdot]_\lambda$ endow T with a δ -Jordan Lie supertriple system structure which is denoted by T_λ , then we call that ψ generates a λ -parameter infinitesimal deformation of the δ -Jordan Lie supertriple system T .

Theorem 30. Let T be a δ -Jordan Lie supertriple system. Then T_λ is a δ -Jordan Lie supertriple system if and only if

- (i) ψ itself defines a δ -Jordan Lie supertriple system structure on T ;
- (ii) ψ is a 3-cocycle of T .

Proof. Assume that T_λ is a δ -Jordan Lie supertriple system. For any $x_1, x_2, x_3 \in T$, we have

$$\begin{aligned}
[x_2, x_1, x_3]_\lambda &= [x_2, x_1, x_3] + \lambda \psi(x_2, x_1, x_3) \\
&= -\delta(-1)^{|x_1||x_2|} [x_1, x_2, x_3] \\
&\quad + \lambda \psi(x_2, x_1, x_3), \\
[x_2, x_1, x_3]_\lambda &= -\delta(-1)^{|x_1||x_2|} [x_1, x_2, x_3]_\lambda \\
&= -\delta(-1)^{|x_1||x_2|} [x_1, x_2, x_3] \\
&\quad - \delta(-1)^{|x_1||x_2|} \lambda \psi(x_1, x_2, x_3).
\end{aligned} \tag{95}$$

It follows that

$$\psi(x_2, x_1, x_3) = -\delta(-1)^{|x_1||x_2|} \psi(x_1, x_2, x_3). \tag{96}$$

For (3), we have

$$\begin{aligned}
&\lambda \left((-1)^{|x_1||x_3|} \psi(x_1, x_2, x_3) + (-1)^{|x_2||x_1|} \psi(x_2, x_3, x_1) \right. \\
&\quad \left. + (-1)^{|x_3||x_2|} \psi(x_3, x_1, x_2) \right) = (-1)^{|x_1||x_3|} \\
&\quad \cdot [x_1, x_2, x_3]_\lambda + (-1)^{|x_2||x_1|} [x_2, x_3, x_1]_\lambda \\
&\quad + (-1)^{|x_3||x_2|} [x_3, x_1, x_2]_\lambda - (-1)^{|x_1||x_3|} [x_1, x_2, x_3] \\
&\quad - (-1)^{|x_2||x_1|} [x_2, x_3, x_1] - (-1)^{|x_3||x_2|} [x_3, x_1, x_2] \\
&= 0,
\end{aligned} \tag{97}$$

as desired. The last equality holds since T and T_λ are both δ -Jordan Lie supertriple systems.

For (3), we take $x_1, x_2, x_3, x_4, x_5 \in T$ and calculate

$$\begin{aligned}
[x_1, x_2, [x_3, x_4, x_5]_\lambda]_\lambda &= [x_1, x_2, [x_3, x_4, x_5] \\
&\quad + \lambda \psi(x_3, x_4, x_5)]_\lambda = [x_1, x_2, [x_3, x_4, x_5] \\
&\quad + \lambda \psi(x_3, x_4, x_5)] + \lambda \psi(x_1, x_2, [x_3, x_4, x_5] \\
&\quad + \lambda \psi(x_3, x_4, x_5)) = [x_1, x_2, [x_3, x_4, x_5]] \\
&\quad + \lambda ([x_1, x_2, \psi(x_3, x_4, x_5)] \\
&\quad + \psi(x_1, x_2, [x_3, x_4, x_5])) \\
&\quad + \lambda^2 \psi(x_1, x_2, \psi(x_3, x_4, x_5)).
\end{aligned} \tag{98}$$

By similar calculation, we have

$$\begin{aligned}
[[x_1, x_2, x_3]_\lambda, x_4, x_5]_\lambda &= [[x_1, x_2, x_3], x_4, x_5] \\
&\quad + \lambda ([\psi(x_1, x_2, x_3), x_4, x_5] \\
&\quad + \psi([x_1, x_2, x_3], x_4, x_5)) \\
&\quad + \lambda^2 \psi(\psi(x_1, x_2, x_3), x_4, x_5),
\end{aligned}$$

$$\begin{aligned}
[x_3, [x_1, x_2, x_4]_\lambda, x_5]_\lambda &= [x_3, [x_1, x_2, x_4], x_5] \\
&\quad + \lambda ([x_3, \psi(x_1, x_2, x_4), x_5] \\
&\quad + \psi(x_3, [x_1, x_2, x_4], x_5)) \\
&\quad + \lambda^2 \psi(x_3, \psi(x_1, x_2, x_4), x_5), \\
[x_3, x_4, [x_1, x_2, x_5]_\lambda]_\lambda &= [x_3, x_4, [x_1, x_2, x_5]] \\
&\quad + \lambda ([x_3, x_4, \psi(x_1, x_2, x_5)] \\
&\quad + \psi(x_3, x_4, [x_1, x_2, x_5])) \\
&\quad + \lambda^2 \psi(x_3, x_4, \psi(x_1, x_2, x_5)).
\end{aligned} \tag{99}$$

It follows that

$$\begin{aligned}
&[[x_1, x_2, x_3]_\lambda, x_4, x_5]_\lambda + (-1)^{|x_3|(|x_1|+|x_2|)} \\
&\quad \cdot [x_3, [x_1, x_2, x_4]_\lambda, x_5]_\lambda + \delta(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)} \\
&\quad \cdot [x_3, x_4, [x_1, x_2, x_5]_\lambda]_\lambda = [[x_1, x_2, x_3], x_4, x_5] \\
&\quad + (-1)^{|x_3|(|x_1|+|x_2|)} [x_3, [x_1, x_2, x_4], x_5] \\
&\quad + \delta(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)} [x_3, x_4, [x_1, x_2, x_5]] \\
&\quad + \lambda \{ [\psi(x_1, x_2, x_3), x_4, x_5] \\
&\quad + \psi([x_1, x_2, x_3], x_4, x_5) + (-1)^{|x_3|(|x_1|+|x_2|)} \\
&\quad \cdot [x_3, \psi(x_1, x_2, x_4), x_5] + (-1)^{|x_3|(|x_1|+|x_2|)} \\
&\quad \cdot \psi(x_3, [x_1, x_2, x_4], x_5) + \delta(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)} \\
&\quad \cdot [x_3, x_4, \psi(x_1, x_2, x_5)] + \delta(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)} \\
&\quad \cdot \psi(x_3, x_4, [x_1, x_2, x_5]) \} \\
&\quad + \lambda^2 \{ \psi(\psi(x_1, x_2, x_3), x_4, x_5) + (-1)^{|x_3|(|x_1|+|x_2|)} \\
&\quad \cdot \psi(x_3, \psi(x_1, x_2, x_4), x_5) \\
&\quad + \delta(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)} \\
&\quad \cdot \psi(x_3, x_4, \psi(x_1, x_2, x_5)) \}.
\end{aligned} \tag{100}$$

Therefore, we have

$$\begin{aligned}
&[x_1, x_2, \psi(x_3, x_4, x_5)] + \psi(x_1, x_2, [x_3, x_4, x_5]) \\
&= [\psi(x_1, x_2, x_3), x_4, x_5] \\
&\quad + \psi([x_1, x_2, x_3], x_4, x_5) + (-1)^{|x_3|(|x_1|+|x_2|)} \\
&\quad \cdot [x_3, \psi(x_1, x_2, x_4), x_5] + (-1)^{|x_3|(|x_1|+|x_2|)} \\
&\quad \cdot \psi(x_3, [x_1, x_2, x_4], x_5) + \delta(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)}
\end{aligned}$$

$$\begin{aligned} & \cdot [x_3, x_4, \psi(x_1, x_2, x_5)] + \delta(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)} \\ & \cdot \psi(x_3, x_4, [x_1, x_2, x_5]), \end{aligned} \quad (101)$$

$$\begin{aligned} \psi(x_1, x_2, \psi(x_3, x_4, x_5)) &= \psi(\psi(x_1, x_2, x_3), x_4, x_5) \\ &+ (-1)^{|x_3|(|x_1|+|x_2|)} \psi(x_3, \psi(x_1, x_2, x_4), x_5) \\ &+ \delta(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)} \psi(x_3, x_4, \psi(x_1, x_2, x_5)). \end{aligned} \quad (102)$$

By (102), ψ satisfies (3). So ψ defines a δ -Jordan Lie supertriple system structure on T .

Since $\theta(x_1, x_2)(x) = (-1)^{|x|(|x_1|+|x_2|)}[x, x_1, x_2]$ and $D(x_1, x_2)(x) = \delta[x_1, x_2, x]$, we can rewrite (97) as follows:

$$\begin{aligned} 0 &= -\delta D(x_1, x_2) \psi(x_3, x_4, x_5) \\ &- \psi(x_1, x_2, [x_3, x_4, x_5]) \\ &+ (-1)^{(|x_1|+|x_2|+|x_3|)(|x_4|+|x_5|)} \theta(x_4, x_5) \psi(x_1, x_2, x_3) \\ &+ \psi([x_1, x_2, x_3], x_4, x_5) - \delta(-1)^{|x_3|(|x_1|+|x_2|)} \\ &\cdot \theta(x_3, e) \psi(x_1, x_2, x_4) + (-1)^{|x_3|(|x_1|+|x_2|)} \\ &\cdot \psi(x_3, [x_1, x_2, x_4], x_5) + \delta(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)} \\ &\cdot D(x_3, x_4) \psi(x_1, x_2, x_5) \\ &+ \delta(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)} \psi(x_3, x_4, [x_1, x_2, x_5]) \\ &= d^3 \psi(x_1, x_2, x_3, x_4, x_5). \end{aligned} \quad (103)$$

The last equality holds since ψ is an even trilinear map. So $d^3 \psi = 0$, as required.

Conversely, if ψ satisfies conditions (i) and (ii), it is easy to see that T_λ is a δ -Jordan Lie supertriple system from (96), (97), (101), and (102). $\square$

Definition 31. A deformation is said to be trivial if there exists a linear map $N : T \rightarrow T$ such that, for all λ , $\varphi_\lambda = id + \lambda N : T_\lambda \rightarrow T$ satisfies

$$\varphi_\lambda [x_1, x_2, x_3]_\lambda = [\varphi_\lambda x_1, \varphi_\lambda x_2, \varphi_\lambda x_3], \quad (104)$$

for any $x_1, x_2, x_3 \in T$.

The left-hand side of (104) is equal to

$$\begin{aligned} & [x_1, x_2, x_3] + \lambda \{\psi(x_1, x_2, x_3) + N[x_1, x_2, x_3]\} \\ & + \lambda^2 N\psi(x_1, x_2, x_3). \end{aligned} \quad (105)$$

The right-hand side of (104) is equal to

$$\begin{aligned} & [x_1, x_2, x_3] + \lambda \{[Nx_1, x_2, x_3] + [x_1, Nx_2, x_3] \\ & + [x_1, x_2, Nx_3]\} + \lambda^2 \{[Nx_1, Nx_2, x_3] \\ & + [x_1, Nx_2, Nx_3] + [Nx_1, x_2, Nx_3]\} \\ & + \lambda^3 [Nx_1, Nx_2, Nx_3]. \end{aligned} \quad (106)$$

Therefore, by (104), we have

$$0 = [Nx_1, Nx_2, Nx_3], \quad (107)$$

$$\begin{aligned} N\psi(x_1, x_2, x_3) &= [Nx_1, Nx_2, x_3] + [x_1, Nx_2, Nx_3] \\ &+ [Nx_1, x_2, Nx_3], \end{aligned} \quad (108)$$

$$\begin{aligned} \psi(x_1, x_2, x_3) &= [Nx_1, x_2, x_3] + [x_1, Nx_2, x_3] \\ &+ [x_1, x_2, Nx_3] - N[x_1, x_2, x_3] \\ &= (-1)^{|x_1|(|x_2|+|x_3|)} \theta(x_2, x_3) Nx_1 \\ &- \delta(-1)^{|x_2||x_3|} \theta(x_1, x_3) Nx_2 \\ &+ \delta D(x_1, x_2) Nx_3 \\ &- N[x_1, x_2, x_3] \\ &= d^1 N(x_1, x_2, x_3). \end{aligned} \quad (109)$$

By (108) and (109), we can deduce that

$$\begin{aligned} N^2[x_1, x_2, x_3] &= N[Nx_1, x_2, x_3] + N[x_1, Nx_2, x_3] \\ &+ N[x_1, x_2, Nx_3] \\ &- [Nx_1, Nx_2, x_3] \\ &- [x_1, Nx_2, Nx_3] \\ &- [Nx_1, x_2, Nx_3]. \end{aligned} \quad (110)$$

Definition 32. A linear operator $N : T \rightarrow T$ is called a Nijenhuis operator if and only if (107) and (110) hold.

Theorem 33. Let N be a Nijenhuis operator for T . Then, a deformation of T can be obtained by putting

$$\begin{aligned} \psi(x_1, x_2, x_3) &= (-1)^{|x_1|(|x_2|+|x_3|)} \theta(x_2, x_3) Nx_1 \\ &- \delta(-1)^{|x_2||x_3|} \theta(x_1, x_3) Nx_2 \\ &+ \delta D(x_1, x_2) Nx_3 - N[x_1, x_2, x_3]. \end{aligned} \quad (111)$$

Moreover, this deformation is trivial.

Proof. The proof is similar to one in the setting of δ -Jordan Lie triple system in [14]. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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