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Research Article

Symmetry, Pulson Solution, and Conservation Laws of the Holm-Hone Equation

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In this paper, we focus on the Holm-Hone equation which is a fifth-order generalization of the Camassa-Holm equation. It was shown that this equation is not integrable due to the nonexistence of a suitable Lagrangian or bi-Hamiltonian structure and negative results from Painlevé analysis and the Wahlquist-Estabrook method. We mainly study its symmetry properties, travelling wave solutions, and conservation laws. The symmetry group and its one-dimensional optimal system are given. Furthermore, preliminary classifications of its symmetry reductions are investigated. Also we derive some solitary pattern solutions and nonanalytic first-order pulson solution via the ansatz-based method. Finally, some conservation laws for the fifth-order equation are presented.

1. Introduction

In the study of shallow water waves, Camassa and Holm derived a nonlinear dispersive shallow water wave equation [1]

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1}$$

which is called Camassa-Holm (CH) equation now. The $u(x,t)$ here denotes the fluid velocity at time t in the x direction [2]. Eq. (1) admits bi-Hamiltonian structure and it is completely integrable [3, 4]. What is more, the equation has infinite conservation laws [5] as well as a spike solitary wave solution

$$u(x,t) = c\exp(-|x-ct|), \tag{2}$$

where c is an arbitrary constant. The solitary wave curve of the solution has a cusp at the peak, and the first derivative of the cusp is not continuous. Accordingly, it is called a peakon [6, 7].

With the further researches on the CH equation, a lot of findings about the equation have been obtained and it is impossible to give a comprehensive overview here. For example, Eq. (1) represents the equation for geodesics on the Bott-Virasoro group and owns the geometric interpretation [8]. The CH equation possesses both global solutions and solutions developing singularities in finite time and the blow-up happens in a way which resembles wave breaking to some extent [9, 10]. The well-developed inverse scattering theory can also be used to integrate the CH flow [11]. In [12], it was shown that the well-known CH equation is included in the negative order CH hierarchy and a class of new algebro-geometric solutions of the CH equation was presented. Moreover, the time evolution of traveling-wave solutions and the interaction of peaked and cusped waves were numerically studied [13].

Recently, one of the latest trends is that researchers are trying to generalize the CH equation to higher order, which is also the subject of this paper. In fact, some higher-order CH equations are well considered. For example, Wazwaz studied the nonlinear fourth-order dispersive variants of the generalized CH equation by using sine-cosine method [14]. The existence of global weak solutions was established for a higher-order CH equation describing exponential curves of the manifold of smooth orientation-preserving diffeomorphisms of the unit circle in the plane [15]. Nonsmooth travelling wave solutions of a generalized fourth-order nonlinear

CH equation were studied in [16]. In [17], the CH model was extended to fifth order and some interesting solutions were obtained including explicit single pseudo-peakons, two-peakon, and N-peakon solutions.

Especially, Holm and Hone discussed a fifth-order partial differential equation (PDE)

$$\begin{aligned} u_{4x,t} - 5u_{xxt} + 4u_t + uu_{5x} + 2u_x u_{4x} - 5uu_{3x} \\ - 10u_x u_{xx} + 12uu_x = 0, \end{aligned} \tag{3}$$

which is a generalization of the integrable CH equation [18]. We will call it the Holm-Hone equation in the following. This fifth-order PDE admits exact solutions in terms of an arbitrary number of superposed pulsons. The pulsons of Eq. (3) are weak solutions with discontinuous second derivatives at isolated points. Numerical simulations show that the pulsons are stable and they dominate the initial value problem and scatter elastically [19]. These characteristics are reminiscent of solitons in integrable systems. However, after demonstrating the nonexistence of a suitable Lagrangian or bi-Hamiltonian structure [20] and obtaining negative results from Painlevé analysis [21] and the Wahlquist-Estabrook method [22], they asserted that the Holm-Hone equation is not integrable. Consequently, most effective methods for integrable systems are not applicable to this equation. However, the Lie symmetry approach originated from Norwegian mathematician Sophus Lie [23] systematically unifies and extends well-known ad hoc techniques to construct explicit solutions for differential equations, no matter the equations are integrable or not.

Actually, there are several important applications of symmetry analysis in the investigation of differential equations [24–27]. Since some solutions of partial differential equations asymptotically tend to solutions of lower-dimensional equations obtained by symmetry reduction, some of these special solutions illustrate important physical phenomena. In particular, the exact solutions arising from symmetry methods can often be used effectively to study properties such as asymptotics and blow-up. Besides, the explicit solutions found by symmetry methods can play important roles in the design and testing of numerical integrators, and these solutions provide an important practical check on the accuracy and reliability of such integrators.

This paper is organized as follows. We first present the common form of the infinitesimal generators of the Holm-Hone equation. Then we establish the optimal system of one-dimensional subalgebras and consider the similarity reductions of the equation. Furthermore, the travelling wave solutions are investigated. At last, some conservation laws of the equation are also derived.

2. Lie Symmetries for the Holm-Hone Equation

In this section, we investigate the Lie symmetries and similarity reductions of the Holm-Hone equation through the classical methods. The infinitesimal generators corresponding to the one-parameter transformation group are presented. Furthermore, the one-dimensional optimal system of the group is derived with the help of the adjoint representation among the vector fields. Finally, the reduced equations are given through the similarity transformation.

2.1. Infinitesimal Generators. We introduce the infinitesimal form of the single parameter transformation group

$$\begin{aligned} x_1 &= x + \epsilon\xi(x,t,u) + O(\epsilon^2), \\ t_1 &= t + \epsilon\tau(x,t,u) + O(\epsilon^2), \\ u_1 &= u + \epsilon\phi(x,t,u) + O(\epsilon^2), \end{aligned} \tag{4}$$

where ϵ is the infinitesimal group parameter and ξ, τ, ϕ are the infinitesimals of the transformation for the independent and dependent variables, respectively. The vector field associated with the above group of transformations can be written as

$$V = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \phi(x,t,u)\partial_u. \tag{5}$$

Using the software package GeM [28], we obtain

$$\begin{aligned} \xi &= c_3, \\ \tau &= c_1 t + c_2, \\ \phi &= -c_1 u, \end{aligned} \tag{6}$$

where c_1, c_2, c_3 are arbitrary constants. The infinitesimal generators of the corresponding Lie algebra are given by

$$\begin{aligned} V_1 &= \partial_x, \\ V_2 &= \partial_t, \\ V_3 &= t\partial_t - u\partial_u. \end{aligned} \tag{7}$$

In order to obtain the group transformation which is generated by the infinitesimal generators V_i for $i = 1, 2, 3$, we need to solve the first-order ordinary differential equations

$$\begin{aligned} \frac{dx_1}{d\epsilon} &= \xi_i(x,t,u), \quad x_1|_{\epsilon=0} = x, \\ \frac{dt_1}{d\epsilon} &= \tau_i(x,t,u), \quad t_1|_{\epsilon=0} = t, \\ \frac{du_1}{d\epsilon} &= \phi_i(x,t,u), \quad u_1|_{\epsilon=0} = u. \end{aligned} \tag{8}$$

Exponentiating the infinitesimal symmetries we get the one-parameter groups $G_i(\epsilon)$ generated by V_i for $i = 1, 2, 3$ as

$$\begin{aligned} G_1 &: (x,t,u) \longmapsto (x+\epsilon, t, u), \\ G_2 &: (x,t,u) \longmapsto (x, t+\epsilon, u), \\ G_3 &: (x,t,u) \longmapsto (x, e^{\epsilon}t, e^{-\epsilon}u). \end{aligned} \tag{9}$$

Accordingly, if $u = f(x,t)$ is a solution of the Holm-Hone equation, so are the functions

$$\begin{aligned} G_1(\epsilon)\cdot f(x,t) &= f(x-\epsilon, t), \\ G_2(\epsilon)\cdot f(x,t) &= f(x, t-\epsilon), \\ G_3(\epsilon)\cdot f(x,t) &= e^{\epsilon} f(x, te^{-\epsilon}). \end{aligned} \tag{10}$$

Table 1: The commutator table.

$[V_i, V_j]$	V_1	V_2	V_3
V_1	0	0	0
V_2	0	0	V_2
V_3	0	$-V_2$	0

Table 2: Adjoint representation table.

$Ad(\exp(\epsilon V_i)V_j)$	V_1	V_2	V_3
V_1	V_1	V_2	V_3
V_2	V_1	V_2	$V_3 - \epsilon V_2$
V_3	V_1	$e^{\epsilon} V_2$	V_3

2.2. One-Dimensional Optimal System. In general, the Lie group has infinite subgroups; it is not usually feasible to list all possible group invariant solutions to the system. We need an effective systematic approach to classify these solutions, so the optimal 1-dimensional subalgebras of group invariant solutions can be derived. In this section, the optimal system is obtained by computing the adjoint representation of the vector fields [25]. We use the Lie series

$$Ad\left(\exp\left(\epsilon V_i\right) V_j\right) = V_j - \epsilon\left[V_i, V_j\right] + \frac{\epsilon^2}{2}\left[V_i, \left[V_i, V_j\right]\right] - \cdots, \tag{11}$$

where $[V_i, V_j]$ is the commutator for the Lie algebra, ϵ is a parameter, and $i, j = 1, 2, 3$. The commutator table of the Lie point symmetries for Eq. (3) and the adjoint representation of the symmetry group on its Lie algebra are presented in Tables 1 and 2, respectively.

Given a nonzero vector

$$V = a_1 V_1 + a_2 V_2 + a_3 V_3, \tag{12}$$

our task is to simplify as many of the coefficients a_i as possible through judicious applications of adjoint maps to V.

Firstly, we suppose that $a_3 \neq 0$. Scaling V if necessary, we can assume that $a_3 = 1$. Referring to Table 2, if we act on such a V by $Ad(\exp(a_2 V_2))$, we can make the coefficient of V_2 vanish and the coefficients of V_1, V_3 cannot be eliminated further, so we can make the coefficient of V_1 either $+1$, -1, or 0. Thus any one-dimensional subalgebra spanned by V with $a_3 \neq 0$ is equivalent to one spanned by either $V_1 + V_3$, $V_3 - V_1$, or V_3.

The remaining one-dimensional subalgebras are spanned by vectors of the above form with $a_3 = 0$. If $a_2 \neq 0$, we scale to make $a_2 = 1$, and then no coefficient vanishes by any action on V, so that V is equivalent to $a_1 V_1 + V_2$.

The remaining case, $a_3 = a_2 = 0$, is equivalent to V_1. And it is impossible to make further simplification.

Until now, we have found the optimal system of one-dimensional subalgebras spanned by

$$\begin{aligned}
&(a_1) \quad V_1 + V_3 = \partial_x + t\partial_t - u\partial_u \\
&(a_2) \quad V_3 - V_1 = t\partial_t - \partial_x - u\partial_u \\
&(a_3) \quad V_3 = t\partial_t - u\partial_u \\
&(b) \quad a_1 V_1 + V_2 = a_1\partial_x + \partial_t \\
&(c) \quad V_1 = \partial_x
\end{aligned} \tag{13}$$

The list can be reduced slightly if we admit the discrete symmetry $(x, t, u) \longmapsto (-x, -t, u)$, which maps $V_3 - V_1$ to $V_3 + V_1$, and thereby the number of inequivalent subalgebras is reduced to four.

2.3. Similarity Reductions. The Holm-Hone equation is expressed in the coordinates (x, t, u), so we try to reduce this equation in order to search for its form in specific coordinates which can be constructed by solving the characteristic equation

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\phi}. \tag{14}$$

Based on the optimal system presented before, we obtain the following four kinds of reductions:

Reduction 1. For $V_1 = \partial_x$, we get the reduction $z = t, u = f(t) = const$.

Reduction 2. For $a_1 V_1 + V_2 = a_1\partial_x + \partial_t$, we get the traveling wave reduction $z = x - a_1 t$, $u = f(z)$, where $f(z)$ satisfies

$$\begin{aligned}
&ff^{(5)} - a_1 f^{(5)} + 5a_1 f^{(3)} - 4a_1 f' + 2f' f^{(4)} - 5ff^{(3)} \\
&\quad - 10 f' f'' + 12 ff' = 0.
\end{aligned} \tag{15}$$

Reduction 3. For $V_3 = t\partial_t - u\partial_u$, we get the reduction $z = x, u = t^{-1} f(z)$, where $f(z)$ satisfies

$$\begin{aligned}
&ff^{(5)} - f^{(4)} + 5f'' - 4f + 2f' f^{(4)} - 5ff^{(3)} - 10 f' f'' \\
&\quad + 12 ff' = 0.
\end{aligned} \tag{16}$$

Reduction 4. For $V_2 + V_3 = \partial_x - u\partial_u + t\partial_t$, we get the reduction $z = te^{-x}, u = f(z)e^{-x}$, where $f(z)$ satisfies

$$\begin{aligned}
&z^4\left(ff^{(5)} + 2f' f^{(4)}\right) + z^3\left(17 ff^{(4)} + 20 f' f^{(3)} - f^{(5)}\right) \\
&\quad + 50z\left(2ff'' - f^{(3)}\right) \\
&\quad + 2z^2\left(40 ff^{(3)} + 20 f' f'' - 7 f^{(4)}\right) - 40 f'' = 0.
\end{aligned} \tag{17}$$

3. Travelling Wave Solutions of the Holm-Hone Equation

The appearance of nonanalytic peakon-type solutions has increased the menagerie of solutions appearing in nonlinear partial differential equations, both integrable and nonintegrable. The pulson solutions for Eq. (3) have a finite jump in second derivative of the solutions. In a series of papers, Wazwaz proposed some schemes to determine new sets of soliton solutions, in addition to the peakon solutions

obtained before, for the family of CH equations [29, 30]. The method rests mainly on some ansatzes that use one hyperbolic function or combine two hyperbolic functions as follows:

(1) A sinh-cosh Ansatz I

$$u(x,t) = \lambda + \alpha \cosh^2[\mu(x-ct)] + \beta \sinh^2[\mu(x-ct)] \tag{18}$$

(2) A sinh-cosh Ansatz II

$$u(x,t) = \lambda + \alpha \cosh[\mu(x-ct)] + \beta \sinh[\mu(x-ct)] \tag{19}$$

(3) A tanh Ansatz or a coth Ansatz

$$u(x,t) = \frac{1}{\lambda + \alpha \tanh[\mu(x-ct)]} \tag{20}$$

or

$$u(x,t) = \frac{1}{\lambda + \alpha \coth[\mu(x-ct)]} \tag{21}$$

(4) The Exponential Peakon Ansatz

$$u(x,t) = \lambda + \alpha e^{-\mu|x-ct|} \tag{22}$$

where λ, μ, α, and β are parameters that will be determined. By this method, solitons, solitary patterns solutions, periodic solutions, compactons, and peakons solutions for a family of CH equations with distinct parameters are obtained. However, we should modify the first ansatz and combine it into the second one since $\cosh^2(\xi) - \sinh(\xi)^2 = 1$ and $\cosh^2(\xi) + \sinh(\xi)^2 = \cosh(2\xi)$. In what follows, we try to describe some specific travelling wave solutions for the Holm-Hone equation via this modified method.

Firstly, we obtain the travelling wave solutions

$$u(x,t) = \alpha \cosh[\mu(x-ct)] + \beta \sinh[\mu(x-ct)] \tag{23}$$

with $\mu = \pm 1$ or ± 2 and α, β are arbitrary constants. Moreover, it can be easily verified that there is no effective result for the third ansatz for the Holm-Hone equation. And in order to take the last exponential peakon ansatz, we are advised to rewrite Eq. (3) as

$$m_t + m_x u + 2u_x m = 0 \tag{24}$$

with

$$m = u_{4x} - 5u_{xx} + 4u. \tag{25}$$

Supposing that $u = U(\xi) = U(x-ct)$, then Eq. (24) becomes

$$(U-c)M' + 2U'M = 0, \tag{26}$$

where

$$M = U^{(4)} - 5U'' + 4U. \tag{27}$$

Let us find all possible nonconstant solutions satisfying $M = 0$ and $U(\pm\infty) = 0$. Since $U^{(4)} - 5U'' + 4U = 0$, the corresponding characteristic equation is $r^4 - 5r^2 + 4 = 0$. Then the characteristic values are ± 1 and ± 2, which yield

$$U = a_1 e^{\xi} + a_2 e^{-\xi} + a_3 e^{2\xi} + a_4 e^{-2\xi}. \tag{28}$$

It is easy to see that if and only if $a_1 = a_3 = 0$

$$\lim_{\xi \longrightarrow +\infty} \left(a_2 e^{-\xi} + a_4 e^{-2\xi}\right) = 0. \tag{29}$$

Nevertheless, when $a_2 = a_4 = 0$

$$\lim_{\xi \longrightarrow -\infty} \left(a_1 e^{\xi} + a_3 e^{2\xi}\right) = 0. \tag{30}$$

Then the solution can be reduced to

$$u(x,t) = \begin{cases} a_1 e^{x-ct} + a_3 e^{2(x-ct)}, & x - ct \le 0, \\ a_2 e^{-(x-ct)} + a_4 e^{-2(x-ct)}, & x - ct > 0. \end{cases} \tag{31}$$

Considering the continuity of solution at $x - ct = 0$, we look for a solution of the following form:

$$u(x,t) = c_1 e^{-|x-ct|} + c_2 e^{-2|x-ct|}, \tag{32}$$

where c_1, c_2 are arbitrary constants. Next we explore the relationship between the coefficients c_1 and c_2. Substituting Eq. (32) into Eq. (27) and using the property $\delta(x)f(x) = \delta(x)f(0)$ we get the travelling wave solution of the original equation

$$u(x,t) = 2ce^{-|x-ct|} - ce^{-2|x-ct|}. \tag{33}$$

This solution is called the first-order pulson solution which is different from the peakons of CH equation. Its first derivative is continuous and the second derivative of the cusp is not continuous.

4. Conservation Laws for the Holm-Hone Equation

Conservation laws are widely applied in the analysis of PDEs, particularly in the study of existence, uniqueness, and stability of solutions. The concept of conservation laws and the relationship between symmetries and conservation laws arise in a wide variety of applications and contexts [25, 26].

A conservation law for (3) is of the form

$$D_t T^1 + D_x T^2 = 0, \tag{34}$$

where D_t and D_x denote the total derivatives as

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{xt} \frac{\partial}{\partial u_x} + \cdots, \tag{35}$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x} + \cdots, \tag{36}$$

and the subscripts denote partial derivatives. The vector $T = (T^1, T^2)$ is a conserved vector for the partial differential

equation. It was shown that Eq. (3) has the conservation law

$$\left(m^{1/2}\right)_t + \left(m^{1/2}u\right)_x = 0, \tag{37}$$

where $m = u_{4x} - 5u_{xx} + 4u$ and it does not own bi-Hamiltonian structure [18]. In this section, we mainly solve the conservation laws of the Holm-Hone equation by multiplier method [31–33].

A multiplier $\wedge$ has the property that

$$\begin{aligned}&\wedge\left(u_{4x,t} - 5u_{xxt} + 4u_t + uu_{5x} + 2u_x u_{4x} - 5uu_{3x}\right.\\&\left.\quad - 10u_x u_{xx} + 12uu_x\right) = D_t T^1 + D_x T^2,\end{aligned} \tag{38}$$

for all solutions $u(x,t)$. Generally speaking, each multiplier is a function as $\wedge(t,x,\partial u,\ldots,\partial^k u)$, where $\partial^k u$ denotes all kth order derivatives of u with respect to all independent variables t, x. Here we only consider multipliers of the form $\wedge = \wedge(t,x,u)$, although multipliers which depend on the first-order and higher-order partial derivatives of u could also be considered, but the calculations become more complicated and we fail to find any result.

The right-hand side of (38) is a divergence expression which leads to the determining equation for the multiplier $\wedge$ as

$$\begin{aligned}&E_u\left[\wedge\left(u_{4x,t} - 5u_{xxt} + 4u_t + uu_{5x} + 2u_x u_{4x} - 5uu_{3x}\right.\right.\\&\left.\left.\quad - 10u_x u_{xx} + 12uu_x\right)\right] = 0,\end{aligned} \tag{39}$$

where E_u is the standard Euler operator.

From the system (39), we can obtain the solution

$$\wedge(x,t,u) = c_1 u + c_2, \tag{40}$$

where c_1 and c_2 are arbitrary constants. Therefore we get that any conserved vector of the Holm-Hone equation with multiplier of the form $\wedge(t,x,u)$ is a linear combination of the two conserved vectors

$$\begin{aligned}(1):\quad & T^1 = 2u^2 - \frac{5}{2}uu_{xx} + \frac{1}{2}uu_{xxxx},\\ & T^2\\ &= -5u^2u_{xx} + 4u^3 + u^2u_{xxxx} + \frac{5}{2}u_x u_t - \frac{5}{2}uu_{tx}\\ &\quad - \frac{1}{2}u_x u_{txx} + \frac{1}{2}uu_{txxx} + \frac{1}{2}u_{xx}u_{tx}\\ &\quad - \frac{1}{2}u_t u_{xxx},\\ (2):\quad & T^1 = 4u - 5u_{xx} + u_{xxxx},\\ & T^2\\ &= 6u^2 - 5uu_{xx} + uu_{xxxx} + u_x u_{xxx} - \frac{1}{2}u_{xx}^2\\ &\quad - \frac{5}{2}u_x^2.\end{aligned} \tag{41}$$

5. Conclusions

In summary, we have found the most general Lie point symmetries group for the nonintegrable Holm-Hone equation which is a fifth-order generalization of the CH equation. Meanwhile, we constructed the optimal system of one-dimensional subalgebras. Afterwards, we created the preliminary classifications of similarity reductions. The Lie invariants and similarity reduced equations corresponding to infinitesimal symmetries have been obtained. In order to obtain the traveling wave solutions of the equation, we adopted the method of ansatz. We also found some conservation laws from the multiplier method.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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