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Research Article

Multiple-Pole Solutions to a Semidiscrete Modified Korteweg-de Vries Equation

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Multiple-pole soliton solutions to a semidiscrete modified Korteweg-de Vries equation are derived by virtue of the Riemann-Hilbert problem with higher-order zeros. A different symmetry condition is introduced to build the nonregular Riemann-Hilbert problem. The simplest multiple-pole soliton solution is presented. The dynamics of the solitons are studied.

1. Introduction

In the theory of inverse scatter transform, soliton solutions are closely related to the poles of the transmission coefficient (we called here the zero of the Riemann-Hilbert problem). In contrast with the study of the multisoliton solutions from simple poles, relatively few multiple-pole solitons are discussed, and mainly on continuous integrable equations [1–8]. It is known that multiple-pole solutions can be obtained directly by other methods, for example, bilinear method and Cauchy matrix approach; these solutions can also be done by coalescing several simple poles with a procedure via taking limits of the poles [9–12].

In this paper, we consider the multiple-pole soliton solutions of a semidiscrete modified Korteweg-de Vries (sdmKdV) equation [13–15]

$$iu_t(n,t) = \left(1 + u^2(n,t)\right)\left[u(n+1,t) - u(n-1,t)\right], \quad (1)$$

in virtue of the Riemann-Hilbert approach under a different symmetry condition.

As we know, for a complex function $f(z)$, if it is analytic inside the circle $|z| < 1$, then by using the Cauchy-Riemann equations $f^*(\overline{z})$ is analytic outside the circle $|z| > 1$, where the star denotes the complex conjugate and $\overline{z} = 1/z^*$. It is worth mentioning that $f(z^{-1})$ is also analytic outside the circle $|z| > 1$, if $f(z)$ is analytic inside the circle $|z| < 1$. With application of the latter fact, for the sdmKdV equation (1) with a modified linear spectral problem, we introduce a new symmetry condition as $f^T(z^{-1}) = f^{-1}(z)$ about the associated eigenfunction. The symmetry condition determines the distribution of the relevant eigenvalues of the discrete spectrum, which are deeply related to the soliton solutions of the integrable equations. In the paper, we derive some multiple-pole soliton solutions according to the new symmetry condition.

The Riemann-Hilbert (RH) problem plays an important role in the scattering theory. For integrable nonlinear equations, the nonregular RH problem will be concerned. After a procedure of regularization of the RH problem with zeros relating to the eigenvalues of discrete spectrum, the so-called soliton matrix will involve a set of relevant zero eigenvectors and is related to the potential functions. The soliton matrix for the case of canonical normalization condition of the RH problem, in general, satisfies the relevant linear spectral problem. Thus the zero eigenvectors can be obtained, and then the soliton matrix and the potential will be given. However, for the sdmKdV equation (1), the normalization condition of the nonregular RH problem is noncanonical, so the soliton matrix will not admit the associated linear spectral problem, which is different from that of the continuous case [7, 8]. In this paper, we choose a special solution of regular RH problem and use the solution of the RH problem (or the sectionally analytic function) to determine the zero eigenvectors.

The paper is organized as follows. In Section 2, we discuss the spectral analysis of the spectral problem of the sdmKdV equation under a new symmetry condition, and in Section 3, we use them to build a nonregular RH problem and then discuss its regularization. Finally, in Section 4, we establish the relationship between the potential and the soliton matrix and then derive the multiple-pole solitons of the sdmKdV equation. The figures of the simplest multiple-pole soliton solution show some novel properties.

2. Spectral Analysis of the Focusing Ablowitz-Ladik Equation

Equation (1) can be represented as the compatibility condition of the following linear system [16]:

$$J(n+1,t) = \gamma_n (I + Q_n) Z J(n,t) Z^{-1}, \tag{2}$$

$$J_t(n,t) = -i\lambda [\sigma_3, J(n,t)] - i\widetilde{Q}_n J(n,t), \tag{3}$$

where z is a spectral parameter and

$$\begin{aligned} \gamma_n &= \sqrt{1+u^2(n,t)}^{\,-1}, \\ \lambda &= \frac{1}{2}(z - z^{-1}), \\ \widetilde{Q}_n &= Q_n + Z^{-1} Q_{n-1} Z, \\ Q_n &= \begin{pmatrix} 0 & u(n,t) \\ -u(n,t) & 0 \end{pmatrix}, \\ Z &= \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{4}$$

To investigate the analyticity of the associated eigenfunction, we introduce the Jost functions $J_\pm(n,z)$ with the following boundary conditions:

$$J_\pm(n,z) = I, \quad n \longrightarrow \infty. \tag{5}$$

Since $J_+(n,z)$ and $J_-(n,z)$ are both solutions of the linear equation (2), they are not independent and are linearly related by the scattering matrix $S(z)$ as

$$\begin{aligned} J_-(n,z) &= J_+(n,z) Z^n S(z) Z^{-n}, \\ S(z) &= \begin{pmatrix} a_+(z) & -b_-(z) \\ b_+(z) & a_-(z) \end{pmatrix}. \end{aligned} \tag{6}$$

By the famous Cauchy-Riemann equations, we know that, for a complex function $f(z)$, $f(z^{-1})$ is analytic in $|z| > 1$, if $f(z)$ is analytic in $|z| < 1$. By the latter fact, we consider a different symmetry condition. In this case, the symmetry conditions can be obtained according to the discrete spectral problem (2) and the asymptotic condition (5) as

$$\begin{aligned} J_\pm^T(n,z^-) &= J_\pm^{-1}(n,z), \\ S^T(z^-) &= S^{-1}(z), \\ z^- &:= z^{-1}. \end{aligned} \tag{7}$$

Note, in (2), $\det(\gamma_n(I+Q_n)) = 1$, then $\det J_\pm(x,z)$ is independent of n. As a result,

$$\begin{aligned} \det J_\pm(n,z) &= 1, \\ \det S(z) &= 1, \end{aligned} \tag{8}$$

in terms of (5) and (6).

It is easy to see that the linear spectral problem (2) is a recurrence equation. Using the above properties of the Jost functions, it follows that the first column of $J_-(n,z)$, denoted by $J_-^{[1]}(n,z)$, and the second column $J_+^{[2]}(n,z)$ of $J_+(n,z)$ are analytic outside the unit circle (i.e., $|z| > 1$) [14]. With these vectors in hand, we define a new matrix function $\Phi_+(n,z)$ by

$$\Phi_+(n,z) = \left(J_-^{[1]}, J_+^{[2]}\right), \tag{9}$$

which is a solution of the spectral problem (2) and is analytic as a whole in the region $|z| > 1$, while another sectionally analytic matrix function $\Phi_-^{-1}(n,z)$ can be defined by virtue of the symmetry condition (7) as

$$\Phi_-^{-1}(n,z) = \begin{pmatrix} \left(J_-^{-1}\right)_{[1]} \\ \left(J_+^{-1}\right)_{[2]} \end{pmatrix}, \tag{10}$$

where $(J_-^{-1})_{[1]}$ denotes the first row of the matrix function J_-^{-1} and $(J_+^{-1})_{[2]}$ is the second row of J_+^{-1}. We note that the function $\Phi_-^{-1}(n,z)$ is a solution of the adjoint spectral problem of (2) and is analytic in the region $|z| < 1$. It follows from (6) that

$$\Phi_+(n,z) = J_\pm(n,z) Z^n S_\pm(z) Z^{-n}, \tag{11}$$

where

$$\begin{aligned} S_+(z) &= \begin{pmatrix} a_+(z) & 0 \\ b_+(z) & 1 \end{pmatrix}, \\ S_-(z) &= \begin{pmatrix} 1 & b_-(z) \\ 0 & a_+(z) \end{pmatrix}. \end{aligned} \tag{12}$$

Similarly, we have

$$\begin{aligned} \Phi_-^{-1}(n,z) &= Z^n T_\pm(z) Z^{-n} J_\pm^{-1}, \\ T_+(z) &= \begin{pmatrix} a_-(z) & b_-(z) \\ 0 & 1 \end{pmatrix}, \\ T_-(z) &= \begin{pmatrix} 1 & 0 \\ b_+(z) & a_-(z) \end{pmatrix}. \end{aligned} \tag{13}$$

Hence, (11)-(13) imply that

$$\begin{aligned} \det \Phi_+(n,z) &= a_+(z), \\ \det \Phi_-^{-1}(n,z) &= a_-(z), \end{aligned} \tag{14}$$

in view of (8). It is worth noting that the symmetry condition about the Jost functions in (7) implies the symmetry condition about the sectionally analytic function:

$$\Phi_+^T(n,z^-) = \Phi_-^{-1}(n,z). \tag{15}$$

Next, we shall discuss the asymptotic behavior of the functions $\Phi_+(n,z)$ and $\Phi_-^{-1}(n,z)$ in the z plane. Let the solution of (2) have the following asymptotic behaviors:

$$J(n,z) = J^{(0)}(n) + z^{-1}J^{(1)}(n) + O\left(z^{-2}\right), \quad z \longrightarrow \infty, \tag{16}$$

and

$$J^{-1}(n,z) = J_{(0)}(n) + zJ_{(1)}(n) + O\left(z^{2}\right), \quad z \longrightarrow 0. \tag{17}$$

Then, on use of the spectral problem (2) and its adjoint problem, the entries of the matrix $J^{(j)}(n)$ and $J_{(j)}(n)$, $j = 0, 1$, satisfy the following equations:

$$\begin{aligned} J_{11}^{(0)}(n+1) &= \gamma_n J_{11}^{(0)}(n), \\ J_{22}^{(0)}(n+1) &= \gamma_n^{-1} J_{22}^{(0)}(n), \\ J_{12}^{(1)}(n) &= -u(n) J_{22}^{(0)}(n), \\ J_{21}^{(0)}(n+1) &= -u(n) J_{11}^{(0)}(n+1), \end{aligned} \tag{18}$$

and $J_{12}^{(0)}(n) = 0$, $J_{(0)21}(n) = 0$,

$$\begin{aligned} J_{(0)11}(n+1) &= \gamma_n J_{(0)11}(n), \\ J_{(0)22}(n+1) &= \gamma_n^{-1} J_{(0)22}(n), \\ J_{(0)12}(n+1) &= -u(n) J_{(0)11}(n+1), \\ J_{(1)21}(n) &= -u(n) J_{(0)22}(n). \end{aligned} \tag{19}$$

We note that the Jost functions $J_\pm(n,z)$ have the same asymptotic behaviors as above.

Hence, asymptotic formulae for the sectionally analytic functions $\Phi_+(n,z)$ and $\Phi_-^{-1}(n,z)$ can be derived from (16) to (19) as

$$\Phi_+(n,z) \longrightarrow \Phi_+^{(0)}(n) = \begin{pmatrix} \nu_-(n) & 0 \\ -u(n-1)\nu_-(n) & \nu_+(n) \end{pmatrix}, \quad z \longrightarrow \infty, \tag{20}$$

and

$$\begin{aligned} \Phi_-^{-1}(n,z) &\longrightarrow \Phi_{-(0)}^{-1}(n) \\ &= \begin{pmatrix} \nu_-(n) & -u(n-1)\nu_-(n) \\ 0 & \nu_+(n) \end{pmatrix}, \quad z \longrightarrow 0, \end{aligned} \tag{21}$$

in view of the symmetry condition (7). Here the functions $\nu_\pm(n)$ are defined as

$$\begin{aligned} \nu_+(n) &= \prod_{l=n}^{\infty} \gamma_l, \\ \nu_-(n) &= \prod_{l=-\infty}^{n-1} \gamma_l, \end{aligned} \tag{22}$$

and γ_l is defined in (4).

It is noted that the potential $u(n)$ can be constructed from (18) as

$$u(n) = -\frac{J_{12}^{(1)}(n)}{J_{22}^{(0)}(n)} = -\lim_{z\longrightarrow\infty} \frac{(z\Phi_+)_{12}}{(\Phi_+)_{22}} = -\frac{\Phi_{+12}^{(1)}(n)}{\Phi_{+22}^{(0)}(n)}, \tag{23}$$

where the expansion

$$\Phi_+(n,z) = \Phi_+^{(0)}(n) + \frac{1}{z}\Phi_+^{(1)}(n) + \cdots, \quad z \longrightarrow \infty \tag{24}$$

has been used.

3. Regularization for the Riemann-Hilbert Problem with Higher-Order Zeros

From the representations (11) and (13) of the sectionally analytic functions, we find that they satisfy the jump condition on the unit circle, which gives

$$\begin{aligned} \Phi_-^{-1}(n,z)\Phi_+(n,z) &= Z^n G(z) Z^{-n}, \quad |z| = 1, \\ G(t,z) &= T_+S_+ = T_-S_- \\ &= \begin{pmatrix} 1 & b_-(z) \\ b_+(z) & 1 \end{pmatrix}. \end{aligned} \tag{25}$$

To obtain a unique solution of the Riemann-Hilbert (RH) problem (25), a normalization condition should be given. Here we take the condition (20) as the normalization condition, where $\det \Phi_+^{(0)}(n) = \nu = \nu_+(n)\nu_-(n) = \prod_{l=-\infty}^{\infty}\gamma_l$ is a constant. Note that this condition is noncanonical.

In order to derive the multiple-pole soliton solutions of the sdmKdV equation (1), we take $G(z) = I$, and suppose that $\det \Phi_+(n,z) = a_+(z)$ has one zero z_1 of order N, ($|z_1| > 1$). Then $\det \Phi_-^{-1}(n,z) = a_-(z)$ has one zero z_1^- of order N, in terms of (7). It is noted that the zero z_1 of the RH problem (25) is an elementary higher-order zero for the 2×2 matrix $\Phi_+(n,z)$, and the RH problem can be solved by means of its regularization. Thus, we have

$$\Phi_\pm(n,z) = \phi_\pm(n,z)\Gamma(n,z), \tag{26}$$

where the soliton matrix and its inverse are given as [7]

$$\begin{aligned} \Gamma(z) &= I + \sum_{l=1}^{N}\sum_{j=1}^{l} \frac{|\overline{q}_j\rangle\langle\overline{p}_{l+1-j}|}{\left(z - z_1^-\right)^{N+1-l}} \\ &= I + \left(|\overline{q}_N\rangle, \ldots, |\overline{q}_1\rangle\right)\overline{D}(z)\begin{pmatrix} \langle\overline{p}_1| \\ \vdots \\ \langle\overline{p}_N| \end{pmatrix}, \end{aligned} \tag{27}$$

$$\begin{aligned} \Gamma^{-1}(z) &= I + \sum_{l=1}^{N}\sum_{j=1}^{l} \frac{|p_{l+1-j}\rangle\langle q_j|}{\left(z - z_1\right)^{N+1-l}} \\ &= I + \left(|p_1\rangle, \ldots, |p_N\rangle\right)D(z)\begin{pmatrix} \langle q_N| \\ \vdots \\ \langle q_1| \end{pmatrix}, \end{aligned} \tag{28}$$

where

$$\overline{D}(z) = \begin{pmatrix} \frac{1}{(z-z_1^-)} & 0 & \cdots & 0 \\ \frac{1}{(z-z_1^-)^2} & \frac{1}{(z-z_1^-)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \frac{1}{(z-z_1^-)^N} & \cdots & \frac{1}{(z-z_1^-)^2} & \frac{1}{(z-z_1^-)} \end{pmatrix}, \quad (29)$$

and

$$D(z) = \begin{pmatrix} \frac{1}{(z-z_1)} & \frac{1}{(z-z_1)^2} & \cdots & \frac{1}{(z-z_1)^N} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \frac{1}{(z-z_1)} & \frac{1}{(z-z_1)^2} \\ 0 & \cdots & 0 & \frac{1}{(z-z_1)} \end{pmatrix}. \quad (30)$$

Here vectors $|p_j\rangle, \langle \overline{p}_j|, \langle q_j|, |\overline{q}_j\rangle, (j = 1\ldots, N)$ are independent of z. We note that, in these vectors, only half of them (i.e., $|p_j\rangle$ and $\langle \overline{p}_j|, (j = 1\ldots, N)$) are independent. It is readily verified that these vectors satisfy the following equations:

$$\mathbf{T}(\Gamma(z_1)) \begin{pmatrix} |p_1\rangle \\ \vdots \\ |p_N\rangle \end{pmatrix} = 0,$$
$$(\langle \overline{p}_1|, \ldots, \langle \overline{p}_N|) \mathbf{T}^T (\Gamma^{-1}(z_1^-)) = 0, \quad (31)$$

where the block Toeplitz matrix function $\mathbf{T}(\Gamma(z))$ is defined as

$$\mathbf{T}(\Gamma(z)) = \begin{pmatrix} \Gamma(z) & 0 & \cdots & 0 \\ \frac{1}{1!}\frac{d}{dz}\Gamma(z) & \Gamma(z) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \frac{1}{(N-1)!}\frac{d^{N-1}}{dz^{N-1}}\Gamma(z) & \cdots & \frac{1}{1!}\frac{d}{dz}\Gamma(z) & \Gamma(z) \end{pmatrix}. \quad (32)$$

For the sake of convenience, we introduce the following transformations:

$$\langle \overline{K}_j(z)| = \sum_{l=j}^{N} \frac{\langle \overline{p}_{l+1-j}|}{(z-z_1^-)^{N+1-l}},$$
$$|K_j(z)\rangle = \sum_{l=j}^{N} \frac{|p_{l+1-j}\rangle}{(z-z_1)^{N+1-l}}, \quad (33)$$

Then (27) and (28) can be rewritten as

$$\Gamma(z) = I - (|p_1\rangle, \ldots, |p_N\rangle)\, \overline{\Omega}^{-1} \overline{D}(z) \begin{pmatrix} \langle \overline{p}_1| \\ \vdots \\ \langle \overline{p}_N| \end{pmatrix}, \quad (34)$$

$$\Gamma^{-1}(z) = I - (|p_1\rangle, \ldots, |p_N\rangle)\, D(z)\, \Omega^{-1} \begin{pmatrix} \langle \overline{p}_1| \\ \vdots \\ \langle \overline{p}_N| \end{pmatrix}, \quad (35)$$

where

$$\Omega = \begin{pmatrix} \langle \overline{p}_1 \mid K_N(z_1^-)\rangle & \cdots & \langle \overline{p}_1 \mid K_1(z_1^-)\rangle \\ \langle \overline{p}_2 \mid K_N(z_1^-)\rangle + \frac{1}{1!}\frac{d}{dz}\langle \overline{p}_1 \mid K_N(z_1^-)\rangle & \cdots & \langle \overline{p}_2 \mid K_1(z_1^-)\rangle + \frac{1}{1!}\frac{d}{dz}\langle \overline{p}_1 \mid K_1(z_1^-)\rangle \\ \vdots & \vdots & \vdots \\ \sum_{l=1}^{N} \frac{1}{(N-l)!}\frac{d^{N-l}}{dz^{N-l}}\langle \overline{p}_l \mid K_N(z_1^-)\rangle & \cdots & \sum_{l=1}^{N} \frac{1}{(N-l)!}\frac{d^{N-l}}{dz^{N-l}}\langle \overline{p}_l \mid K_1(z_1^-)\rangle \end{pmatrix}, \quad (36)$$

$$\overline{\Omega} = \begin{pmatrix} \langle \overline{K}_N(z_1) \mid p_1\rangle & \langle \overline{K}_N(z_1) \mid p_2\rangle + \frac{1}{1!}\frac{d}{dz}\langle \overline{K}_N(z_1) \mid p_1\rangle & \cdots & \sum_{l=1}^{N} \frac{1}{(N-l)!}\frac{d^{N-l}}{dz^{N-l}}\langle \overline{K}_N(z_1) \mid p_1\rangle \\ \vdots & \vdots & & \vdots \\ \langle \overline{K}_1(z_1) \mid p_1\rangle & \langle \overline{K}_1(z_1) \mid p_2\rangle + \frac{1}{1!}\frac{d}{dz}\langle \overline{K}_1(z_1) \mid p_1\rangle & \cdots & \sum_{l=1}^{N} \frac{1}{(N-l)!}\frac{d^{N-l}}{dz^{N-l}}\langle \overline{K}_1(z_1) \mid p_1\rangle \end{pmatrix}. \quad (37)$$

From (28) and (34), we find

$$\Gamma^{(1)}(n) = -\left(|p_1\rangle, \dots, |p_N\rangle\right)\overline{\Omega}^{-1}\begin{pmatrix}\langle\overline{p}_1| \\ \vdots \\ \langle\overline{p}_N|\end{pmatrix}, \tag{38}$$

where $\Gamma^{(1)}(n)$ is defined by the expansion

$$\Gamma(n,z) = I + \frac{1}{z}\Gamma^{(1)}(n) + \cdots, \quad z \longrightarrow \infty. \tag{39}$$

4. Multiple-Pole Solitons

In this section, we shall construct the soliton solution of the sdmKdV equation by means of the soliton matrix in (34). To this end, we find, from (24), (39), and (26), that

$$\begin{gathered}\phi_+ = \Phi_+^{(0)}(n), \\ \Phi_+^{(1)}(n) = \Phi_+^{(0)}(n)\,\Gamma^{(1)}(n),\end{gathered} \tag{40}$$

The potential $u(n)$ can be rewritten as

$$u(n) = -\frac{\nu_-(n)}{\nu_+(n)}\Gamma_{12}^{(1)}(n), \tag{41}$$

in view of (20). From (22), we find that $\nu_+(n)$ and $\nu_-(n)$ are still representation of the potential $u(n)$. Hence, we need to establish the relationship between $\nu_\pm(n)$ and the soliton matrix Γ. Note that $G(z) = I$ implies $\Phi_+ = \Phi_-$, from which we can consider the asymptotic behavior of Φ_+ near $z = 0$. Indeed, the asymptotic formulae (20) and (21) imply that

$$\Phi_+ \longrightarrow \Phi_{-(0)}(n) = \begin{pmatrix}\nu_-^{-1}(n) & u(n-1)\,\nu_+^{-1}(n) \\ 0 & \nu_+^{-1}(n)\end{pmatrix}, \quad z \longrightarrow 0. \tag{42}$$

Thus from (26) we obtain

$$\begin{aligned}\Gamma(n,z)\big|_{z=0} &= \left(\Phi_+^{(0)}\right)^{-1}(n)\,\Phi_-\Big|_{z=0} \\ &= \frac{1}{\nu}\begin{pmatrix}\dfrac{\nu_+(n)}{\nu_-(n)} & u(n-1) \\ u(n-1) & \dfrac{\nu_-(n-1)}{\nu_+(n-1)}\end{pmatrix},\end{aligned} \tag{43}$$

which implies that $\nu_+(n)/\nu_-(n) = \nu\Gamma_{11}(n, z = 0)$. Here $\nu = \nu_+(n)\nu_-(n)$. As a result, the potential $u(n)$ takes the form

$$u(n) = -\frac{1}{\nu}\frac{\Gamma_{12}^{(1)}(n)}{\Gamma_{11}(n, z = 0)}. \tag{44}$$

On the other hand, $G(z) = I$ implies $a_+(z)a_+(z^-) = 1$, and from the zeros of $a_+(z)$, we know that

$$a_+(z) = \frac{(z - z_1)^N}{(zz_1 - 1)^N}. \tag{45}$$

Because $a_+(z) \longrightarrow \nu, z \longrightarrow \infty$, then we have

$$\frac{1}{\nu} = z_1^N. \tag{46}$$

Now we need to determine the vectors $|p_j\rangle$ and $\langle\overline{p}_j|$ in (34). It is noted that the soliton matrix $\Gamma(z)$ does not satisfy the linear system (2) and (3), if the normalization condition of the RH problem is noncanonical. Because $\Phi_+(n,z) = \Phi_+^{(0)}(n)\Gamma(n,z)$ and $\Phi_+^{(0)}(n)$ is independent of the spectral variable z, we obtain

$$\begin{gathered}\mathbf{\Phi}(z_1)\begin{pmatrix}|p_1\rangle \\ \vdots \\ |p_N\rangle\end{pmatrix} = 0, \\ \mathbf{\Phi}(z_1) = \mathbf{T}(\Phi(z_1)),\end{gathered} \tag{47}$$

in terms of (32). It is easy to verify that $\mathbf{\Phi}(z)$ admits

$$\begin{gathered}\mathbf{\Phi}(n+1,z) = \mathbf{U_n}\mathbf{Z}(z)\,\mathbf{\Phi}(n,z)\,\mathbf{Z}^{-1}(z), \\ \mathbf{\Phi}_t(n,z) = -i\left[\mathbf{\Lambda}(z), \mathbf{\Phi}(n,z)\right] - i\mathbf{V_n}(z)\,\mathbf{\Phi}(n,z),\end{gathered} \tag{48}$$

where $\mathbf{U_n} = \mathrm{diag}(U_n, \dots, U_n)_{(2N)\times(2N)}$, $(U_n = \gamma_n(\mathrm{I} + Q_n))$, is a diagonal block matrix, and

$$\begin{gathered}\mathbf{Z}(z) = \mathbf{T}(Z(z)), \\ \mathbf{Z}^{-1}(z) = \mathbf{T}\left(Z^{-1}(z)\right), \\ \mathbf{\Lambda}(z) = \mathbf{T}(\sigma_3\lambda(z)), \\ \mathbf{V_n}(z) = \mathbf{T}\left(\widetilde{Q}(z)\right).\end{gathered} \tag{49}$$

Equations (47) and (48) imply

$$\begin{gathered}\mathbf{\Phi}(n+1,z_1)\begin{pmatrix}|p_1(n+1)\rangle \\ \vdots \\ |p_N(n+1)\rangle\end{pmatrix} = 0, \\ \mathbf{\Phi}_t(n,z_1)\begin{pmatrix}|p_1(n)\rangle \\ \vdots \\ |p_N(n)\rangle\end{pmatrix} + \mathbf{\Phi}(n,z_1)\begin{pmatrix}|p_1(n)\rangle \\ \vdots \\ |p_N(n)\rangle\end{pmatrix}_t = 0.\end{gathered} \tag{50}$$

Thus, we take, for simplicity,

$$\begin{pmatrix}|p_1\rangle \\ \vdots \\ |p_N\rangle\end{pmatrix} = \mathbf{Z}^n(z_1)\exp\{-i\mathbf{\Lambda}(z_1)\,t\}\begin{pmatrix}|p_1\rangle_0 \\ \vdots \\ |p_N\rangle_0\end{pmatrix}, \tag{51}$$

or equivalently

$$\begin{pmatrix}|p_1\rangle \\ \vdots \\ |p_N\rangle\end{pmatrix} = \mathbf{T}(E(z_1))\begin{pmatrix}|p_1\rangle_0 \\ \vdots \\ |p_N\rangle_0\end{pmatrix}. \tag{52}$$

Here $|p_j\rangle_0$, $j = 1, \ldots, N$ are some constant vectors, and

$$E(z_1) = Z^n(z_1)\exp\{-i\lambda(z_1)t\sigma_3\}. \tag{53}$$

In addition, from the symmetry condition (15), we obtain

$$(\langle\overline{p}_1|, \ldots, \langle\overline{p}_N|) = (|p_1\rangle^T, \ldots, |p_N\rangle^T). \tag{54}$$

With the vectors $|p_j\rangle$ and $\langle\overline{p}_j|$ in hand, we shall construct the multiple-pole solitons of sdmKdV (1) by virtue of equations (44), (38), and (34). As an application, we consider the simplest multiple-pole soliton, that is, $N = 2$. In this case, (52) reduces to

$$\begin{aligned} |p_1\rangle &= E(z_1)|p_1\rangle_0, \\ |p_2\rangle &= \frac{d}{dz}E(z_1)|p_1\rangle_0 + E(z_1)|p_2\rangle_0. \end{aligned} \tag{55}$$

Let

$$\begin{aligned} |p_1\rangle_0 &= \begin{pmatrix} \alpha_1 \\ 1 \end{pmatrix}, \\ |p_2\rangle_0 &= \begin{pmatrix} \alpha_0\alpha_1 \\ 1 \end{pmatrix}, \end{aligned} \tag{56}$$

and (55) can be rewritten as

$$\begin{aligned} |p_1\rangle &= f(n,0)\begin{pmatrix} f(n,t) \\ f^{-1}(n,t) \end{pmatrix}, \\ |p_2\rangle &= f(n,0)\begin{pmatrix} \beta_1 f(n,t) \\ \beta_2 f^{-1}(n,t) \end{pmatrix}, \end{aligned} \tag{57}$$

where

$$\begin{aligned} f(n,t) &= \alpha_1^{1/2} z_1^{n/2} e^{-i\lambda(z_1)t}, \\ \beta_1 &= \frac{n}{z_1} - i\frac{d\lambda}{dz}(z_1)t + \alpha_0, \\ \beta_2 &= -i\frac{d\lambda}{dz}(z_1)t. \end{aligned} \tag{58}$$

Thus,

$$\begin{aligned} \langle\overline{p}_1| &= f_0(f, f^{-1}), \\ \langle\overline{p}_2| &= f_0(\beta_1 f, \beta_2 f^{-1}), \end{aligned} \tag{59}$$

where the abbreviations $f = f(n,t)$ and $f_0 = f(n,0)$ have been used. It is easy to find that

$$\begin{aligned} \overline{\Omega}_{11} &= f_0^2\frac{f^2 + f^{-2}}{z_1 - z_1^-}, \\ \overline{\Omega}_{12} &= f_0^2\left(\frac{\beta_1^2 f^2 + \beta_2^2 f^{-2}}{z_1 - z_1^-} - \frac{f^2 + f^{-2}}{(z_1 - z_1^-)^2}\right), \\ \overline{\Omega}_{12} &= f_0^2\left(\frac{\beta_1^2 f^2 + \beta_2^2 f^{-2}}{z_1 - z_1^-} + \frac{f^2 + f^{-2}}{(z_1 - z_1^-)^2}\right), \\ \overline{\Omega}_{22} &= f_0^2\left(\frac{\beta_1^2 f^2 + \beta_2^2 f^{-2}}{z_1 - z_1^-} - 2\frac{f^2 + f^{-2}}{(z_1 - z_1^-)^3}\right). \end{aligned} \tag{60}$$

Hence, we have

$$\det\overline{\Omega} = f_0^4\left(\frac{(\beta_1 - \beta_2)^2}{(z_1 - z_1^-)^2} - \frac{(f^2 + f^{-2})^2}{(z_1 - z_1^-)^4}\right). \tag{61}$$

Furthermore, we have

$$\Gamma_{12}^{(1)}(n) = \Delta\left[\beta_{12}(z_1 - z_1^-)^2 + 2(z_1 - z_1^-)\right](f^2 + f^{-2}), \tag{62}$$

$$\begin{aligned} \Gamma_{11}(n, z = 0) = {} & -z_1^2\Delta\left[\beta_{12}(z_1 - z_1^-)^3 - \beta_{12}^2(z_1 - z_1^-)^2\right. \\ & \left. + z_1^{-2}(f^2 + f^{-2})^2 + (z_1^2 - z_1^{-2})f^2(f^2 + f^{-2})\right], \end{aligned} \tag{63}$$

where

$$\begin{aligned} \Delta &= \frac{f_0^4}{(z_1 - z_1^-)^4\det\overline{\Omega}}, \\ \beta_{12} &= \beta_1 - \beta_2. \end{aligned} \tag{64}$$

We now take $z_1 = e^k$, ($|k| > 0$) and $f = e^{\theta/2}$; the multiple-pole soliton for $N = 2$ has the form

$$u = \frac{2(\beta_{12}\sinh^2 k + \sinh k)\cosh(\theta)}{2\beta_{12}\sinh^3 k - \beta_{12}^2\sinh^2 k + \cosh(\theta)\cosh(\theta + 2k)}, \tag{65}$$

in terms of $\nu = z_1^{-2}$. Here θ and β_{12} are defined by

$$\begin{aligned} \theta(n,t) &= nk - 2it\sinh k + \theta_0, \\ \beta_{12} &= e^{-k}(n - 2it\cosh k + \beta_0), \end{aligned} \tag{66}$$

and θ_0 and β_0 are taken to be real constants. We note that the parameter k is a complex number. In particular, if we take $k = k_1 + i\pi/2$, ($k_1 > 0$) (i.e., $z_1 = ie^{k_1}$), then

$$\begin{aligned} \sinh k &= i\sinh k_1, \\ \cosh k &= i\cosh k_1, \\ e^{-2k} &= -e^{-2k_1}, \\ \sinh 2k &= -\sinh 2k_1, \end{aligned} \tag{67}$$

and

$$\begin{aligned} \theta(n,t) &= \theta_1(n,t) + i\frac{n\pi}{2}, \\ \theta_1(n,t) &= nk_1 + 2t\sinh k_1 + \theta_0, \\ \beta_{12}(n,t) &= -i\rho(n,t), \\ \rho(n,t) &= e^{-k_1}(n + 2t\cosh k_1 + \beta_0). \end{aligned} \tag{68}$$

In this case, the multiple-pole soliton (65) has a different form

$$\begin{aligned} u(2n,t) &= \frac{-iW(2n,t)}{V(2n,t)}, \\ u(2n+1,t) &= \frac{W(2n+1,t)}{V(2n+1,t)}, \end{aligned} \tag{69}$$

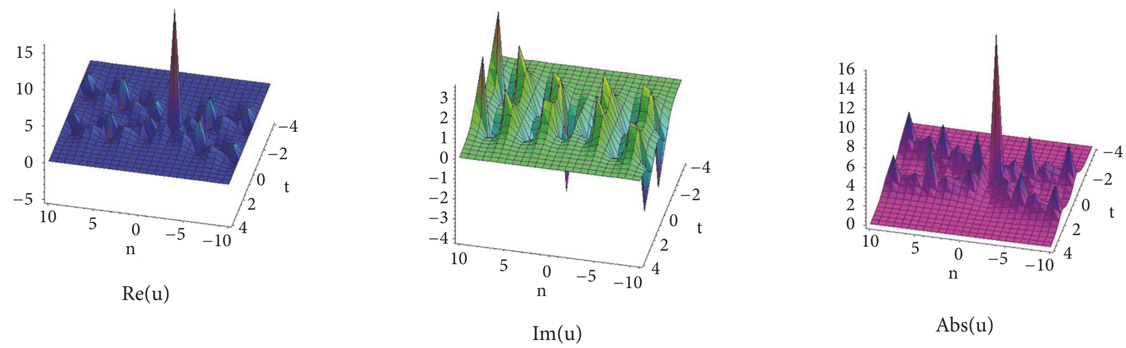

Figure 1: Solution (65) with $z_1 = ie^{0.1}, \theta_0 = \beta_0 = 0$.

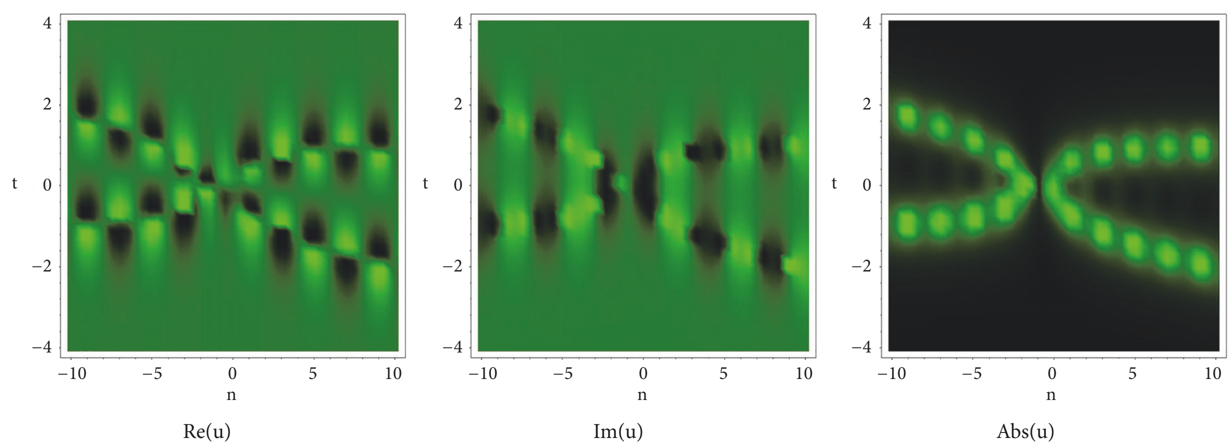

Figure 2: Densityplot of solution (65) with $z_1 = ie^{0.1}, \theta_0 = \beta_0 = 0$.

where

$$\begin{aligned}
W(2n,t) &= 2(-1)^n \left(\rho(2n,t)\sinh^2 k_1 + \sinh k_1\right) \\
&\quad \cdot \cosh\theta_1(2n,t), \\
W(2n+1,t) &= 2(-1)^n \\
&\quad \cdot \left(\rho(2n+1,t)\sinh^2 k_1 + \sinh k_1\right) \\
&\quad \cdot \sinh\theta_1(2n+1,t), \\
V(2n,t) &= 2\rho(2n,t)\sinh^3 k_1 + \rho^2(2n,t)\sinh^2 k_1 \\
&\quad + \cosh\{\theta_1(2n,t)\}\cosh\{\theta_1(2n,t)+2k_1\}, \\
V(2n+1,t) &= 2\rho(2n+1,t)\sinh^3 k_1 + \rho^2(2n+1,t) \\
&\quad \cdot \sinh^2 k_1 - \sinh\{\theta_1(2n+1,t)\} \\
&\quad \cdot \sinh\{\theta_1(2n+1,t)+2k_1\}.
\end{aligned} \tag{70}$$

From (69), one finds that if $z_1 = ie^{k_1}, (k_1 > 0)$, solution (65) is a real function for odd n or an imaginary function for even n (see Figure 2) and reaches its peak near $n = t = 0$ for the parameters chosen as $k_1 = 0.1, \theta_0 = \beta_0 = 0$ (see Figure 1).

We note that the multiple-pole soliton (65) possesses the character of two-soliton type solution (see the densityplot of (65) in Figure 2). The collision of the "two-soliton" is illustrated in Figures 1 and 2. However, each "single-soliton" in Figure 2 shows the character of a breather. In addition, one "single-soliton" is a transverse wave, and another is a longitudinal wave. Figure 3 shows the local pictures of the collision.

5. Conclusion

In this paper, we have considered a focusing Ablowitz-Ladik equation associated with a modified linear spectral problem and its adjoint problem. By means of the dressing method, we have constructed the nonregular Riemann-Hilbert problem with one zero of order N, which is closely related to the spectrum of discrete eigenvalues. The multiple-pole soliton solutions have been obtained in virtue of the regularization of the Riemann-Hilbert problem. For application, we have presented the simplest multiple-pole soliton solution and studied the relevant dynamics. In view of the analysis, we clearly see that the adaptive method is efficient for solving the nonregular Riemann-Hilbert problem with more higher-order zeros.

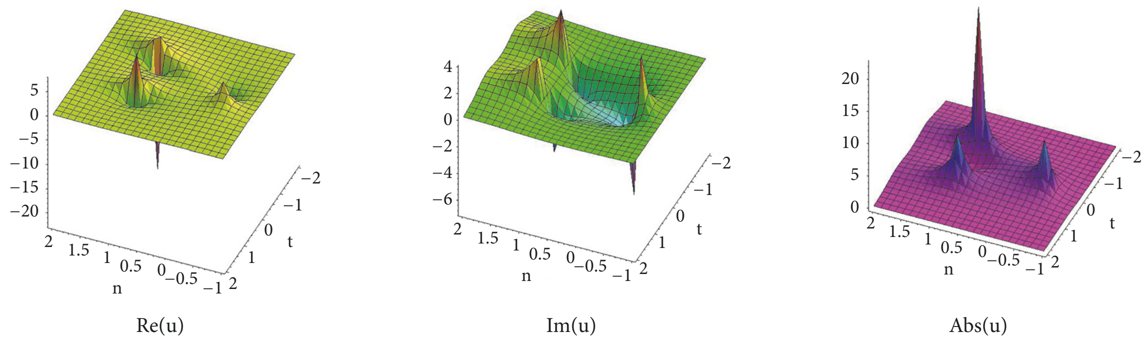

Figure 3: Collision of solution (65) with $z_1 = ie^{0.1}, \theta_0 = \beta_0 = 0$.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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