Research Article

A New Approach for the Approximate Analytical Solution of Space-Time Fractional Differential Equations by the Homotopy Analysis Method

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The motivation of this study is to construct the truncated solution of space-time fractional differential equations by the homotopy analysis method (HAM). The first space-time fractional differential equation is transformed into a space fractional differential equation or a time fractional differential equation before the HAM. Then the power series solution is constructed by the HAM. Finally, taking the illustrative examples into consideration we reach the conclusion that the HAM is applicable and powerful technique to construct the solution of space-time fractional differential equations.

1. Introduction

The fractional calculus is a very attractive concept for scientists in various fields since it has a wide application in mathematics, physics, biology, and so on [1–4]. Various fractional derivatives are defined to get the solution of fractional differential equations [5, 6]. In order to get the solution, different methods are introduced and applied [7–11]. These methods include the finite difference method [12], the homotopy perturbation method (HPM) [13, 14], and the generalized differential transform method [15].

Since the mathematical models include space and time variables in the modelling of mathematical and physical problems, fractional derivatives are required. The HAM provides as simple way to construct truncated solution of various differential equations. The advantage of the HAM is that the accuracy of the HAM does not depend on small parameter in the considered equation [16–18]. Therefore, compare the other technique HAM has many advantage in the application [19–21].

The motivation of this paper is to determine the truncated solution of the following space-time FPDE by implementing the HAM with a new algorithm by which space-time

fractional differential equation is reduced to either a space fractional differential equation or a time fractional differential equation making the problem easier to deal with by the HAM:

$$D_t^\alpha u = D_x^\beta u + F, \quad (1)$$

$$u(x, 0) = \varphi(x), \quad 0 < x \leq L \quad (2)$$

$$u(0, t) = \mu_1(t), \quad 0 < t \leq T \quad (3)$$

$$u_x(0, t) = \mu_2(t), \quad 0 < t \leq T \quad (4)$$

where $0 < \alpha \leq 1$, $1 < \beta \leq 2$ and the source function F depends on either x or t .

2. Preliminaries

This section is about basic definitions and fundamental properties of the fractional calculus.

Riemann-Liouville fractional integral operator of order α ($\alpha \geq 0$) is introduced as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha > 0, \quad x > 0 \quad (5)$$

$$J^0 f(x) = f(x). \quad (6)$$

Some of the properties of the operator J^α , which we will need in our work, are as follows [1, 2]:

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x) \quad (7)$$

$$J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x) \quad (8)$$

$$J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+1+\gamma)} x^{\gamma+\alpha} \quad (9)$$

If $m-1 < \alpha \leq m$, $m \in N$, then

$$D^\alpha J^\alpha f(x) = f(x) \quad (10)$$

$$J^\alpha D^\alpha f(x) = f(x) - \sum_{i=0}^{m-1} f^{(i)}(0^+) \frac{x^i}{i!}, \quad x > 0 \quad (11)$$

The Caputo fractional derivative of order α is defined as

$$\begin{aligned} D^\alpha f(x) &= J^{m-\alpha} D^m f(x) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt, \end{aligned} \quad (12)$$

$$m-1 < \alpha \leq m, \quad x > 0$$

where D^m is the classical differential operator of order m .

The definitions of the Mittag-Leffler function and the generalized Mittag-Leffler function are given, respectively, as follows:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0 \quad (13)$$

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0 \quad (14)$$

3. Basic Ideas of the Homotopy Analysis Method (HAM)

Let us consider the differential equation

$$N[u(x, t)] = 0, \quad (15)$$

where N is a nonlinear operator, x and t are independent variables, and $u(x, t)$ is an unknown function. Liao [17] introduced the so-called zero-order deformation equation by the homotopy method as follows:

$$\begin{aligned} (1-p)L[\phi(x, t; p) - u_0(x, t)] \\ = p\hbar H(x, t)N[\phi(x, t; p)], \end{aligned} \quad (16)$$

with the initial conditions

$$\phi^{(k)}(x, 0; p) = g_k(x), \quad k = 0, 1, 2, \dots, m-1, \quad (17)$$

where $p \in [0, 1]$, $\hbar \neq 0$ are parameters, $H(x, t)$ is nonzero auxiliary function, and $L = D_t^\alpha$, ($n-1 < \alpha \leq n$) is an auxiliary linear operator with the following feature:

$$L[\phi(x, t)] = 0, \quad \phi(x, t) = 0 \quad (18)$$

Obviously, for $p = 0$ and $p = 1$, we have

$$\phi(x, t; 0) = u_0(x, t), \quad (19)$$

$$\phi(x, t; 1) = u(x, t),$$

where $u_0(x, t)$ is an initial guess of $u(x, t)$ and $\phi(x, t; p)$ is an unknown function. As p goes to 1, the solution $\phi(x, t; p)$ changes from $u_0(x, t)$ to $u(x, t)$. By Taylor series of $\phi(x, t; p)$ based on p is written as

$$\phi(x, t; p) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t) p^m, \quad (20)$$

where

$$u_m(x, t) = \frac{1}{m!} \left. \frac{\partial^m \phi(x, t; p)}{\partial p^m} \right|_{p=0} \quad (21)$$

If the auxiliary linear operator, the initial guess, the auxiliary operator h , and the auxiliary function are properly chosen, the series (20) converges at $p = 1$, then we have

$$u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t). \quad (22)$$

Define the vector

$$u_n^{\rightarrow} = \{u_0(x, t), u_1(x, t), \dots, u_n(x, t)\}. \quad (23)$$

Differentiating (16) m -times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$, we obtain the m th-order deformation equation, with assumption $H(x, t) = 1$

$$L[u_m(x, t) - \chi_m u_{m-1}(x, t)] = \hbar R_m(u_{m-1}^{\rightarrow}) \quad (24)$$

subject to the following initial conditions:

$$u_m^{(k)}(x, 0) = 0, \quad k = 0, 1, 2, \dots, m-1, \quad (25)$$

where

$$R_m(u_{m-1}^{\rightarrow}) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} N[\phi(x, t; p)]}{\partial p^{m-1}} \right|_{p=0} \quad (26)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \quad (27)$$

We can apply the operator J^α to both sides of (24) to obtain

$$J^\alpha D^\alpha [u_m(x, t) - \chi_m u_{m-1}(x, t)] = \hbar J^\alpha R_m(u_{m-1}^-) \quad (28)$$

Using the Caputo derivative and the initial conditions, we obtain

$$u_m(x, t) = \chi_m u_{m-1}(x, t) + \hbar J^\alpha R_m(u_{m-1}^-) \quad (29)$$

Finally, we will approximate the HAM solution (22) by the truncated series

$$\phi_m(x, t) = \sum_{k=0}^{m-1} u_k(x, t) \quad (30)$$

The m th-order deformation (24) is linear and thus can be easily solved, especially by means of symbolic computation software such as Matlab and Maple.

4. Space-Time FPDE by the HAM

To construct the solution of problem (1)-(4) by the HAM, it is transformed into a space fractional differential problem or time fractional differential problem. The transformation we apply in this paper is motivated by the properties (7)-(9). Therefore let us take the problems with and without source functions into consideration:

(A) *Transforming into Time-Fractional Differential Equation*

(A1) *No Source Function.* In order to make the boundary condition (4) equal to zero, the transformation $v(x, t) = u(x, t) - u(x, 0)$ is applied to (1)-(2):

$$D_t^\alpha v = D_x^\beta v + D_x^\beta \varphi(x), \quad (31)$$

$$v(x, 0) = 0. \quad (32)$$

In order to make the order of derivative on the right-hand side, integer, the transformation $v = D_x^{2-\beta} w$ is applied to (31)-(32):

$$D_t^\alpha (D_x^{2-\beta} w) = D_{xx} w + D_x^\beta \varphi(x), \quad (33)$$

$$w(x, 0) = D_x^{\beta-2} v(x, 0) = 0. \quad (34)$$

Substituting (33) into (26), $R_m(u_{m-1}^-)$ can be given by

$$R_m(w_{m-1}^-) = D_t^\alpha (D_x^{2-\beta} w_{m-1}) - D_{xx} w_{m-1} - D_x^\beta \varphi(x) \quad (35)$$

The solution of m th-order deformation (26) for $m \geq 1$ becomes

$$w_m(x, t) = \chi_m w_{m-1}(x, t) + \hbar J^\alpha R_m(w_{m-1}^-) \quad (36)$$

By using (36) with the initial condition given by (34), we now successively obtain

$$w_0(x, t) = w(x, 0) = 0 \quad (37)$$

$$w_1(x, t) = -\hbar D_x^\beta \varphi(x) \frac{t^\alpha}{\Gamma(1+\alpha)} \quad (38)$$

$$w_2(x, t) = -\hbar D_x^\beta \varphi(x) \frac{t^\alpha}{\Gamma(1+\alpha)} - \hbar^2 D_x^2 \varphi(x) \frac{t^\alpha}{\Gamma(1+\alpha)} \quad (39)$$

$$+ \hbar^2 D_x^{\beta+2} \varphi(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$

$$w_3(x, t) = -\hbar D_x^\beta \varphi(x) \frac{t^\alpha}{\Gamma(1+\alpha)} - \hbar^2 D_x^2 \varphi(x) \frac{t^\alpha}{\Gamma(1+\alpha)} \quad (40)$$

$$+ \hbar^2 D_x^{\beta+2} \varphi(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$

$$- \hbar^2 D_x^2 \varphi(x) \frac{t^\alpha}{\Gamma(1+\alpha)}$$

$$- \hbar^3 D_x^{4-\beta} \varphi(x) \frac{t^\alpha}{\Gamma(1+\alpha)} \quad (40)$$

$$+ \hbar^3 D_x^4 \varphi(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$

$$+ \hbar^2 D_x^{\beta+2} \varphi(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$

$$+ \hbar^3 D_x^4 \varphi(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$

$$- \hbar^3 D_x^{\beta+4} \varphi(x) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}$$

and so on. Hence, by using first four terms of the approximate solution $w(x, t)$ can be obtained in the form of the series below:

$$w(x, t) = \sum_{k=0}^3 w_k(x, t) \quad (41)$$

By the transformations $u(x, t)$, the solution of problem (1)-(2) is obtained.

(A2) *Including the Source Function.* In order to make the boundary condition (4) equal to zero, the transformation $v(x, t) = u(x, t) - u(x, 0)$ is applied to (1)-(2):

$$D_t^\alpha v = D_x^\beta v + D_x^\beta \varphi(x) + f(x), \quad (42)$$

$$v(x, 0) = 0. \quad (43)$$

In order to make the order of derivative on the right-hand side, integer, the transformation $v = D_x^{2-\beta} w$ is applied to (42)-(43):

$$D_t^\alpha (D_x^{2-\beta} w) = D_{xx} w + D_x^\beta \varphi(x) + f(x), \quad (44)$$

$$w(x, 0) = D_x^{\beta-2} v(x, 0) = 0. \quad (45)$$

Substituting (26) into (44), $R_m(u_{m-1}^{\rightarrow})$ can be given by

$$R_m(w_{m-1}^{\rightarrow}) = D_t^\alpha (D_x^{2-\beta} w_{m-1}) - D_{xx} w_{m-1} - (1 - \chi_m) F(x) \quad (46)$$

The solution of m th-order deformation (46) for $m \geq 1$ becomes

$$w_m(x, t) = \chi_m w_{m-1}(x, t) + \hbar J^\alpha R_m(w_{m-1}^{\rightarrow}) \quad (47)$$

By using (47) with the initial condition given by (45), we now successively obtain

$$w_0(x, t) = 0 \quad (48)$$

$$w_1(x, t) = -\hbar (D_x^\beta \varphi(x) + f(x)) \frac{t^\alpha}{\Gamma(1+\alpha)} \quad (49)$$

$$\begin{aligned} w_2(x, t) = & -\hbar (D_x^\beta \varphi(x) + f(x)) \frac{t^\alpha}{\Gamma(1+\alpha)} \\ & - \hbar^2 D_x^2 \varphi(x) \frac{t^\alpha}{\Gamma(1+\alpha)} \\ & - \hbar^2 D_x^{2-\beta} f(x) \frac{t^\alpha}{\Gamma(1+\alpha)} \\ & + \hbar^2 D_x^{\beta+2} \varphi(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ & + \hbar^2 D_x^2 f(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \end{aligned} \quad (50)$$

$$\begin{aligned} w_3(x, t) = & -\hbar (D_x^\beta \varphi(x) + f(x)) \frac{t^\alpha}{\Gamma(1+\alpha)} \\ & - \hbar^2 D_x^2 \varphi(x) \frac{t^\alpha}{\Gamma(1+\alpha)} \\ & - \hbar^2 D_x^{2-\beta} f(x) \frac{t^\alpha}{\Gamma(1+\alpha)} \\ & + \hbar^2 D_x^{\beta+2} \varphi(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ & + \hbar^2 D_x^2 f(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ & - \hbar^2 D_x^2 \varphi(x) \frac{t^\alpha}{\Gamma(1+\alpha)} \\ & - \hbar^2 D_x^{2-\beta} f(x) \frac{t^\alpha}{\Gamma(1+\alpha)} \\ & - \hbar^3 D_x^{4-\beta} \varphi(x) \frac{t^\alpha}{\Gamma(1+\alpha)} \\ & - \hbar^3 D_x^{4-2\beta} f(x) \frac{t^\alpha}{\Gamma(1+\alpha)} \\ & + \hbar^3 D_x^4 \varphi(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ & - \hbar^2 D_x^{4-\beta} f(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ & + \hbar^2 D_x^{\beta+2} \varphi(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ & + \hbar^2 D_x^2 f(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ & + \hbar^3 D_x^4 \varphi(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ & + \hbar^3 D_x^{4-\beta} f(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ & - \hbar^3 D_x^{\beta+4} \varphi(x) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \\ & - \hbar^3 D_x^4 f(x) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \end{aligned} \quad (51)$$

and so on. Hence, the truncated solutions of $w(x, t)$ can be constructed in the series form by using the first four terms as follows:

$$w(x, t) = \sum_{k=0}^3 w_k(x, t) \quad (52)$$

By the transformations $u(x, t)$, the solution of problem (1)-(2) is determined.

(B) Transforming into Space-Fractional Differential Equation

(B1) No Source Function. By the transformation $u = D_t^{1-\alpha} v$ to (1), (3), and (4) are transformed into the problem below:

$$D_t v = D_x^\beta (D_t^{1-\alpha} v), \quad (53)$$

$$v(0, t) = D_t^{\alpha-1} \mu_1(t), \quad (54)$$

$$v_x(0, t) = D_t^{\alpha-1} \mu_2(t). \quad (55)$$

Substituting (26) into (53), $R_m(u_{m-1}^{\rightarrow})$ can be given by

$$R_m(v_{m-1}^{\rightarrow}) = D_x^\beta (D_t^{1-\alpha} v_{m-1}) - D_t v_{m-1} \quad (56)$$

The solution of m th-order deformation (56) for $m \geq 1$ becomes

$$v_m(x, t) = \chi_m v_{m-1}(x, t) + \hbar J^\beta R_m(v_{m-1}^{\rightarrow}) \quad (57)$$

By using (54)-(55), we now successively obtain

$$v_0(x, t) = v(x, 0) = D_t^{\alpha-1} \mu_1(t) + x D_t^{\alpha-1} \mu_2(t) \quad (58)$$

$$v_1(x, t) = -\hbar D_t^\alpha \mu_1(t) \frac{x^\beta}{\Gamma(1+\beta)} - \hbar D_t^\alpha \mu_2(t) \frac{x^{1+\beta}}{\Gamma(2+\beta)} \quad (59)$$

$$\begin{aligned} v_2(x, t) = & -\hbar D_t^\alpha \mu_1(t) \frac{x^\beta}{\Gamma(1+\beta)} - \hbar D_t^\alpha \mu_2(t) \frac{x^{1+\beta}}{\Gamma(2+\beta)} \\ & - \hbar^2 D_t \mu_1(t) \frac{x^\beta}{\Gamma(1+\beta)} - \hbar^2 D_t \mu_2(t) \frac{x^{1+\beta}}{\Gamma(2+\beta)} \\ & + \hbar^2 D_t^{1+\alpha} \mu_1(t) \frac{x^{2\beta}}{\Gamma(1+2\beta)} + \hbar^2 D_t^{1+\alpha} \mu_2(t) \frac{x^{1+2\beta}}{\Gamma(2+2\beta)} \end{aligned} \quad (60)$$

$$\begin{aligned} v_3(x, t) = & -\hbar D_t^\alpha \mu_1(t) \frac{x^\beta}{\Gamma(1+\beta)} - \hbar D_t^\alpha \mu_2(t) \frac{x^{1+\beta}}{\Gamma(2+\beta)} \\ & - \hbar^2 D_t \mu_1(t) \frac{x^\beta}{\Gamma(1+\beta)} - \hbar^2 D_t \mu_2(t) \frac{x^{1+\beta}}{\Gamma(2+\beta)} \\ & + \hbar^2 D_t^{1+\alpha} \mu_1(t) \frac{x^{2\beta}}{\Gamma(1+2\beta)} + \hbar^2 D_t^{1+\alpha} \mu_2(t) \frac{x^{1+2\beta}}{\Gamma(2+2\beta)} \\ & - \hbar^3 D_t \mu_1(t) \frac{x^\beta}{\Gamma(1+\beta)} - \hbar^3 D_t \mu_2(t) \frac{x^{1+\beta}}{\Gamma(2+\beta)} \\ & - \hbar^3 D_t \mu_1(t) \frac{x^\beta}{\Gamma(1+\beta)} - \hbar^3 D_t \mu_2(t) \frac{x^{1+\beta}}{\Gamma(2+\beta)} \\ & + \hbar^3 D_t^2 \mu_1(t) \frac{x^{2\beta}}{\Gamma(1+2\beta)} + \hbar^3 D_t^2 \mu_2(t) \frac{x^{1+2\beta}}{\Gamma(2+2\beta)} \\ & + \hbar^3 D_t^{\alpha+1} \mu_1(t) \frac{x^{2\beta}}{\Gamma(1+2\beta)} + \hbar^3 D_t^{\alpha+1} \mu_2(t) \frac{x^{1+2\beta}}{\Gamma(2+2\beta)} \\ & + \hbar^3 D_t^2 \mu_1(t) \frac{x^{2\beta}}{\Gamma(1+2\beta)} + \hbar^3 D_t^2 \mu_2(t) \frac{x^{1+2\beta}}{\Gamma(2+2\beta)} \\ & - \hbar^3 D_t^{\alpha+2} \mu_1(t) \frac{x^{3\beta}}{\Gamma(1+3\beta)} - \hbar^3 D_t^{\alpha+2} \mu_2(t) \frac{x^{1+3\beta}}{\Gamma(2+3\beta)} \end{aligned} \quad (61)$$

and so on. Hence, the truncated solutions of $v(x, t)$ can be constructed in the series form by using the first four terms below:

$$v(x, t) = \sum_{k=0}^3 v_k(x, t) \quad (62)$$

By the transformation $u(x, t) = D_t^{1-\alpha} v(x, t)$, the solution of problem (1), (3), and (4) is determined.

(B2) *Including Source Function.* Let us consider the problem below:

$$D_t v = D_x^\beta (D_t^{1-\alpha} v) + g(t), \quad (63)$$

$$v(0, t) = D_t^{\alpha-1} \mu_1(t), \quad (64)$$

$$v_x(0, t) = D_t^{\alpha-1} \mu_2(t). \quad (65)$$

Substituting (26) into (63), $R_m(u_{m-1}^{\rightarrow})$ can be given by

$$\begin{aligned} R_m(v_{m-1}^{\rightarrow}) &= D_x^\beta (D_t^{1-\alpha} v_{m-1}) - D_t v_{m-1} \\ &\quad + (1 - \chi_m) g(t) \end{aligned} \quad (66)$$

The solution of m th-order deformation (66) for $m \geq 1$ becomes

$$v_m(x, t) = \chi_m v_{m-1}(x, t) + \hbar J^\beta R_m(v_{m-1}^{\rightarrow}) \quad (67)$$

By using (64)-(65), we now successively obtain

$$v_0(x, t) = v(x, 0) = D_t^{\alpha-1} \mu_1(t) + x D_t^{\alpha-1} \mu_2(t) \quad (68)$$

$$v_1(x, t) = -\hbar D_t^\alpha \mu_1(t) \frac{x^\beta}{\Gamma(1+\beta)} \quad (69)$$

$$-\hbar D_t^\alpha \mu_2(t) \frac{x^{1+\beta}}{\Gamma(2+\beta)} + \hbar g(t) \frac{x^\beta}{\Gamma(1+\beta)}$$

$$\begin{aligned} v_2(x, t) &= -\hbar D_t^\alpha \mu_1(t) \frac{x^\beta}{\Gamma(1+\beta)} \\ &\quad -\hbar D_t^\alpha \mu_2(t) \frac{x^{1+\beta}}{\Gamma(2+\beta)} + \hbar g(t) \frac{x^\beta}{\Gamma(1+\beta)} \\ &\quad -\hbar^2 D_t \mu_1(t) \frac{x^\beta}{\Gamma(1+\beta)} \end{aligned}$$

$$\begin{aligned} &-\hbar^2 D_t \mu_2(t) \frac{x^{1+\beta}}{\Gamma(2+\beta)} \\ &+\hbar^2 D_t^{1-\alpha} g(t) \frac{x^\beta}{\Gamma(1+\beta)} \\ &+\hbar^2 D_t^{\alpha+1} \mu_1(t) \frac{x^{2\beta}}{\Gamma(1+2\beta)} \\ &+\hbar^2 D_t^{\alpha+1} \mu_2(t) \frac{x^{1+2\beta}}{\Gamma(2+2\beta)} \\ &-\hbar^2 D_t g(t) \frac{x^{2\beta}}{\Gamma(1+2\beta)} \end{aligned} \quad (70)$$

$$\begin{aligned} v_3(x, t) &= -\hbar D_t^\alpha \mu_1(t) \frac{x^\beta}{\Gamma(1+\beta)} \\ &\quad -\hbar D_t^\alpha \mu_2(t) \frac{x^{1+\beta}}{\Gamma(2+\beta)} + \hbar g(t) \frac{x^\beta}{\Gamma(1+\beta)} \\ &\quad -\hbar^2 D_t \mu_1(t) \frac{x^\beta}{\Gamma(1+\beta)} \\ &\quad -\hbar^2 D_t \mu_2(t) \frac{x^{1+\beta}}{\Gamma(2+\beta)} \\ &\quad +\hbar^2 D_t^{1-\alpha} g(t) \frac{x^\beta}{\Gamma(1+\beta)} \\ &\quad +\hbar^2 D_t^{\alpha+1} \mu_1(t) \frac{x^{2\beta}}{\Gamma(1+2\beta)} \\ &\quad +\hbar^2 D_t^{\alpha+1} \mu_2(t) \frac{x^{1+2\beta}}{\Gamma(2+2\beta)} \\ &\quad -\hbar^2 D_t g(t) \frac{x^{2\beta}}{\Gamma(1+2\beta)} \\ &\quad -\hbar^2 D_t \mu_1(t) \frac{x^\beta}{\Gamma(1+\beta)} \\ &\quad -\hbar^2 D_t \mu_2(t) \frac{x^{1+\beta}}{\Gamma(2+\beta)} \\ &\quad +\hbar^2 D_t^{1-\alpha} g(t) \frac{x^\beta}{\Gamma(1+\beta)} \\ &\quad -\hbar^3 D_t^{2-\alpha} \mu_1(t) \frac{x^\beta}{\Gamma(1+\beta)} \\ &\quad -\hbar^3 D_t^{2-\alpha} \mu_2(t) \frac{x^{1+\beta}}{\Gamma(2+\beta)} \\ &\quad +\hbar^3 D_t^{2-2\alpha} g(t) \frac{x^\beta}{\Gamma(1+\beta)} \end{aligned}$$

$$+ \hbar^3 D_t^2 \mu_1(t) \frac{x^{2\beta}}{\Gamma(1+2\beta)}$$

$$+ \hbar^3 D_t^2 \mu_2(t) \frac{x^{1+2\beta}}{\Gamma(2+2\beta)}$$

$$- \hbar^3 D_t^{2-\alpha} g(t) \frac{x^{2\beta}}{\Gamma(1+2\beta)}$$

$$+ \hbar^2 D_t^{\alpha+1} \mu_1(t) \frac{x^{2\beta}}{\Gamma(1+2\beta)}$$

$$+ \hbar^2 D_t^{\alpha+1} \mu_2(t) \frac{x^{1+2\beta}}{\Gamma(2+2\beta)}$$

$$- \hbar^2 D_t g(t) \frac{x^{2\beta}}{\Gamma(1+2\beta)}$$

$$+ \hbar^3 D_t^2 \mu_1(t) \frac{x^{2\beta}}{\Gamma(1+2\beta)}$$

$$+ \hbar^3 D_t^2 \mu_2(t) \frac{x^{1+2\beta}}{\Gamma(2+2\beta)}$$

$$+ \hbar^3 D_t^{2-\alpha} g(t) \frac{x^{2\beta}}{\Gamma(1+2\beta)}$$

$$- \hbar^3 D_t^{\alpha+2} \mu_1(t) \frac{x^{3\beta}}{\Gamma(1+3\beta)}$$

$$- \hbar^3 D_t^{\alpha+2} \mu_2(t) \frac{x^{1+3\beta}}{\Gamma(2+3\beta)}$$

$$+ \hbar^3 D_t^2 g(t) \frac{x^{3\beta}}{\Gamma(1+3\beta)}$$

(71)

and so on. Hence, the truncated solutions of $v(x, t)$ can be constructed in the series form by using the first four terms below:

$$v(x, t) = \sum_{k=0}^3 v_k(x, t) \quad (72)$$

By the transformation $u(x, t) = D_t^{1-\alpha} v(x, t)$, the solution of problem (1), (3), and (4) is determined.

5. Examples

Example 1. Let us take the homogeneous problem.

(A1) Including No Source Function

$$D_t^\alpha u = D_x^\beta u, \quad (73)$$

$$u(x, 0) = \cos(x), \quad (74)$$

$$u(0, t) = \exp(-t), \quad (75)$$

$$u_x(0, t) = 0. \quad (76)$$

Transforming into time-fractional differential equation, (73)-(74) are reduced as follows:

$$D_t^\alpha (D_x^{2-\beta} w) = D_{xx} w \quad (77)$$

$$- \frac{1}{2} x^{2-\beta} (E_{1,3-\beta}(ix) + E_{1,3-\beta}(-ix)),$$

$$w(x, 0) = 0. \quad (78)$$

Using the HAM and by the transformations $u(x, t)$, we obtain the following solution of problem (73)-(76):

$$u(x, t) = \cos(x) + \left(3\hbar \cos(x) \right. \\ + \frac{3\hbar^2 i}{2} x^{\beta-1} (E_{1,\beta}(ix) - E_{1,\beta}(-ix)) \\ - \frac{\hbar^3}{2} x^{2\beta-2} (E_{1,2\beta-1}(ix) + E_{1,2\beta-1}(-ix)) \Big) \\ \cdot \frac{t^\alpha}{\Gamma(1+\alpha)} \left(3\hbar^2 \cos(x) \right. \\ + \frac{2\hbar^3 i}{2} x^{\beta-1} (E_{1,\beta}(ix) - E_{1,\beta}(-ix)) \Big) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ + \hbar^3 \cos(x) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \quad (79)$$

In a particular case for $\alpha = 1$ and $\beta = 2$, the HAM solution of (73)-(76) coincides with the exact solution $u(x, t) = \exp(-t) \cos(x)$.

(B1) Including No Source Function

$$D_t^\alpha u = D_x^\beta u, \quad (80)$$

$$u(x, 0) = \exp(x), \quad (81)$$

$$u(0, t) = \exp(t), \quad (82)$$

$$u_x(0, t) = \exp(t). \quad (83)$$

Transforming into space-fractional differential equation, (80), (82), and (83) are reduced to the problem below:

$$D_t v = D_x^\beta (D_t^{1-\alpha} v), \quad (84)$$

$$v(0, t) = D_t^{\alpha-1} \exp(t) = t^{-\alpha} E_{1,1-\alpha}(t), \quad (85)$$

$$v_x(0, t) = D_t^{\alpha-1} \exp(t) = t^{-\alpha} E_{1,1-\alpha}(t), \quad (86)$$

Using the HAM and applying the transformation $u(x, t) = D_t^{1-\alpha} v(x, t)$, we obtain the following solution of problem (80), (82), and (83):

$$\begin{aligned}
 u_0(x, t) &= \exp(t) (1 + x) \\
 u_1(x, t) &= -\hbar \exp(t) \left(\frac{x^\beta}{\Gamma(1+\beta)} + \frac{x^{1+\beta}}{\Gamma(2+\beta)} \right) \\
 u_2(x, t) &= -\hbar \exp(t) \left(\frac{x^\beta}{\Gamma(1+\beta)} + \frac{x^{1+\beta}}{\Gamma(2+\beta)} \right) \\
 &\quad - \hbar^2 t^{2-\alpha} E_{1,3-\alpha}(t) \left(\frac{x^\beta}{\Gamma(1+\beta)} + \frac{x^{1+\beta}}{\Gamma(2+\beta)} \right) \\
 &\quad + \hbar^2 \exp(t) \left(\frac{x^{2\beta}}{\Gamma(1+2\beta)} + \frac{x^{1+2\beta}}{\Gamma(2+2\beta)} \right) \\
 u_3(x, t) &= -\hbar \exp(t) \left(\frac{x^\beta}{\Gamma(1+\beta)} + \frac{x^{1+\beta}}{\Gamma(2+\beta)} \right) \\
 &\quad - \hbar^2 t^{2-\alpha} E_{1,3-\alpha}(t) \left(\frac{x^\beta}{\Gamma(1+\beta)} + \frac{x^{1+\beta}}{\Gamma(2+\beta)} \right) \\
 &\quad + \hbar^2 \exp(t) \left(\frac{x^{2\beta}}{\Gamma(1+2\beta)} + \frac{x^{1+2\beta}}{\Gamma(2+2\beta)} \right) \\
 &\quad - \hbar^2 t^{2-\alpha} E_{1,3-\alpha}(t) \left(\frac{x^\beta}{\Gamma(1+\beta)} + \frac{x^{1+\beta}}{\Gamma(2+\beta)} \right) \\
 &\quad - \hbar^3 t^{2-\alpha} E_{1,3-\alpha}(t) \left(\frac{x^\beta}{\Gamma(1+\beta)} + \frac{x^{1+\beta}}{\Gamma(2+\beta)} \right) \\
 &\quad + \hbar^3 t^{3-\alpha} E_{1,4-\alpha}(t) \left(\frac{x^{2\beta}}{\Gamma(1+2\beta)} + \frac{x^{1+2\beta}}{\Gamma(2+2\beta)} \right) \\
 &\quad + \hbar^2 \exp(t) \left(\frac{x^{2\beta}}{\Gamma(1+2\beta)} + \frac{x^{1+2\beta}}{\Gamma(2+2\beta)} \right) \\
 &\quad + \hbar^3 t^{3-\alpha} E_{1,4-\alpha}(t) \left(\frac{x^{2\beta}}{\Gamma(1+2\beta)} + \frac{x^{1+2\beta}}{\Gamma(2+2\beta)} \right) \\
 &\quad - \hbar^3 \exp(t) \left(\frac{x^{3\beta}}{\Gamma(1+3\beta)} + \frac{x^{1+3\beta}}{\Gamma(2+3\beta)} \right)
 \end{aligned} \tag{87}$$

which coincides with the exact solution.

In Tables 1 and 2, the absolute errors have been worked out for various values of α and β . As can be seen from the tables, for the fixed values of α and β as x increases, the absolute error decreases, while as t increases, the absolute error also increases. Moreover as α increases to 1, β increases to 2 and the absolute error decreases. In addition, the absolute error

takes the minimum values at the auxiliary values $\hbar = -0.94$, $\hbar = -1.02$ for case A and case B, respectively.

Example 2. Let us take the nonhomogeneous problem.

(A2) Including Source Function $f(x)$

$$D_t^\alpha u = D_x^\beta u + 3 \cos(x), \tag{88}$$

$$u(x, 0) = 4 \cos(x), \tag{89}$$

$$u(0, t) = 3 + \exp(-t), \tag{90}$$

$$u_x(0, t) = 0. \tag{91}$$

Reduction to time-fractional differential equation, (88)-(89) transform to the following:

$$\begin{aligned}
 D_t^\alpha (D_x^{2-\beta} w) &= D_{xx} w \\
 &\quad - \frac{1}{2} x^{2-\beta} (E_{1,3-\beta}(ix) + E_{1,3-\beta}(-ix)) \\
 &\quad + 3 \cos(x), \\
 w(x, 0) &= 0.
 \end{aligned} \tag{92}$$

Using the HAM and by the transformations $u(x, t)$, we obtain the following solution of problem (88)-(91):

$$\begin{aligned}
 u(x, t) &= 4 \cos(x) + \left(12\hbar \cos(x) \right. \\
 &\quad - \frac{9\hbar i}{2} x^{\beta-1} (E_{1,\beta}(ix) - E_{1,\beta}(-ix)) \\
 &\quad + 6\hbar^2 ix^{\beta-1} (E_{1,\beta}(ix) - E_{1,\beta}(-ix)) \\
 &\quad + \frac{9\hbar^2}{2} x^{2\beta-2} (E_{1,2\beta-1}(ix) + E_{1,2\beta-1}(-ix)) \\
 &\quad - 2\hbar^2 x^{2\beta-2} (E_{1,2\beta-1}(ix) + E_{1,2\beta-1}(-ix)) \\
 &\quad + \frac{3\hbar^3 i}{2} x^{3\beta-2} (E_{1,3\beta-1}(ix) - E_{1,3\beta-1}(-ix)) \Big) \\
 &\quad \cdot \frac{t^\alpha}{\Gamma(1+\alpha)} \left(12\hbar^2 \cos(x) - \frac{9\hbar^2 i}{2} x^{\beta-1} (E_{1,\beta}(ix) \right. \\
 &\quad - E_{1,\beta}(-ix)) + \frac{8\hbar^3 i}{2} x^{\beta-1} (E_{1,\beta}(ix) - E_{1,\beta}(-ix)) \\
 &\quad + 3\hbar^3 x^{2\beta-2} (E_{1,2\beta-1}(ix) + E_{1,2\beta-1}(-ix)) \Big) \\
 &\quad \cdot \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \left(4\hbar^3 \cos(x) - \frac{3\hbar^3 i}{2} x^{\beta-1} (E_{1,\beta}(ix) \right. \\
 &\quad - E_{1,\beta}(-ix)) \Big) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}
 \end{aligned} \tag{94}$$

TABLE 1: The absolute errors for case A in Example 1.

t	x	$\hbar = -0.94$			$\hbar = -1$		
		$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
		$\beta = 1.5$	$\beta = 1.75$	$\beta = 2$	$\beta = 1.5$	$\beta = 1.75$	$\beta = 2$
0.3	0.2	0.68841	0.03769	0.00004	0.81871	0.04127	0.00031
	0.4	0.27514	0.01312	0.00004	0.37117	0.00200	0.00029
	0.6	0.08483	0.04706	0.00003	0.16617	0.03354	0.00026
	0.8	0.03374	0.07006	0.00003	0.03721	0.05572	0.00022
0.6	0.2	1.02204	0.10261	0.00044	1.16631	0.08697	0.00472
	0.4	0.55654	0.06402	0.00041	0.67960	0.07240	0.00443
	0.6	0.33977	0.02800	0.00037	0.45762	0.04762	0.00397
	0.8	0.19646	0.00200	0.00031	0.31112	0.02465	0.00335
0.9	0.2	1.21882	0.11304	0.00636	1.34963	0.05854	0.02261
	0.4	0.76521	0.12078	0.00598	0.89617	0.11277	0.02125
	0.6	0.55660	0.10408	0.00536	0.69660	0.12060	0.01904
	0.8	0.41428	0.08328	0.00452	0.56229	0.11704	0.01607

TABLE 2: The absolute errors for case B in Example 1.

t	x	$\hbar = -1.02$			$\hbar = -1$		
		$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
		$\beta = 1.5$	$\beta = 1.75$	$\beta = 2$	$\beta = 1.5$	$\beta = 1.75$	$\beta = 2$
0.3	0.2	0.07111	0.02540	0.00000	0.07111	0.02540	0.00000
	0.4	0.19034	0.07464	0.00000	0.19032	0.07464	0.00000
	0.6	0.34423	0.14093	0.00000	0.34412	0.14092	0.00000
	0.8	0.53422	0.22399	0.00000	0.53372	0.22390	0.00001
0.6	0.2	0.09598	0.03428	0.00000	0.09598	0.03428	0.00000
	0.4	0.25693	0.10075	0.00000	0.25691	0.10076	0.00000
	0.6	0.46467	0.19024	0.00000	0.46452	0.19023	0.00000
	0.8	0.72113	0.30235	0.00000	0.72045	0.30223	0.00001
0.9	0.2	0.12957	0.04627	0.00000	0.12957	0.04627	0.00000
	0.4	0.34681	0.13600	0.00000	0.34679	0.13601	0.00000
	0.6	0.62724	0.25680	0.00000	0.62703	0.25678	0.00000
	0.8	0.97342	0.40813	0.00000	0.97250	0.40797	0.00001

For $\alpha = 1$ and $\beta = 2$, the HAM solution of (88)-(91) coincides with the exact solution $u(x, t) = (\exp(-t) + 3) \cos(x)$.

(B2) Including Source Function $g(t)$

$$D_t^\alpha u = D_x^\beta u + 3t^2, \quad (95)$$

$$u(x, 0) = \exp(x), \quad (96)$$

$$u(0, t) = \exp(t) + t^3, \quad (97)$$

$$u_x(0, t) = \exp(t). \quad (98)$$

Transforming (95), (97), and (98) into the following problem, we have

$$D_t v = D_x^\beta (D_t^{1-\alpha} v) + 3t^2, \quad (99)$$

$$\begin{aligned} v(0, t) &= D_t^{\alpha-1} (\exp(t) + t^3) \\ &= t^{-\alpha} E_{1,1-\alpha}(t) + 6 \frac{t^{4-\alpha}}{\Gamma(5-\alpha)}, \end{aligned} \quad (100)$$

$$v_x(0, t) = D_t^{\alpha-1} \exp(t) = t^{-\alpha} E_{1,1-\alpha}(t). \quad (101)$$

Using the HAM and the transformation $u(x, t) = D_t^{1-\alpha} v(x, t)$, we obtain the following solution of problem (95), (97), and (98):

$$u_0(x, t) = (\exp(t) + t^3) + x \exp(t)$$

$$u_1(x, t)$$

$$\begin{aligned} &= -\hbar (\exp(t) + 3t^2) \frac{x^\beta}{\Gamma(1+\beta)} \\ &\quad - \hbar \exp(t) \frac{x^{1+\beta}}{\Gamma(2+\beta)} + 3\hbar t^2 \frac{x^\beta}{\Gamma(1+\beta)} \end{aligned}$$

$$\begin{aligned}
u_2(x, t) &= -\hbar (\exp(t) + 3t^2) \frac{x^\beta}{\Gamma(1+\beta)} \\
&\quad - \hbar \exp(t) \frac{x^{1+\beta}}{\Gamma(2+\beta)} + 3\hbar t^2 \frac{x^\beta}{\Gamma(1+\beta)} \\
&\quad - \hbar^2 \left(t^{2-\alpha} E_{1,3-\alpha}(t) + 6 \frac{t^{1+\alpha}}{\Gamma(2+\alpha)} \right) \frac{x^\beta}{\Gamma(1+\beta)} \\
&\quad - \hbar^2 t^{2-\alpha} E_{1,3-\alpha}(t) + 6\hbar^2 \frac{t^{1+\alpha}}{\Gamma(2+\alpha)} \frac{x^\beta}{\Gamma(1+\beta)} \\
&\quad + \hbar^2 (\exp(t) + 6t) \frac{x^{2\beta}}{\Gamma(1+2\beta)} \\
&\quad + \hbar^2 \exp(t) \frac{x^{1+2\beta}}{\Gamma(2+2\beta)} - 6\hbar^2 t \frac{x^{2\beta}}{\Gamma(1+2\beta)} \\
u_3(x, t) &= -\hbar (\exp(t) + 3t^2) \frac{x^\beta}{\Gamma(1+\beta)} \\
&\quad - \hbar \exp(t) \frac{x^{1+\beta}}{\Gamma(2+\beta)} + 3\hbar t^2 \frac{x^\beta}{\Gamma(1+\beta)} \\
&\quad - \hbar^2 \left(t^{2-\alpha} E_{1,3-\alpha}(t) + 6 \frac{t^{1+\alpha}}{\Gamma(2+\alpha)} \right) \frac{x^\beta}{\Gamma(1+\beta)} \\
&\quad - \hbar^2 t^{2-\alpha} E_{1,3-\alpha}(t) \frac{x^{1+\beta}}{\Gamma(2+\beta)} \\
&\quad + 6\hbar^2 \frac{t^{1+\alpha}}{\Gamma(2+\alpha)} \frac{x^\beta}{\Gamma(1+\beta)} \\
&\quad + \hbar^2 (\exp(t) + 6t) \frac{x^{2\beta}}{\Gamma(1+2\beta)} \\
&\quad + \hbar^2 \exp(t) \frac{x^{1+2\beta}}{\Gamma(2+2\beta)} - 6\hbar^2 t \frac{x^{2\beta}}{\Gamma(1+2\beta)} \\
&\quad - \hbar^2 \left(t^{2-\alpha} E_{1,3-\alpha}(t) + 6 \frac{t^{1+\alpha}}{\Gamma(2+\alpha)} \right) \frac{x^\beta}{\Gamma(1+\beta)} \\
&\quad - \hbar^2 t^{2-\alpha} E_{1,3-\alpha}(t) \frac{x^{1+\beta}}{\Gamma(2+\beta)} \\
&\quad + 6\hbar^2 \frac{t^{1+\alpha}}{\Gamma(2+\alpha)} \frac{x^\beta}{\Gamma(1+\beta)} \\
&\quad - \hbar^3 \left(t^{3-\alpha} E_{1,4-\alpha}(t) + 6 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \right) \frac{x^\beta}{\Gamma(1+\beta)} \\
&\quad - \hbar^3 t^{3-\alpha} E_{1,4-\alpha}(t) \frac{x^{1+\beta}}{\Gamma(2+\beta)}
\end{aligned}$$

$$\begin{aligned}
&+ 6\hbar^3 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \frac{x^\beta}{\Gamma(1+\beta)} \\
&+ \hbar^3 \left(t^{3-\alpha} E_{1,4-\alpha}(t) + 6 \frac{t^\alpha}{\Gamma(1+\alpha)} \right) \frac{x^{2\beta}}{\Gamma(1+2\beta)} \\
&+ \hbar^3 t^{3-\alpha} E_{1,4-\alpha}(t) \frac{x^{1+2\beta}}{\Gamma(2+2\beta)} \\
&- 6\hbar^3 \frac{t^\alpha}{\Gamma(1+\alpha)} \frac{x^{2\beta}}{\Gamma(1+2\beta)} \\
&+ \hbar^2 (\exp(t) + 6t) \frac{x^{2\beta}}{\Gamma(1+2\beta)} \\
&+ \hbar^2 \exp(t) \frac{x^{1+2\beta}}{\Gamma(2+2\beta)} - 6\hbar^2 t \frac{x^{2\beta}}{\Gamma(1+2\beta)} \\
&+ \hbar^3 \left(t^{3-\alpha} E_{1,4-\alpha}(t) + 6 \frac{t^\alpha}{\Gamma(1+\alpha)} \right) \frac{x^{2\beta}}{\Gamma(1+2\beta)} \\
&+ \hbar^3 t^{3-\alpha} E_{1,4-\alpha}(t) \frac{x^{1+2\beta}}{\Gamma(2+2\beta)} \\
&+ 6\hbar^3 \frac{t^\alpha}{\Gamma(1+\alpha)} \frac{x^{2\beta}}{\Gamma(1+2\beta)} \\
&- \hbar^3 (\exp(t) + 6) \frac{x^{3\beta}}{\Gamma(1+3\beta)} \\
&- \hbar^3 \exp(t) \frac{x^{1+3\beta}}{\Gamma(2+3\beta)} + 6\hbar^3 \frac{x^{3\beta}}{\Gamma(1+3\beta)}
\end{aligned} \tag{102}$$

which is the exact solution.

In Tables 3 and 4, the absolute errors have been worked out for various values of α and β . As is obvious from the tables, as α increases to 1, β increases to 2 and the absolute error decreases. In addition, the absolute error takes the minimum values at the auxiliary values $\hbar = -0.94$, $\hbar = -0.57$ for case A and case B, respectively.

6. Conclusion

In this research, the truncated solution of space-time FPDEs is constructed by the HAM and special transformations which allows us to implement the HAM to deal with harder problems. It is obvious that applying the HAM to the equation including two different fractional derivatives is harder than applying the HAM to the equation including one fractional derivative. Therefore, the transformation in this paper allows us to find the solution of the problem by the HAM much more easier. Examples manifest that when the order of space and time derivatives is fractional, the result of the HAM method gives better results for smaller values of $\hbar$ in some interval unlike the integer order of space and time fractional derivatives.

TABLE 3: The absolute errors for case A in Example 2.

t	x	$\hbar = -0.94$			$\hbar = -1$		
		$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
		$\beta = 1.5$	$\beta = 1.75$	$\beta = 2$	$\beta = 1.5$	$\beta = 1.75$	$\beta = 2$
0.3	0.2	0.80914	0.14812	0.00004	0.34918	0.08528	0.00031
	0.4	1.16838	0.03354	0.00004	1.17892	0.01225	0.00029
	0.6	0.71218	0.09107	0.00003	0.82383	0.09475	0.00026
	0.8	0.28227	0.19786	0.00003	0.43360	0.19139	0.00022
0.6	0.2	0.23377	0.10803	0.00044	1.19358	0.03953	0.00472
	0.4	1.06874	0.01958	0.00041	0.89593	0.05309	0.00443
	0.6	0.80431	0.11677	0.00037	0.83515	0.15221	0.00397
	0.8	0.44477	0.24245	0.00031	0.57549	0.25288	0.00335
0.9	0.2	1.63325	0.07079	0.00636	3.10231	0.32344	0.02261
	0.4	0.65994	0.08755	0.00598	0.25862	0.23737	0.02125
	0.6	0.67465	0.27502	0.00536	0.57511	0.28500	0.01904
	0.8	0.46155	0.47543	0.00452	0.52316	0.34909	0.01607

TABLE 4: The absolute errors for case B in Example 2.

t	x	$\hbar = -0.57$			$\hbar = -1$		
		$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
		$\beta = 1.5$	$\beta = 1.75$	$\beta = 2$	$\beta = 1.5$	$\beta = 1.75$	$\beta = 2$
0.3	0.2	0.10133	0.03198	0.00243	0.13890	0.04687	0.00825
	0.4	0.29402	0.10097	0.01034	0.39635	0.15200	0.03468
	0.6	0.57660	0.20716	0.02621	0.75465	0.31273	0.08418
	0.8	0.96440	0.35981	0.05427	1.22052	0.53736	0.16431
0.6	0.2	0.17372	0.06593	0.01563	0.30196	0.10909	0.03275
	0.4	0.50798	0.21636	0.06467	0.87541	0.36506	0.13518
	0.6	0.99839	0.45298	0.15447	1.68492	0.76466	0.31946
	0.8	1.67026	0.79509	0.29737	2.75087	1.32983	0.60520
0.9	0.2	0.27067	0.11810	0.03777	0.48542	0.20136	0.07345
	0.4	0.78697	0.39088	0.15460	1.40930	0.67743	0.30029
	0.6	1.53126	0.81511	0.36227	2.71061	1.41561	0.69954
	0.8	2.53281	1.41600	0.68048	4.41814	2.44668	1.30201

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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