We study a dynamic three-dimensional (3D) field localized states in a medium with percolation disorder, where the percolation cluster is filled by the active nanoemitters. In such a system, the incipient percolating cluster generates a fractal radiating structure in which the field is radiated and scattered by the anisotropic inhomogeneity. Our numerical 3D simulations show that such a nonlinear system with noninteger fractal dimension has well-defined localized solutions for fields (3D speckles). The statistics of speckles is studied too.

1. Introduction

Disordered mediums can diffuse or localize the light waves due to random multiple scattering that leads to formation of the electromagnetic modes depending on the structural correlations, scattering strength, and dimensionality of the system [1–7]. The Anderson localization was predicted as a noninteracting linear interference effect [8]. However, in real systems the nonnegligible interactions between light and medium can take place. Therefore, an important aspect of the optical localization is the interplay between nonlinear interactions [6]. Nonlinear interactions appear in optics, due to nonlinear responses of a disordered medium that normally gives rise to indirect interactions between the photons through various mechanisms. In case of classical waves the localization can be interpreted as interference between the various amplitudes associated with the scattering paths of optical waves propagating among the diffusers. The study of transition in three-dimensional (3D) optical systems still is an object of active investigation. However, the localization transition may be difficult to reach in 3D system due to various effects in disordered media required to achieve a strong scattering. The strong fluctuation of the wave function at the optical localization that leads to nontrivial length-scale dependence of the intensity distribution (multifractality) is observed below the transition in 3D disordered medium [9]. Here we study a different dynamic field localization that may occur in strong disordered systems. Such localization happens in 3D active percolating medium, where the clusters are filled by the light nanoemitters in the excited state. This 3D percolating structure has a noninteger fractal (Hausdorff) dimensionality due to the percolating transition [10–12] that breaks the spatial homogeneity of the medium. In similar disordered systems the light is both emitted and scattered by the inhomogeneity of the clusters. We note that the optical transport in 3D disordered percolating system already was registered [13]. Besides, it is found that the random lasing assisted by the emitters incorporated into disordered structure can occur [14]. The linear theory of the localization in a percolating system that was considered in [15] is valid only for small times and it does not allow investigating the nonlinear dynamics that occur for longer times. The following questions are investigated in such a study: (i) whether the field localization can appear for noninteger 3D fractal dimensionality for homogeneous initial states and (ii) if the field localization (3D speckles) can be found in 3D active and nonlinear system where the dynamics are not a time-reversal invariant. This paper is organized as follows. In Section 2 we formulate the nonlinear equations for our model. In Section 3 we numerically study the statistical properties of the patterns of dynamic field excitations that arise in 3D disordered...
2. Basic Equations

Various random optical processes may generate the laser speckle patterns when the granular points appear in the scattered light. In this paper we consider the properties of field in materials with random percolating clusters. We studied the field dynamics in a cubical sample \((x, y, z) \in [0, l_0] \) (where \( l_0 = 10 \mu m \)) in 3D medium containing the percolating clusters that are filled by the light emitters placed in random positions \( R_k \) \((k = 1, \ldots, N)\) of a cluster, where \( N \) is number of emitters. The Maxwell equations for electrical \( E = E(r, t) \) and magnetic \( H = H(r, t) \) fields in the system read

\[
\nabla \times E = -\mu_0 \frac{\partial H}{\partial t},
\]
\[
\nabla \times H = \frac{\partial E}{\partial t} + \sum_k j(R_k, t) \delta(r - R_k),
\]

where \( j(R_k, t) \) \((k = 1, \ldots, N)\) is an electric current produced by radiated emitters in positions \( R_k \). The equation for polarization density \( P_k = P(R_k, t) \) in such a cluster filled by the emitters (4-level atoms) with occupations of the levels \( N_{j,k} \) \((j = 0 \ldots 3)\) is

\[
\frac{\partial^2 P_k}{\partial t^2} + \Delta \omega_a \frac{\partial P_k}{\partial t} + \omega_a^2 P_k = \frac{6\pi e_0^2 c^3}{\tau_{21}\omega_a^3} (N_{1,k} - N_{2,k}) E_k. \quad (2)
\]

To complete the model we add the rate equations for the occupation levels of emitters \( N_{j,k} = N_j(R_k, t) \):

\[
\frac{\partial N_{0,k}}{\partial t} = -A_r N_{0,k} + \frac{N_{1,k}}{\tau_{13}},
\]
\[
\frac{\partial N_{3,k}}{\partial t} = A_r N_{0,k} - \frac{N_{1,k}}{\tau_{12}},
\]
\[
\frac{\partial N_{1,k}}{\partial t} = \frac{N_{2,k}}{\tau_{52}} - M_k - \frac{N_{1,k}}{\tau_{13}},
\]
\[
\frac{\partial N_{2,k}}{\partial t} = \frac{N_{1,k}}{\tau_{12}} + M_k - \frac{N_{2,k}}{\tau_{12}},
\]

\[
M_k = (j \cdot E)_k \frac{\hbar \omega_a}{\tau_{13}}, \quad (5)
\]

Here \( \Delta \omega_a = \frac{1}{\tau_{21}} + 2T^2 \), where \( T \) is the mean time between dephasing events, \( \tau_{21} \) is the decay time from the second atomic level to the first one, and \( \omega_a \) is the frequency of radiation (see, e.g., [16]). It is difficult to obtain an analytical solution of the nonlinear system (1)–(4) even for weak disorder or small number of emitters \( N \). The main difficulty comes here from the fact that for such 3D case with percolating randomness in the strongly scattering regime the perturbative expansions on the disorder strength fail (in zone of the percolating transition) and one has to apply the numerical simulations. In the following section we use the numerical technique (FDTD [17]) to obtain the exact 3D solutions to system (1)–(4).

3. Numerics

In this section we study the temporal dynamics of the field structures described by the system of (1)–(4) formulated in the previous section, with the use of numerical 3D simulations. In our calculation the following dimensionless variables are used: \( r \rightarrow r/l_0, t \rightarrow t(c/l_0) \), where \( l_0 = 10 \mu m \) is size of the system, \( c \) is velocity of light in vacuum, and the electromagnetic wavelength is \( \lambda = 1 \mu m \). In what follows, we consider an active medium with the parameters similar to [18, 19]. Initially, the field emission appears from small-scaled uncoupled domains of disordered emitters that give rise to weak incoherent radiation with random phases. However, later the radiation from nearest patterns is being synchronized which leads to formation of macroscopically large areas (patterns) of field. To study the properties of such a field we first calculate a representative dependence of the specific field energy \( W_s = W/l_{SN} \) (\( W \) is total energy) lengthwise of the optimal distance \( l_{SN} \). Such a distance \( l_{SN} \) is based on the Fermat principle that allows calculating the optimal (shortest) optical path (see [14, 15]) where a photon can travel between \( N \) emitters inside a finite size 3D-sample without visiting the same region twice. Figure 1 shows such a specific energy \( W_s \) as function of the time \( t \) for family of

![Figure 1](image-url)
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Figure 2: The properties of the speckle patterns generated in 3D cube with $100 \times 100 \times 100$ grids by random scatters (a), (c), and random optical emitters in percolating clusters (b), (d). (a) Central slice of the computer generated speckle pattern consisting of large number of independent scatters with the phases uniformly distributed over $[-\pi, \pi]$; (b) the field speckle pattern generated by radiated emitters (depicted as point-like spots) placed into a random percolating cluster with the population probability $p = 0.25$ at small time $t = 1$; (c) the histogram of intensity $V$ and the probability density function (dashed red line) $F_{3D}(q) = \exp(-q)/\langle V \rangle$ for the pattern shown in (a); (d) histogram of intensity $V$ (in exponential scale) and the (fitting) probability density function (dashed red line) $F_p(q)$ (see (6)) for the emitter pattern shown in (b). One can see that the probability density functions for (a) and (b) are different functions.

From Figure 1 we observe that the family $W_s$ is separated in two completely different groups. In first group at $p \leq 0.26$ the value $W_s$ conserves very small initial values $\sim 10^{-9}$ and practically is stationary in time. However in second group when $p \geq 0.27 \ldots 0.30$ the values $W_s(t)$ increase very fast and attain a maximal value $\sim 10^{-7}$ at $t = 8$; for longer time $W_s(t)$ oscillates around $10^{-2}$. Such a weak disorder level at $p = 0.26$ in Figure 1 can be interpreted as an effective mobility edge: below $p = 0.26$ there exist only extended modes with drop of the energy (see Figure 2(b)). The case with $p >= 0.27$ in Figure 1 may have a meaning as a strong disorder level, when the field localized modes with sharp rise of the energy are established in the system (see Figures 4(a) and 4(b)).

Equations (1)–(4) determine the dynamics and spatial structure of radiated fields in vicinity of nanoemitters. But the localization effects produced by random wave rereflections normally can be observed in form of speckles. The latter produce an interference pattern generated by the random scattering and can be described as a superposition of large

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The probability populations $p$ of percolating clusters $p = 0.25 \ldots 0.3$. (The 3D optical localization criterion shows [15] that the localization effect occurs closely to the critical value of the occupation probability of percolating, that is, $p_c \leq 0.318$.) To simulate the noise in such a system the initial seed for the electromagnetic field has been created with random phases for each emitter.
number of random plane waves. The statistical properties of conventional optical speckles [20] are well studied for 2D geometry [21, 22]). Therefore, it is interesting to compare the properties of such optical speckles (but for the 3D case) with the speckles formed in result of radiating emitters located in disordered percolation clusters.

The conventional optical speckles normally have stationary statistics; therefore, firstly we compare the latter with statistical properties of emitters for \( p = 0.25 \) that exhibit a field branch having very weak time dependence (see Figure 1). Speckle patterns are identified by their characteristic statistical properties and Fourier components that are limited by well-defined spatial frequency range. Figures 2–4 display the speckle pattern statistics for conventional and also for localized fields generated by radiated emitters.

Figure 2 shows the properties of the speckle patterns in 3D cube (central slice) with \( 100 \times 100 \times 100 \) grids by random scatters (a,c) and the random emitters in a percolating clusters (b,d). Figure 2(a) displays the computer generated speckle pattern consisting of large number of independent scatters, with the phases uniformly distributed over \([−\pi, \pi]\). Panel (b) shows field speckle pattern generated by radiated emitters placed into a random percolating cluster with the population probability \( p = 0.25 \) at small time \( t = 1 \). Panel (c) exhibits the histogram of intensity \( V = |F^{-1}(\hat{A}_f\tilde{F}(E))|^2 \), where \( F \) is 3D Fourier transformation, \( \hat{A}_f \) is the low-pass filter (see details for 2D in [22]), and \( F_{3D} = \exp(-V/(\langle V \rangle))/\langle V \rangle \) is the probability density function (dashed red line) for the pattern shown in panel (a). Panel (d) shows (in exponential scale) the histogram of intensity and the approximate probability density function \( F_p(q) \) (that fit well the numerical results) (see dashed red line) with

\[
F_p(q) = \frac{\alpha x_m^\alpha}{q^{\alpha+1}}, \tag{6}
\]

where \( \alpha = 2.1, x_m = 5.6.10^{-6} \), and \( F_p(q) = 0 \) if \( q < x_m \). The probability density in (6) can be associated with the stationary Pareto distribution [23] for the pattern shown in panel (b). Such a distribution in Figure 2(d) considerably differs from the negative exponential distribution for random scatters shown in Figure 2(c). This is a natural result because in this case the incoherent radiation of emitters with random phases is being synchronized (stimulated radiation) with time and finally leads to formation of macroscopically large areas (patterns) of field; see Figures 3(a), 3(b), 4(a), and 4(b).

The formation of such patterns for longer time \( t = 5 \) is shown in Figure 3 for radiated emitters in a percolating clusters at \( p = 0.28 \) (a,c) and \( p = 0.30 \) (b,d), respectively. Figure 3(a) displays the speckle pattern in a central slice of 3D cube with \( 100^3 \) grids consisting of large number of emitters with the nearly synchronized phases; panel (b) shows the same as (a) but for percolating cluster with larger population.

Figure 3: The initial formation of field patterns for time \( t = 5 \) for optical emitters in a percolating clusters for cases \( p = 0.28 \) (a,c) and \( p = 0.30 \) (b,d), respectively. (a) The speckle pattern in a central slice of 3D cube similar to Figure 2(b), (b) the same as (a) but for percolating cluster with larger population \( p = 0.30 \). We observe in (a), (b) the radiated emitters (depicted as point-like spots) surrounded by smoothly shaped areas that correspond to weak localized fields; (c) and (d) exhibit corresponding histograms of intensity \( V \) and the probability density functions (dashed red line) with \( F_p(q) \) defined by (6).
Figure 4: The same as in Figure 3 but for long time $t = 10$ that corresponds to a developed state when the emitters already acquire well-synchronized phases. From (a), (b) we observe that the field amplitudes already have large values (see colorbar images in (a), (b)). Besides, one can see that in this case the speckles migrate to rather long distance from the initial positions, where they were generated (see Figure 4(a)); (c), (d) the intensity $V$ and the stationary probability function (dashed red line) (6) that, however, already does not fit well the histograms (cf. Figure 3(d)). Such a mismatch arises because of a dynamic (nonstationary) transport of the field speckles.
Except seeing of the field structure in central 2D slices of 3D system (shown in Figures 2–4) one should gain a deeper insight into the general behavior of field patterns in considered 3D percolating system. Such a 3D field structure is displayed in Figure 5. We observe from Figure 5 that near the entrance of percolating cluster (plane $x = 0$ in Figure 5) the transport of speckles to the left is suppressed because of strong disorder level and scattering of the optical fields by the cluster inhomogeneity in this a region. However, such a field dynamics becomes possible at the output zone where number of emitters is small and the transport of speckles is only governed by the field equations (1).

4. Conclusion

We investigated a dynamic three-dimensional (3D) field localized states (speckles) in a medium with percolation disorder, where the percolation cluster is filled by the active nanoemitters. In such a system, the incipient percolating cluster represents a fractal radiating structure where the field is radiated and scattered by the anisotropic nonhomogeneity of a cluster. Analysis of the statistics for field patterns shows that the speckle distribution in such a dynamic system for short times is close to the Pareto distribution. Our numerical 3D simulations show that such a nonlinear model with a noninteger fractal dimension has smooth well-defined localized solutions for fields. That allows recognizing such dynamic field domains as 3D field speckles that are formed in the nonlinear active fractal medium with a large disorder.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that they have no conflicts of interest.

References


