Research Article

Krasnoselskii N-Tupled Fixed Point Theorem with Applications to Fractional Nonlinear Dynamical System

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In this paper, N-tupled fixed point theorems for two monotone nondecreasing mappings in complete normed linear space are established. The extension of Krasnoselskii fixed point theorem for a version of N-tupled fixed point is given. Our theoretical results are applied to prove the existence of a mild solution of the system of N-nonlinear fractional evolution equations. Finally, an example of a nonlinear fractional dynamical system is given to illustrate the results.

1. Introduction

It is outstanding that the fixed point theorem of Krasnoselskii [1] might be joined with Banach and Schauder's fixed point theorems. The sum of operators is obviously found in neutral functional equations and delay integral equations, which have been talked about widely in [2, 3], for instance. In 1964, Krasnoselskii studied the existence of fixed point for a sum of two operators, one being a continuous and compact and the other being a strict contraction. A lot of generalizations of Krasnoselskii's theorem have been done, mostly by relaxing the continuous and compact conditions and sometimes by withdrawing the requirement of the strict contraction or even both.

In 2006, Bhaskar and Lskshmikanthan [4] proposed new approach of the fixed point theorem which was called coupled fixed point. They gave some useful applications for mixed monotone operators.

In 2011, Luong and Thuan [5] extend the coupled fixed point approach to study the unique solution of the integral equations. Recently in 2013, Dhage [6] proposed new a version of the Krasnoselskii fixed point theorem and used it to prove the existence of the solution of nonlinear fractional differential equation.

More recently, in 2017, Yang et al. [7] proved the Krasnoselskii coupled fixed point theorem under some or certain

conditions and used their results to study the existence of the solution of a nonlinear coupled system of fractional differential equations. Imdad et al. [8] in 2013, gave a new approach of the fixed point theory which was called N-tupled fixed point. This approach is useful to study a fixed point of mappings $F : \underbrace{X \times X \times \cdots \times X}_{n\text{-times}} \rightarrow X$, where X is a

nonempty set. In this paper, we extend the results of Yang et al. [7] to prove a generalized version of N-tupled fixed point theorems. We give an application to the study existence of a solution of a system of nonlinear fractional differential evolution equations.

We organize the paper as follows: In Section 2, some basic definitions are introduced. In Section 3, Krasnoselskii N-tupled fixed point (KNTFP) is proved. In Section 4, the existence of the solution of a nonlinear N-system of fractional evolution differential equations is considered. In Section 5, we give an example of a nonlinear fractional dynamical system. Finally, in Section 6 the conclusion is presented.

2. Preliminaries

Let $X \neq \emptyset$ and $F : X \times X \rightarrow X$ be a mapping. A point $(x, y) \in X \times X$ is said to be a coupled fixed point of F in $X \times X$ if $F(x, y) = x$ and $F(y, x) = y$. Throughout this paper, let E be a nonempty set and $(E, \leq, \|\cdot\|)$ be a partially ordered normed

linear space. If $Q : E \rightarrow E$ is a mapping. Then Q is said to be monotone nondecreasing if $a \leq b$ implies $Q(a) \leq Q(b)$, for all $a, b \in E$. Two elements $a, b \in E$ are said to be comparable if $a \leq b$ or $b \leq a$. If C is a nonempty subset of E , C is said to be a chain if each two elements $a, b \in C$ are comparable. We recall the following definitions which are given in [6, 7].

Definition 1. Let $Q : E \rightarrow E$ be a mapping. Q is called partially compact if for each chain C subset of E , $Q(C)$ is a relatively compact subset of E .

Definition 2. Let $Q : E \rightarrow E$ be a mapping. Given an element $a \in E$, define the orbit $Y(a; Q)$ as

$$Y(a; Q) = \{a, Qa, Q^2a, Q^3a, \dots, Q^n a, \dots\}. \quad (1)$$

If for any sequence $\{a_n\} \in Y(a; Q)$ such that $a_n \rightarrow a^*$ as $n \rightarrow \infty$ implies $Qa_n \rightarrow Qa^*$, for each $a \in E$, then Q is called Y -orbitally continuous in E . Furthermore, $(E, \leq, \|\cdot\|)$ is said to be Y -orbitally complete if each sequence $\{a_n\} \in Y(a; Q)$ converges to an element $a^* \in E$.

Definition 3. A mapping $\phi : E \rightarrow E$ is said to be D -function if it is upper semicontinuous and monotone nondecreasing such that $\phi(0) = 0$.

Definition 4. A mapping $Q : E \rightarrow E$ is called partially nonlinear D -contraction in E , if for each comparable elements $a, b \in E$, there exists a D -function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that the following conditions are satisfied:

- (i) $\|Qa - Qb\| \leq \phi(\|a - b\|)$,
- (ii) $\phi(r) < r$ for all $r > 0$.

Definition 5. Let $(E, \leq, \|\cdot\|)$ be a Y -orbitally complete linear space. The positive cone $K \in E$ is defined as $K = \{x \in E : x \geq 0\}$.

The following theorem was given in [7] and will be used as a tool to prove the main results.

Theorem 6. Let E be a partially ordered Y -orbitally complete normed linear space and K be the positive cone of E . Let K be normal and C be a nonempty closed subset of E . Consider the two monotone nondecreasing mappings $P, Q : C \rightarrow C$ such that the following conditions are satisfied:

- (1) P is Y -orbitally continuous and a partially nonlinear D -contraction,
- (2) Q is Y -orbitally continuous and partially compact,
- (3) there exists an element $a \in C$ such that $a \leq Pa + Qy$ for all $y \in C$,
- (4) every pair of elements in C has an upper and lower bound.

Then $F(x, y) = Px + Qy$ has a coupled fixed point in $E \times E$.

Now, we recall the definition about the fractional derivative; for more detail about the fractional calculus, please see [9, 10]. The fractional derivative is defined via the fractional integral operator. So, we will start by the definition of the fractional integral operator. The fractional integral operator

of order $\alpha (> 0) \in (0, \infty)$ of a function $g \in L^1([0, b], E)$ such that $b > 0$ is defined as

$$I^\alpha g(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s) ds. \quad (2)$$

Also, the Riemann-Liouville derivative of order $\alpha (> 0)$ is defined as

$${}^R D^\alpha g(t) = D^m (I^{m-\alpha} g(t)), \quad (3)$$

where $m = [\alpha]$ and $D = d/dt$.

The Caputo fractional derivative of order $\alpha (> 0)$ is denoted by D_*^α and is defined as

$$D_*^\alpha g(t) = I^{n-\alpha} (D^n g(t)), \quad (4)$$

where $n = [\alpha]$, $g \in L^1([0, b], E)$ such that $b > 0$ and $t > 0$ [11].

3. Krasnoselshii N-Tupled Fixed Point Theorem

In this section we give the main N-tupled fixed point theorem which is used in the proposed applications. Consider $E^n = \underbrace{E \times E \times \dots \times E}_{n\text{-times}}$ and define the two operations (sum and scalar multiplication in E^n) as follows: if $W = (w_1, w_2, \dots, w_n)$, $V = (v_1, v_2, \dots, v_n) \in E^n$, and $\lambda \in \mathbb{R}$ then, we have that

$$\begin{aligned} W + V &= (w_1 + v_1, w_2 + v_2, \dots, w_n + v_n) \\ \lambda V &= (\lambda v_1, \lambda v_2, \dots, \lambda v_n) \end{aligned} \quad (5)$$

Therefore, the positive cone in E^n can be defined as

$$\begin{aligned} K_{E^n} \\ = \{W = (w_1, w_2, \dots, w_n) \in E^n : w_1, w_2, \dots, w_n \geq 0\}. \end{aligned} \quad (6)$$

Also the norm in E^n can be defined as

$$\begin{aligned} \|W\|_{E^n} &= \|(w_1, w_2, \dots, w_n)\| \\ &= \|w_1\| + \|w_2\| + \dots + \|w_n\|, \end{aligned} \quad (7)$$

where $\|\cdot\|$ is the norm defined on E . It is easy to prove that $(E^n, \|\cdot\|_{E^n})$ is partially ordered normed linear space with order relation $\leq$ induced by K_{E^n} . If D is a nonempty closed subset of E , then $\underbrace{D \times D \times \dots \times D}_{n\text{-times}}$ is also nonempty closed subset of E^n .

On the other hand, if K is normal cone, the K_{E^n} is normal.

Let $Q : E^n \rightarrow E$. The point $(x_1, x_2, \dots, x_n) \in E^n$ is said to be N-tupled fixed point of Q if and only if

$$\begin{aligned} x_1 &= Q(x_1, x_2, \dots, x_n) \\ x_2 &= Q(x_2, x_3, \dots, x_n, x_1) \\ &\vdots \\ x_n &= Q(x_n, x_1, \dots, x_{n-1}) \end{aligned} \quad (8)$$

Now, we prove the main fixed point theorem.

Theorem 7. Let E be a partially ordered Y -orbitally complete normed linear space and K be the positive cone of E . Let K be normal and D be a nonempty closed subset of E . Consider $T_1, T_2, \dots, T_n : D \rightarrow D$ are n monotone nondecreasing mappings such that the following are satisfied:

- (1) T_1 is Y -orbitally continuous and a partially nonlinear D -contraction,
- (2) $T_2, \dots, T_n$ are Y -orbitally continuous and a partially compact,
- (3) there exists an element $a \in D$ such that $a \leq T_1 a + T_2 y_2 + T_3 y_3 + \dots + T_n y_n$ for all $y_2, y_3, \dots, y_n \in D$,
- (4) every pair of elements in D has an upper and lower bounded.

Then $Q(x_1, x_2, \dots, x_n) = T_1 x_1 + T_2 x_2 + \dots + T_n x_n$ has a N -tupled fixed point in E^n .

Proof. According to the fact that $(E, \leq, \|\cdot\|)$ is a partially ordered Y -complete normed linear space and K is normal, then $(E^n, \leq, \|\cdot\|_{E^n})$ is also a partially ordered Y -complete normed linear space and K_{E^n} is normal. Define $D^n = \underbrace{D \times D \times \dots \times D}_{n\text{-times}}$, then D^n is closed subset of E^n . Consider $P_1, P_2, \dots, P_n : D^n \rightarrow D^n$ such that, for $u = (x_1, x_2, \dots, x_n) \in D^n$,

$$\begin{aligned} P_1 u &= (T_1 x_1, T_1 x_2, \dots, T_1 x_n) \\ P_2 u &= (T_2 x_2, T_2 x_3, \dots, T_2 x_n, T_2 x_1) \\ &\vdots \\ P_n u &= (T_n x_n, T_n x_1, T_n x_2, \dots, T_n x_{n-1}) \end{aligned} \quad (9)$$

Therefore, if we prove that the operator equation $u = P_1 u + P_2 u + \dots + P_n u$ has a solution $u = (x_1, x_2, \dots, x_n) \in D^n$, then we have that

$$\begin{aligned} (x_1, x_2, \dots, x_n) &= (T_1 x_1, T_1 x_2, \dots, T_1 x_n) \\ &\quad + (T_2 x_2, T_2 x_3, \dots, T_2 x_n, T_2 x_1) \\ &\quad + \dots \\ &\quad + (T_n x_n, T_n x_1, T_n x_2, \dots, T_n x_{n-1}). \end{aligned} \quad (10)$$

This implies that $(x_1, x_2, \dots, x_n)$ is a N -tupled fixed point of Q in E^n . Our proof of the theorem is divided into 4 steps. $\square$

Step 1. P_1 is Y -orbitally continuous and $\|P_1 W - P_2 V\|_{E^n} \leq \|W - V\|_{E^n}$, for all comparable $W \neq V \in D^n$. Since T_1 is Y -orbitally continuous then by the definition of P_1 we get that P_1 is Y -orbitally continuous.

Let $W \neq V \in D^n$ be comparable elements; we have that

$$\begin{aligned} \|P_1 W - P_2 V\|_{E^n} &= \|(T_1 w_1, T_1 w_2, \dots, T_1 w_n) \\ &\quad - (T_1 v_1, T_1 v_2, \dots, T_1 v_n)\|_{E^n} \leq \|T_1 w_1 - T_1 v_1\| + \dots \\ &\quad + \|T_1 w_n - T_1 v_n\| \leq \phi(\|w_1 - v_1\|) + \dots \\ &\quad + \phi(\|w_n - v_n\|) \leq \|w_1 - v_1\| + \dots + \|w_n - v_n\| \\ &= \|W - V\|_{E^n}, \end{aligned} \quad (11)$$

Step 2. $P_2, P_3, \dots, P_n$ are Y -orbitally continuous and partially compact.

Since $T_2, \dots, T_n$ are Y -orbitally continuous, it is easy to prove that $P_2, \dots, P_n$ are Y -orbitally continuous. Consider $C \subset D$ is bounded chain. According to the fact that T_2 is partially compact in D , then we get that $T_2(C)$ is equicontinuous and uniformly bounded in D . Let $C^n = \underbrace{C \times C \times \dots \times C}_{n\text{-times}}$, then $C^n \subset D^n$ and C^n is bounded chain in D^n . Now, we prove that $P_2(C^n)$ is equicontinuous and uniformly bounded in D^n . Since $P_2(C) \subset D$ is uniformly bounded, there is a constant $M_2 > 0$ satisfying $\|T_2 z\| \leq \varepsilon$ for all $z \in C$. For any $Z \in P_2(C^n)$, we can find $y_1, y_2, \dots, y_n \in C$ such that $u = (y_1, y_2, \dots, y_n) \in C^n$ and $Z = P_2 u$. Therefore, we have that

$$\begin{aligned} \|Z\|_{E^n} &= \|P_2 u\|_{E^n} = \|(T_2 y_2, T_2 y_3, \dots, T_2 y_n, T_2 y_1)\|_{E^n} \\ &= \|T_2 y_2\| + \|T_2 y_3\| + \dots + \|T_2 y_n\| + \|T_2 y_1\| \\ &\leq n\varepsilon. \end{aligned} \quad (12)$$

Thus, $P_2(C^n)$ is uniformly bounded in D^n . By doing the same steps we get that $P_3, \dots, P_n$ are uniformly bounded in D^n . According to the fact that $T_2(C)$ is equicontinuous in D , for any $z \in C$ and $t_2 > t_1$, we have that $\|(T_2 z)(t_2) - (T_2 z)(t_1)\| \rightarrow 0$, as $t_2 - t_1 \rightarrow 0$. Thus, for any $Z = P_2 u \in P_2(C^n)$, where $t_2 - t_1 \rightarrow 0$, we get the following:

$$\begin{aligned} \|Z(t_2) - Z(t_1)\|_{E^n} &= \|(P_2 u)(t_2) - (P_2 u)(t_1)\|_{E^n} \\ &= \|(T_2 y_2)(t_2) - (T_2 y_2)(t_1)\| \\ &\quad + \|(T_2 y_3)(t_2) - (T_2 y_3)(t_1)\| + \dots \\ &\quad + \|(T_2 y_1)(t_2) - (T_2 y_1)(t_1)\| \rightarrow 0. \end{aligned} \quad (13)$$

Thus, $P_2(C^n)$ is equicontinuous in D^n . Therefore, $P_2(C^n)$ is relatively compact in D^n . Hence, $P_2 : D^n \rightarrow D^n$ is partially compact. By doing the same steps, we get that $P_3, P_3, \dots, P_n : D^n \rightarrow D^n$ are partially compact.

Step 3. There is an element $u_0 \in D$ satisfying: $u_0 \leq P_1 u_0 + P_2 u_0 + \dots + P_n u_0$ for all $u \in D^n$.

Let $u_0 = (a, a, \dots, a)$ and $u = (x_1, x_2, \dots, x_n) \in D^n$. By condition (3) in the theorem, we get that

$$\begin{aligned} P_1 u_0 + P_2 u_0 + \dots + P_n u_0 &= (T_1 a, T_1 a, \dots, T_1 a) + (T_2 x_2, T_2 x_3, \dots, T_2 x_1) \\ &\quad + \dots + (T_n x_n, T_n x_1, \dots, T_n x_{n-1}) \geq (a, a, \dots, a). \end{aligned} \quad (14)$$

Step 4. Every pair of elements in D^n has an upper bound and lower bound. For every pair of elements $W = (w_1, w_2, \dots, w_n), V = (v_1, v_2, \dots, v_n) \in D^n$, from condition

(4) of the theorem, there exist $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ such that, for all $i = 1, 2, \dots, n$, we have that

$$\begin{aligned} a_i &\leq w_i, \\ a_i &\leq v_i, \\ b_i &\geq w_i, \\ b_i &\geq v_i. \end{aligned} \quad (15)$$

Hence, we find that

$$\begin{aligned} (a_1, a_2, \dots, a_n) &\leq (w_1, w_2, \dots, w_n) \leq (b_1, b_2, \dots, b_n), \\ (a_1, a_2, \dots, a_n) &\leq (v_1, v_2, \dots, v_n) \leq (b_1, b_2, \dots, b_n). \end{aligned} \quad (16)$$

Thus, we get that every pair $W, V \in D^n$ has an upper and lower bound.

Define $B : D^n \rightarrow D^n$ as $BU = P_2U + P_3U + \dots + P_nU$, for all $U \in D^n$. Thus B is Y -orbitally continuous and a partially compact. Also there exists an element $u_0 \in D^n$ such that $u_0 \leq P_1u_0 + BV$ for all $V \in D^n$. Hence $U = P_1U + BU$ has a solution in E^n . Then if the solution is $(x_1, x_2, \dots, x_n)$, we get that

$$\begin{aligned} (x_1, x_2, \dots, x_n) &= (T_1x_1, T_1x_2, \dots, T_1x_n) \\ &\quad + (T_2x_2, T_2x_3, \dots, T_2x_n, T_2x_1) \\ &\quad + \dots \\ &\quad + (T_nx_n, T_nx_1, T_nx_2, \dots, T_nx_{n-1}). \end{aligned} \quad (17)$$

Therefore, we have that

$$\begin{aligned} x_1 &= T_1x_1 + T_2x_2 + \dots + T_nx_n = Q(x_1, x_2, \dots, x_n) \\ x_2 &= T_1x_2 + T_2x_3 + \dots + T_nx_1 = Q(x_2, x_3, \dots, x_1) \\ &\vdots \\ x_n &= T_1x_n + T_2x_1 + \dots + T_nx_{n-1} \\ &= Q(x_n, x_1, \dots, x_{n-1}) \end{aligned} \quad (18)$$

Hence $Q(x_1, x_2, \dots, x_n)$ has a N -tupled fixed point. By changing condition (3) due to the fact that there exists an element $a \in D$ such that $a \geq T_1a + T_2y_2 + T_3y_3 + \dots + T_ny_n$ for all $y_2, y_3, \dots, y_n \in D$, the theoretical results are also correct.

Theorem 8. Let E be a partially ordered Y -orbitally complete normed linear space and K be the positive cone of E . Let K be normal and D be a nonempty closed subset of E . Consider $T_1, T_2, \dots, T_n : D \rightarrow D$ are n monotone nondecreasing mappings such that the following are satisfied:

(1) T_1 is Y -orbitally continuous and a partially nonlinear D -contraction,

(2) $T_2, \dots, T_n$ are Y -orbitally continuous and a partially compact,

(3) there exists an element $a \in D$ such that $a \geq T_1a + T_2y_2 + T_3y_3 + \dots + T_ny_n$ for all $y_2, y_3, \dots, y_n \in D$,

(4) every pair of elements in D has an upper and lower bounded.

Then $Q(x_1, x_2, \dots, x_n) = T_1x_1 + T_2x_2 + \dots + T_nx_n$ has a N -tupled fixed point in E^n .

4. Existence of Mild Solution of the System of N- Fractional Evolution Equations

Let $\tilde{K}$ be the positive convex cone of Y -orbitally Banach space $(X, \|\cdot\|)$. Suppose that $\tilde{K}$ is normal. It is clear that X is partially order by the poset $\leq$ induced by $\tilde{K}$. In this section, we prove the existence of mild solution of the system of N -fractional evolution equations:

$$\begin{aligned} D_*^\alpha x_1(t) + Ax_1(t) &= f_1(t, x_1(t)) + f_2(t, x_2(t)) + \dots \\ &\quad + f_n(t, x_n(t)), \\ D_*^\alpha x_2(t) + Ax_2(t) &= f_1(t, x_2(t)) + f_2(t, x_3(t)) + \dots \\ &\quad + f_n(t, x_1(t)), \\ &\vdots \end{aligned} \quad (19)$$

$$\begin{aligned} D_*^\alpha x_n(t) + Ax_n(t) &= f_1(t, x_n(t)) + f_2(t, x_1(t)) + \dots \\ &\quad + f_n(t, x_{n-1}(t)), \end{aligned}$$

$$x_i(0) = \zeta_i \in X, \quad \forall i = 1, 2, \dots, n,$$

$$\alpha \in (0, 1), \quad t \in [0, b], \quad b > 0,$$

where $-A$ generates a C_0 -semigroup $\{S(t) : t \geq 0\}$ of uniformly linear and bounded operators in X . Also, $f_1, f_2, \dots, f_n$ are given functions. According to the fact that $S(t)$ are bounded, then there exists $d \in (0, \infty)$ such that $\|S(t)\| \leq d$ for all $t \geq 0$.

Define two families of operator $\{\Psi_\alpha(t) : t \geq 0\}$ and $\{\Phi_\alpha(t) : t \geq 0\}$

$$\begin{aligned} \Psi_\alpha(t)x &= \int_0^\infty \eta_\alpha(s) S(t^\alpha s) x ds, \\ \Phi_\alpha(t)x &= \alpha \int_0^\infty s \eta_\alpha(s) S(t^\alpha s) x ds, \end{aligned} \quad (20)$$

where $\alpha \in (0, 1)$ and

$$\begin{aligned} \eta_\alpha(s) &= \frac{1}{\alpha} s^{-1-1/\alpha} \gamma_\alpha(s^{-1/\alpha}), \\ \gamma_\alpha(s) &= \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} s^{-\alpha n-1} \frac{\Gamma(n\alpha + 1)}{n!} \sin(n\pi\alpha), \end{aligned} \quad (21)$$

$$s \in (0, \infty).$$

According to [12, 13], the family of operators $\{\Psi_\alpha(t) : t \geq 0\}$ and $\{\Phi_\alpha(t) : t \geq 0\}$ has the following properties:

(Pr1) For any $x \in X$ and $t \geq 0$ fixed, we have that

$$\begin{aligned} \|\Psi_\alpha(t)x\| &\leq d \|x\|, \\ \|\Phi_\alpha(t)x\| &\leq \frac{d}{\Gamma(\alpha)} \|x\|, \end{aligned} \quad (22)$$

(Pr2) $\{S(t) : t \geq 0\}$ is an equicontinuous semigroup which implies that $\{\Phi_\alpha(t) : t \geq 0\}$ is equicontinuous in X ,

(Pr3) $\{S(t) : t \geq 0\}$ is positive C_0 -semigroup which implies that $\{\Psi_\alpha(t) : t \geq 0\}$ and $\{\Phi_\alpha(t) : t \geq 0\}$ are positive operators.

Define $C_{[0,b]}(X) = \{f : [0, b] \rightarrow X : f \text{ is continuous}\}$ and $\tilde{K}_C = \{x \in C_{[0,b]}(X) : x(t) \in \tilde{K}, t \in [0, b]\}$. It is easy to show $C_{[0,b]}(X)$ is a partially order Y -orbitally complete normed space with the norm: $\|x\|_{C_{[0,b]}} = \sup_{t \geq 0} \|x(t)\|$, and a poset $\leq$ induced by $\tilde{K}_C$. It is easy to show that $\tilde{K}_C$ is normal because $\tilde{K}$ is normal.

Let $E = C_{[0,b]}(X)$ and $K = \tilde{K}_C$. Define, n -operators $T_1, T_2, \dots, T_n : E \rightarrow E$ as

$$\begin{aligned} (T_1 x)(t) &= \Psi_\alpha(t) \xi_0 \\ &+ \int_0^t (t-s)^{\alpha-1} \Phi_\alpha(t-s) f_1(s, x(s)) ds, \\ &t \in [0, b], \\ (T_2 x)(t) &= \int_0^t (t-s)^{\alpha-1} \Phi_\alpha(t-s) f_2(s, x(s)) ds, \\ &t \in [0, b], \\ (T_3 x)(t) &= \int_0^t (t-s)^{\alpha-1} \Phi_\alpha(t-s) f_3(s, x(s)) ds, \\ &t \in [0, b], \\ &\vdots \\ (T_n x)(t) &= \int_0^t (t-s)^{\alpha-1} \Phi_\alpha(t-s) f_n(s, x(s)) ds, \\ &t \in [0, b]. \end{aligned} \quad (23)$$

Then we can say that an element $(x_1, x_2, \dots, x_n) \in E_n$ is a N -tupled mild solution of the system (19) if and only if $(x_1, x_2, \dots, x_n)$ is the solution of the system of the operator equations:

$$\begin{aligned} x_1(t) &= (T_1 x_1)(t) + (T_2 x_2)(t) + \dots + (T_n x_n)(t), \\ &t \in [0, b], \\ x_2(t) &= (T_1 x_2)(t) + (T_2 x_3)(t) + \dots + (T_n x_1)(t), \\ &t \in [0, b], \\ &\vdots \\ x_n(t) &= (T_1 x_n)(t) + (T_2 x_1)(t) + \dots + (T_n x_{n+1})(t), \\ &t \in [0, b]. \end{aligned} \quad (24)$$

We will apply Theorem 7 to prove that the system of operator equations (24) has a N -tupled fixed point in E_n under the following important conditions:

(C₁) The positive C_0 -semigroups $\{S(t) : t \geq 0\}$ is equicontinuous.

(C₂) The function $f_1 : [0, b] \times X \rightarrow X$ is continuous in X for all $t \in [0, b]$ and there exist a constant $\varrho \in \mathbb{R}$ with $0 < \varrho < \Gamma(\alpha + 1)/db^\alpha$ and a D -function $\phi : [0, \infty) \rightarrow [0, \infty)$ and satisfied the fact that $\phi(t) < r$ for any $r \in (0, \infty)$ and

$$0 \leq f_1(t, e_1) - f_1(t, e_2) \leq \varrho \phi(\|e_1 - e_2\|), \quad (25)$$

for all $e_1, e_2 (e_1 \geq e_2) \in X$ and $t \in [0, b]$.

(C₃) The functions $f_2(t, x), f_3(t, x), \dots, f_n(t, x) : [0, b] \times X \rightarrow X$ are continuous, nondecreasing, and bounded in x .

(C₄) There exists an element $\tilde{x} \in E$ which satisfies the fact that

$$\begin{aligned} D_*^\alpha \tilde{x}(t) + A \tilde{x}(t) &\leq f_1(t, \tilde{x}) + f_2(t, y_2(t)) + \dots \\ &+ f_n(t, y_n(t)), \\ &\forall y_2, \dots, y_n \in E, \end{aligned} \quad (26)$$

$$\tilde{x} \leq \xi_0 \in X,$$

$$\alpha \in (0, 1), \quad t \in [0, b], \quad b > 0.$$

Theorem 9. If conditions (C₁), (C₂), (C₃), and (C₄) are satisfied, the system of fractional evolution equations (19) has a N -tupled mild solution on $[0, b]$.

Proof. Since condition (C₃) holds, then $f_2(t, x), f_3(t, x), \dots, f_n(t, x)$ are bound for all $t \in [0, b]$, and there are constants $N_2, N_3, \dots, N_n$ such that, for all $t \in [0, b]$, we get that

$$\begin{aligned} \|f_2(t, x)\| &\leq N_2, \\ \|f_3(t, x)\| &\leq N_3, \\ &\vdots, \\ \|f_n(t, x)\| &\leq N_n, \\ &\forall x \in X. \end{aligned} \quad (27)$$

For all $i = 2, \dots, n$, we have that

$$\begin{aligned} \tilde{r} \geq \max \left\{ \frac{dN_i b^\alpha}{\Gamma(\alpha + 1)}, \left(1 - \frac{d\varrho b^\alpha}{\Gamma(\alpha + 1)} \right)^{-1} \right. \\ \left. \cdot \left(d\|\xi_0\| + \frac{d\tilde{G}b^\alpha}{\Gamma(\alpha + 1)} \right) \right\} + 1, \end{aligned} \quad (28)$$

where $\tilde{G} = \sup\{\|f_1(t, 0)\| : t \in [0, b]\}$. We can define the following open balls in E as

$$B(x_0, r) = \{x \in E : \|x_0 - x\|_{C_{[0,b]}} < r\} \quad (29)$$

where $r = \|x_0\|_{C_{[0,b]}} + \tilde{r}$. Define $D = \overline{B}(x_0, r)$. It is clear that D is bounded and closed subset in E . Now we apply 3 steps to prove the theorem.

Step 1. The operator $T_1 : D \rightarrow D$ is Y -orbitally continues and a partially nonlinear D -contraction in E .

Since condition (C_2) holds, we get that, for any $x \in E$ with $\|x\|_{C_{[0,b]}} \leq \tilde{r}$, we have

$$\begin{aligned} \|(T_1 x) t\| &\leq d \|\xi_0\| + \frac{d}{\Gamma(\alpha)} \int_0^t (s-t)^{\alpha-1} \\ &\cdot [\|f_1(s, x(s)) - f_1(s, 0)\| + \|f_1(s, 0)\|] ds \\ &\leq d \|\xi_0\| + \frac{d}{\Gamma(\alpha)} \int_0^t (s-t)^{\alpha-1} \\ &\cdot [\varrho \phi(\|x\|_{C_{[0,b]}}) + \tilde{G}] ds \leq d \|\xi_0\| + \frac{d \varrho b^\alpha}{\Gamma(\alpha+1)} \tilde{r} \\ &+ \frac{d \tilde{G} b^\alpha}{\Gamma(\alpha+1)} \leq \tilde{r}. \end{aligned} \quad (30)$$

By taking the sup, we get that $\|(T_1 x) t\|_{C_{[0,b]}} \leq \tilde{r}$ for all $x \in E$, such that $\|x\|_{C_{[0,b]}} \leq \tilde{r}$. Therefore, we have the following:

$$\begin{aligned} \|x_0 - T_1 x\|_{C_{[0,b]}} &\leq \|x_0\|_{C_{[0,b]}} + \|T_1 x\|_{C_{[0,b]}} \\ &\leq \|x_0\|_{C_{[0,b]}} + \tilde{r} = r_1. \end{aligned} \quad (31)$$

Hence, T_1 is a function from D to D . According to the fact that $\{S(t) : t \geq 0\}$ is positive C_0 - semigroup and the properties $(Pr1)$, $(Pr2)$, and $(Pr3)$, it is provided that $T_1 : D \rightarrow D$ is nondecreasing. Let $\{x_m\} \in Y(x; T_1)$ for any $x \in D$ such that $x_m \rightarrow \hat{x}$ as $m \rightarrow \infty$. Thus by condition (C_2) , we get that T_1 is continuous in X for all $t \in [0, b]$, and therefore we have the following:

$$\begin{aligned} \lim_{m \rightarrow \infty} (T_1 x_m)(t) &= \Psi_\alpha(t) \xi_0 \\ &+ \int_0^t (t-s)^{\alpha-1} \Phi_\alpha(t-s) \lim_{m \rightarrow \infty} f_1(s, x_m(s)) ds \\ &= \Psi_\alpha(t) \xi_0 + \int_0^t (t-s)^{\alpha-1} \Phi_\alpha(t-s) f_1(s, \hat{x}(s)) ds \\ &= (T_1 \hat{x})(t), \quad t \in [0, b]. \end{aligned} \quad (32)$$

Thus T_1 is Y -orbitally continuous.

Let $x_1, x_2 \in D$ be comparable such that $x_1 \geq x_2$. We have the following:

$$\begin{aligned} \|(T_1 x_1)(t) - (T_1 x_2)(t)\| &\leq \int_0^t (s-t)^{\alpha-1} \\ &\cdot \|\Phi_\alpha(t-s) [f_1(s, x_1(s)) - f_1(s, x_2(s))]\| ds \\ &\leq \frac{d \varrho}{\Gamma(\alpha)} \int_0^t (s-t)^{\alpha-1} \phi(\|x_1(s) - x_2(s)\|) ds \\ &\leq \frac{d \varrho b^\alpha}{\Gamma(\alpha+1)} \phi(\|x_1(s) - x_2(s)\|_{C_{[0,b]}}). \end{aligned} \quad (33)$$

Since $0 < \varrho < \Gamma(\alpha+1)/db^\alpha$, we get that

$$\begin{aligned} \|(T_1 x_1)(t) - (T_1 x_2)(t)\|_{C_{[0,b]}} &\leq \phi(\|x_1(s) - x_2(s)\|_{C_{[0,b]}}). \end{aligned} \quad (34)$$

Hence, T_1 is partially nonlinear D - contraction operator in E .

Step 2. The operators $T_2, T_3, \dots, T_n : D \rightarrow D$ are Y - orbitally continuous, nondecreasing, and partially compact operators in E .

First, we prove that $T_2, T_3, \dots, T_n$ are mappings from D to D . By the condition (C_3) , for any $x \in E$ such that $\|x\|_{C_{[0,b]}} \leq \tilde{r}$, we get that

$$\begin{aligned} \|(T_2 x)(t)\| &\leq \frac{d}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f_2(s, x(s))\| ds \\ &\leq \frac{d N_2 b^\alpha}{\Gamma(\alpha+1)} \leq r_2. \end{aligned} \quad (35)$$

Similarly, we have that

$$\begin{aligned} \|(T_3 x)(t)\| &\leq r_3, \\ &\vdots \\ \|(T_n x)(t)\| &\leq r_n. \end{aligned} \quad (36)$$

Thus, we get the following:

$$\|T_2 x\|_{C_{[0,b]}} \leq r_2, \dots, \|T_n x\|_{C_{[0,b]}} \leq r_n. \quad (37)$$

Therefore, we get that

$$\begin{aligned} \|x_0 - T_2 x\|_{C_{[0,b]}} &\leq \|x_0\|_{C_{[0,b]}} + \|T_2 x\|_{C_{[0,b]}} \leq \|x_0\|_{C_{[0,b]}} \\ &+ r_2 = \tilde{r}_2 \\ &\vdots \\ \|x_0 - T_n x\|_{C_{[0,b]}} &\leq \|x_0\|_{C_{[0,b]}} + \|T_n x\|_{C_{[0,b]}} \leq \|x_0\|_{C_{[0,b]}} \\ &+ r_n = \tilde{r}_n. \end{aligned} \quad (38)$$

Hence, $T_2, \dots, T_n$ are mappings from D to D . By using the same method as in Step 1, we can prove $T_2, \dots, T_n$ are Y - orbitally continuous and nondecreasing E . Consider $C \in D$ is any arbitrary chain subset of D . For any $Y \in T_2(C)$, there is an element $x \in D$ such that $Y(t) = (T_2 x)(t)$, for all $t \in [0, b]$. For any $t_1, t_2 \in [0, b]$ such that $t_1 < t_2$, we get that

$$\begin{aligned} \|Y(t_2) - Y(t_1)\| &\leq \|(T_2 x)(t_2) - (T_2 x)(t_1)\| \\ &\leq H_1 + H_2 + H_3, \end{aligned} \quad (39)$$

where

$$\begin{aligned}
 H_1 &= \left\| \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] \Phi_\alpha(t_2 - s) \right. \\
 &\quad \cdot f_2(s, x(s)) ds \Big\|, \\
 H_2 &= \left\| \int_0^{t_1} (t_1 - s)^{\alpha-1} [\Phi_\alpha(t_2 - s) - \Phi_\alpha(t_1 - s)] \right. \\
 &\quad \cdot f_2(s, x(s)) ds \Big\|, \\
 H_3 &= \left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \Phi_\alpha(t_2 - s) f_2(s, x(s)) ds \right\| \\
 &\leq \frac{dN_2 b^\alpha}{\Gamma(\alpha + 1)} (t_2 - t_1)^\alpha.
 \end{aligned} \tag{40}$$

Hence, when $t_2 - t_1 \rightarrow 0$, we get that

$$H_3 \rightarrow 0, \tag{41}$$

and also,

$$H_1 \leq \frac{dN_2}{\Gamma(\alpha)} \int_0^{t_1} |(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}| ds \rightarrow 0. \tag{42}$$

Let $t_1 = 0$ and $0 < t_2 < b$; then we get that $H_2 \rightarrow 0$. For $t_1 > 0$ and $\varepsilon \in (0, t_1)$ where ε is very small, then we get that

$$\begin{aligned}
 H_2 &\leq \left\| \int_0^{t_1-\varepsilon} (t_1 - s)^{\alpha-1} [\Phi_\alpha(t_2 - s) - \Phi_\alpha(t_1 - s)] \right. \\
 &\quad \cdot f_2(s, x(s)) ds \Big\| + \left\| \int_{t_1-\varepsilon}^{t_1} (t_1 - s)^{\alpha-1} \right. \\
 &\quad \cdot [\Phi_\alpha(t_2 - s) - \Phi_\alpha(t_1 - s)] f_2(s, x(s)) ds \Big\|. \\
 &\leq \frac{N_2 (t_1^\alpha - \varepsilon^\alpha)}{\alpha} \sup_{s \in [0, t_1-\varepsilon]} \|\Phi_\alpha(t_2 - s) - \Phi_\alpha(t_1 - s)\| \\
 &\quad + \frac{2dN_2 \varepsilon^\alpha}{\Gamma(\alpha + 1)}
 \end{aligned} \tag{43}$$

By using condition (C_1) , we get that $H_2 \rightarrow 0$ as $t_2 - t_1 \rightarrow 0$. Therefore, $\|Y(t_2) - Y(t_1)\| \rightarrow 0$ as $t_2 - t_1 \rightarrow 0$. Thus, $T_2(C)$ is equicontinuous on $[0, b]$. Also, using condition (C_3) , we get that

$$\begin{aligned}
 \|Y(t)\| &= \|(T_2 x)(t)\| \\
 &\leq \int_0^t (t - s)^{\alpha-1} \|\Phi_\alpha(t - s) f_2(s, x(s))\| ds \\
 &\leq \frac{dN_2 b^\alpha}{\Gamma(\alpha + 1)}, \quad \forall t \in [0, b].
 \end{aligned} \tag{44}$$

Hence, $T_2(C)$ is uniformly bounded operator in E and T_2 is partially compact operator from D into itself. Similarly, by

doing the same steps we get that $T_3, \dots, T_n : D \rightarrow D$ are partially compact operators.

Step 3. Prove the existence of the N-tupled fixed point.

By condition (C_4) , we get that

$$\begin{aligned}
 \tilde{x} &= D_*^\alpha \tilde{x}(t) + A \tilde{x}(t) \\
 &\leq \Psi_\alpha(t) \xi_0 + \int_0^t (t - s)^{\alpha-1} \Phi_\alpha(t - s) f_1(t, \tilde{x}(t)) ds \\
 &\quad + \int_0^t (t - s)^{\alpha-1} \Phi_\alpha(t - s) f_2(t, y_2(t)) ds + \dots \\
 &\quad + \int_0^t (t - s)^{\alpha-1} \Phi_\alpha(t - s) f_n(t, y_n(t)) ds \\
 &= (T_1 \tilde{x})(t) + (T_2 y_2)(t) + \dots + (T_n y_n)(t), \\
 &\quad \forall y_2, \dots, y_n \in D, \quad t \in [0, b] \quad b > 0.
 \end{aligned} \tag{45}$$

By applying Theorem 7, we get that the system of fractional evolution equations (19) has N-tuple mild solution in E_n . $\square$

5. An Application to Fractional Dynamical System

Consider the following fractional dynamical system:

$$\begin{aligned}
 D_*^{1/2} x_1(z, t) + \frac{\partial x_1(z, t)}{\partial z} &= f_1(z, t, x_1(z, t)) + f_2(z, t, x_2(z, t)) \\
 &\quad + f_3(z, t, x_3(z, t)), \quad (z, t) \in [0, 1] \times [0, 1], \\
 D_*^{1/2} x_2(z, t) + \frac{\partial x_2(z, t)}{\partial z} &= f_1(z, t, x_2(z, t)) + f_2(z, t, x_3(z, t)) \\
 &\quad + f_1(z, t, x_3(z, t)), \quad (z, t) \in [0, 1] \times [0, 1], \\
 D_*^{1/2} x_3(z, t) + \frac{\partial x_3(z, t)}{\partial z} &= f_1(z, t, x_3(z, t)) + f_2(z, t, x_1(z, t)) \\
 &\quad + f_3(z, t, x_2(z, t)), \quad (z, t) \in [0, 1] \times [0, 1], \\
 x_1(0, t) &= x_1(1, t) = x_2(0, t) = x_2(1, t) = x_3(0, t) \\
 &= x_3(1, t) = 0, \quad t \in [0, 1], \\
 x_1(z, 0) &= x_2(z, 0) = x_3(z, 0) = \xi_0(z), \quad z \in (0, 1),
 \end{aligned} \tag{46}$$

such that $f_2 : [0, 1] \times [0, 1] \times R$ is defined as

$$f_2(z, t, x) = \begin{cases} 5, & \text{if } x \leq 0 \\ 5 + \frac{5x}{1+2x}, & \text{if } 0 < x < 2, \\ 7, & \text{if } x \geq 2, \end{cases} \tag{47}$$

and $f_3 : [0, 1] \times [0, 1] \times R$ is defined as

$$f_3(z, t, x) = \begin{cases} 1, & \text{if } x \leq 0 \\ 1 + 2x, & \text{if } 0 < x < 5, \\ 11, & \text{if } x \geq 5, \end{cases} \quad (48)$$

Thus f_2, f_3 are continuous and nondecreasing, $|f_1(z, t, x)| \leq 7$ and $|f_3(z, t, x)| \leq 11$. Hence, condition (C_3) is satisfied.

Theorem 10. *Let the following two conditions hold:*

$(C_1)'$ *The function $f_1 : [0, 1] \times [0, 1] \times R \rightarrow R$ is continuous and there exists a constant $\varrho \in R$ with $0 < \varrho < \sqrt{\pi}/2$ and a D -function $\phi : [0, \infty) \rightarrow [0, \infty)$ and satisfied the fact that $\phi(t) < r$ for any $r \in (0, \infty)$ and*

$$0 \leq f_1(z, t, y_1(z, t)) - f_1(z, t, y_2(z, t)) \leq \varrho \phi(|y_1 - y_2|), \quad (49)$$

for all $(z, t) \in [0, 1] \times [0, 1]$ and $y_1, y_2 \in C_{[0,1]}(R)$ such that $y_1 \geq y_2$.

$(C_2)'$ *There exists an element $\tilde{x} \in C_{[0,1]}(R)$ which satisfies the fact that*

$$\begin{aligned} D_*^\alpha \tilde{x}(z, t) + \frac{\partial \tilde{x}(z, t)}{\partial z} &\leq f_1(z, t, \tilde{x}) + f_2(t, y_2(t)) \\ &\quad + f_3(t, y_3(t)), \\ \forall y_2, y_3 &\in C_{[0,1]}(R), \quad (50) \\ \tilde{x} &\leq \xi_0(z, 0), \quad z \in (0, 1), \\ \alpha &\in (0, 1), \\ t &\in [0, 1]. \end{aligned}$$

Then the fractional dynamical system (46) has a N -tupled mild solution.

Proof. It is clear that if $X = C_{[0,1]}(R)$, X is Y -orbitally complete normed linear space with $\|x(t)\|_X = \max_{t \in [0,1]} |x(t)|$. Let $K = \{x \in X : x \geq 0\}$; the K is closed convex positive cone in X . Thus K is normal. Let $F : D(F) \rightarrow X$, which $D(F) \subset X$, be defined as

$$\begin{aligned} Fy &= y', \\ y \in D(F) &= \{y = X : y' \in X, y(0) = y(1) = 0\}. \end{aligned} \quad (51)$$

Thus we can find A - generates a C_0 - semigroup $\{S(t) : t \geq 0\}$ which is defined as

$$S(t)y(z) = y(t+z), \quad t \geq 0, y \in X. \quad (52)$$

Hence, $\{S(t) : t \geq 0\}$ is an equicontinuous C_0 - semigroup. Therefore, it is clear that $\sup_{t \in [0,1]} \|S(t)\| \leq 1$. Thus, condition $(C_1)'$ holds. Define,

$$\begin{aligned} x_1(t)(z) &= x_1(z, t), \\ x_2(t)(z) &= x_2(z, t), \\ x_3(t)(z) &= x_3(z, t), \\ f_1(t, x_1(t))(z) &= f_1(z, t, x_1(z, t)); \\ f_2(t, x_1(t))(z) &= f_2(z, t, x_1(z, t)); \\ f_3(t, x_1(t))(z) &= f_3(z, t, x_1(z, t)); \end{aligned} \quad (53)$$

Thus, the fractional dynamical system (46) can be written as the fractional system (19). From Theorem 9 system (46) has a N -tupled mild solution. $\square$

6. Conclusions

This paper introduced a new version of Krasnoselskii fixed point theorem. This version is more general because by it we can study the N -tupled fixed point. We applied the abstract proposed fixed point theorem to prove the existence of mild solution of the system of N -fractional evolution equations. We gave an example of the fractional dynamical system to illustrate the abstract fixed point proposed theorem. The results are interesting and can be directly applied to some other interesting problems such as the nonlocal problems for fuzzy implicit fractional differential systems [14] and fuzzy-valued equations [15].

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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