Hindawi
Advances in Mathematical Physics
Volume 2019, Article ID 7528937, 7 pages
https://doi.org/10.1155/2019/7528937

Research Article

Nonlocal Telegraph Equation in Frame of the Conformable Time-Fractional Derivative

Mohamed Bouaouid, Khalid Hilal, and Said Melliani

Department of Mathematics, Sultan Moulay Slimane University, BP 523, 23000 Beni Mellal, Morocco

Correspondence should be addressed to Mohamed Bouaouid; bouaouidfst@gmail.com

Received 15 August 2018; Revised 23 December 2018; Accepted 31 January 2019; Published 3 March 2019

Academic Editor: Sergey Shmarev

We use the cosine family of linear operators to prove the existence, uniqueness, and stability of the integral solution of a nonlocal telegraph equation in frame of the conformable time-fractional derivative. Moreover, we give its implicit fundamental solution in terms of the classical trigonometric functions.

1. Introduction and Statement of the Problem

The telegraph equation is better than the heat equation in modeling of physical phenomena, which have a parabolic behavior [1]. The one-dimensional telegraph equation can be written as follows:

$$\frac{\partial^2 u(t,\ell)}{\partial t^2} = \frac{1}{LC}\frac{\partial^2 u(t,\ell)}{\partial \ell^2} - \frac{RG}{LC}u(t,\ell) - \left(\frac{R}{L}+\frac{G}{C}\right)\frac{\partial u(t,\ell)}{\partial t}, \tag{1}$$

where R and G are, respectively, the resistance and the conductance of resistor, C is the capacitance of capacitor, and L is the inductance of coil. Many concrete applications amount to replacing the time derivative in the telegraph equation with a fractional derivative. For example, in the works [2–8], the authors have extensively studied the time-fractional telegraph equation with Caputo fractional derivative. For more details about the good effect of the fractional derivative, we refer to monographs [9–13].

Recently, a new definition of fractional derivative, named "fractional conformable derivative," is introduce by Khalil et al. [14]. This novel fractional derivative is compatible with the classical derivative and it is excellent for study nonregular solutions. Since the subject of the fractional conformable derivative has attracted the attention of many authors in domains such as mechanics [15], electronic [16], and anomalous diffusion [17]. We are interested in studying in this paper the telegraph model (1) in framework of the time-fractional conformable derivative. Precisely, we will propose the following transformations:

$$\frac{\partial^2}{\partial t^2} \longrightarrow \frac{\partial^\alpha}{\partial t^\alpha}\frac{\partial^\alpha}{\partial t^\alpha}, \quad 0<\alpha<1, \tag{2}$$

$$\frac{\partial}{\partial t} \longrightarrow \frac{\partial^\gamma}{\partial t^\gamma}, \quad 0<\gamma\le\alpha<1, \tag{3}$$

where $\partial^\alpha/\partial t^\alpha$ and $\partial^\gamma/\partial t^\gamma$ are the time-fractional conformable derivative operators [14]. Then, we get the fractional conformable telegraph model associated with the transformations (2) and (3) as follows:

$$\frac{\partial^\alpha}{\partial t^\alpha}\frac{\partial^\alpha u(t,\ell)}{\partial t^\alpha} = \frac{1}{LC}\frac{\partial^2 u(t,\ell)}{\partial \ell^2} - \frac{RG}{LC}u(t,\ell) - \left(\frac{R}{L}+\frac{G}{C}\right)\frac{\partial^\gamma u(t,\ell)}{\partial t^\gamma}, \tag{4}$$

where the time-parameter t belongs to an interval $[0,\tau]$, with τ is a fixed positive real number. The spatial parameter ℓ belongs to the interval $[0,\pi]$.

We associate to (4) the boundary and the nonlocal initial conditions:

$$u(t,0)=0, \quad t\in[0,\tau], \tag{5}$$

$$u(t,\pi)=0, \quad t\in[0,\tau], \tag{6}$$

$$u(0,\ell)=x_0, \quad \ell\in[0,\pi], \tag{7}$$

$$\frac{\partial^\alpha u(0,\ell)}{\partial t^\alpha}=x_1+\sum_{i=1}^{p}\varepsilon_i u(t_i,\ell), \quad \ell\in[0,\pi], \tag{8}$$

where p is a nonnull fixed integer and $0<t_1<\cdots<t_p<\tau$. The quantities x_0, x_1, and $\varepsilon_1,\ldots,\varepsilon_p$ are physical measures. The condition appearing in (8) means the nonlocal condition [18]. For physical interpretations of this condition, we refer to works [19, 20]. For example, in [19] the author used a nonlocal condition of the form (8) to describe the diffusion phenomenon of a small amount of gas in a transparent tube.

We note that it is not easy to find the fundamental solution of (4) by using the Laplace transform method if $\alpha\neq\gamma$. For this reason, we will propose the integral solution concept based on an operator theory approach. When $\alpha=\gamma$, we investigate an implicit fundamental solution.

The content of this paper is organized as follows. In Section 2, we recall some needed results of the conformable fractional derivative and cosine family of linear operators. In Section 3, we prove the existence, uniqueness, and stability of the integral solution of (4) by using of an operator theory approach. Section 4 is devoted to an implicit fundamental solution of (4) in terms of the classical trigonometric functions.

2. Preliminaries

We start recalling some concepts on the conformable fractional calculus [14].

The conformable fractional derivative of a function x of order α at $t>0$ is defined by the following limit:

$$\frac{d^\alpha x(t)}{dt^\alpha}=\lim_{\varepsilon\longrightarrow 0}\frac{x\left(t+\varepsilon t^{1-\alpha}\right)-x(t)}{\varepsilon}. \tag{9}$$

When this limit exists, we say that x is (α)-differentiable at t.

If x is (α)-differentiable at some $t\sim 0$ and the limit $\lim_{t\longrightarrow 0^+}(d^\alpha x(t)/dt^\alpha)$ exists. Then, we define the conformable fractional derivative of x at 0 by

$$\frac{d^\alpha x(0)}{dt^\alpha}=\lim_{t\longrightarrow 0^+}\frac{d^\alpha x(t)}{dt^\alpha}. \tag{10}$$

The (α)-fractional integral of a function x is defined by

$$I^\alpha(x)(t)=\int_0^t s^{\alpha-1}x(s)\,ds. \tag{11}$$

If x is a continuous function in the domain of I^α, then we have

$$\frac{d^\alpha\left(I^\alpha(x)(t)\right)}{dt^\alpha}=x(t). \tag{12}$$

According to [21], if x is differentiable, then we have

$$I^\alpha\left(\frac{d^\alpha x}{dt^\alpha}\right)(t)=x(t)-x(0). \tag{13}$$

We remark that the classical Laplace transform is not compatible with the conformable fractional derivative. For this reason, the adapted transform is defined as [21]

$$\mathscr{L}_\alpha(x(t))(\lambda):=\int_0^{+\infty}t^{\alpha-1}e^{-\lambda t^\alpha/\alpha}x(t)\,dt. \tag{14}$$

The above transformation is called fractional Laplace transform of order α of the function x.

If x is differentiable, then the action of the fractional Laplace transform on the conformable fractional derivative is given as follows:

$$\mathscr{L}_\alpha\left(\frac{d^\alpha x(t)}{dt^\alpha}\right)(\lambda)=\lambda\mathscr{L}_\alpha(x(t))(\lambda)-x(0). \tag{15}$$

Now, we introduce some results concerning the cosine family theory [22].

A one-parameter family $(C(t))_{t\in\mathbb{R}}$ of bounded linear operators on the Banach space X is called a strongly continuous cosine family if and only if

(1) $C(0)=I$, (I is the identity operator),

(2) $C(s+t)+C(s-t)=2C(s)C(t)$, for all $t,s\in\mathbb{R}$,

(3) $t\longmapsto C(t)x$ is continuous for each fixed $x\in X$.

We define also the sine family by

$$S(t):=\int_0^t C(s)\,ds. \tag{16}$$

The infinitesimal generator A of a strongly continuous cosine family $((C(t))_{t\in\mathbb{R}},(S(t))_{t\in\mathbb{R}})$ on X is defined by

$$\begin{aligned} D(A)=\{x\in X,t&\longmapsto C(t)\\ &\cdot x \text{ is a twice continuously differentiable function}\},\\ A=\left.\frac{d^2C(t)}{dt^2}\right|_{t=0}&. \end{aligned} \tag{17}$$

If A is the infinitesimal generator of a strongly cosine family $((C(t))_{t\in\mathbb{R}},(S(t))_{t\in\mathbb{R}})$ on X, then there exists a constant $\omega\geq 0$ such that, for all λ with $Re(\lambda)>\omega$, we have

$$\begin{aligned} &\lambda^2\in\rho(A),\\ &\qquad(\rho(A)\colon \text{is the resolvent set of } A),\\ &\lambda\left(\lambda^2I-A\right)^{-1}=\int_0^{+\infty}e^{-\lambda t}C(t)\,dt,\\ &\left(\lambda^2I-A\right)^{-1}=\int_0^{+\infty}e^{-\lambda t}S(t)\,dt, \end{aligned} \tag{18}$$

where $Re(\lambda)$ is the real part of the complex number λ.

Before presenting the main results, we introduce the following notations:

$$a = \frac{RG}{LC}, \quad b = \frac{R}{L} + \frac{G}{C}, \quad c = \frac{1}{LC} \quad \text{and } \varepsilon = \sum_{i=1}^{p} |\varepsilon_i|. \tag{19}$$

3. Integral Solution by Using an Operator Theory Approach

Define the operator $A : L^2([0,\pi], d\ell) \longrightarrow L^2([0,\pi], d\ell)$ by

$$A = c\frac{\partial^2 (.)}{\partial \ell^2} \quad \text{and } D(A) = \left\{u \in H^2((0,\pi), d\ell),\ u(0) = u(\pi) = 0\right\}. \tag{20}$$

According to [23], the operator A generates a cosine family $((C(t))_{t\in\mathbb{R}}, (S(t))_{t\in\mathbb{R}})$ on $L^2([0,\pi], d\ell)$. Moreover, $|C(t)| \le 1$ and $|S(t)| \le 1$, for all $t \in [0,\tau]$.

Next, we consider the following transformations:

$$x(t)(\ell) = u(t,\ell), \quad f\left(t, x(t), \frac{d^\gamma x(t)}{dt^\gamma}\right) = -ax(t) - b\frac{d^\gamma x(t)}{dt^\gamma}, \quad g(x) = \sum_{i=1}^{p} \varepsilon_i x(t_i). \tag{21}$$

Then, we get the following nonlocal fractional ordinary differential equation:

$$\frac{d^\alpha}{dt^\alpha}\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + f\left(t, x(t), \frac{d^\gamma x(t)}{dt^\gamma}\right), \tag{22}$$

$$x(0) = x_0, \tag{23}$$

$$\frac{d^\alpha x(0)}{dt^\alpha} = x_1 + g(x). \tag{24}$$

We denote $\mathscr{C}^\alpha$ the Banach space of continuously (α)-differentiable functions from $[0,\tau]$ into $L^2([0,\pi], d\ell)$ with the norm $|x| = \sup_{t\in[0,\tau]}\|x(t)\| + \sup_{t\in[0,\tau]}\|d^\alpha x(t)/dt^\alpha\|$. Here, $\|.\|$ is the classical norm in the space $L^2([0,\pi], d\ell)$.

3.1. Existence and Uniqueness of the Integral Solution

3.1. Existence and Uniqueness of the Integral Solution. To explain integral Duhamel's formula, we apply the fractional Laplace transform to (22), obtaining

$$x(t) = C\left(\frac{t^\alpha}{\alpha}\right)x_0 + S\left(\frac{t^\alpha}{\alpha}\right)[x_1 + g(x)] + \int_0^t s^{\alpha-1} S\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f\left(s, x(s), \frac{d^\gamma x(s)}{dt^\gamma}\right) ds. \tag{25}$$

We remark that, for $\alpha = \gamma = 1$, we have classical Duhamel's formula [22]. Hence, we can introduce the following definition.

Definition 1. We say that $x \in \mathscr{C}^\alpha$ is an integral solution of the equation (22) if the following assertion is true:

$$x(t) = C\left(\frac{t^\alpha}{\alpha}\right)x_0 + S\left(\frac{t^\alpha}{\alpha}\right)[x_1 + g(x)] + \int_0^t s^{\alpha-1} S\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f\left(s, x(s), \frac{d^\gamma x(s)}{dt^\gamma}\right) ds, \quad t \in [0,\tau]. \tag{26}$$

Theorem 2. *The Cauchy problem (22) has a unique integral solution, provided that*

$$\varepsilon + \max\left(\frac{a\tau^\alpha}{\alpha}, \frac{b\tau^{2\alpha-\gamma}}{2\alpha-\gamma}\right) < \frac{1}{2}. \tag{27}$$

Proof. Define the operator $\Gamma : \mathscr{C}^\alpha \longrightarrow \mathscr{C}^\alpha$ by

$$\Gamma(x)(t) = C\left(\frac{t^\alpha}{\alpha}\right)x_0 + S\left(\frac{t^\alpha}{\alpha}\right)[x_1 + g(x)] + \int_0^t s^{\alpha-1} S\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f\left(s, x(s), \frac{d^\gamma x(s)}{dt^\gamma}\right) ds. \tag{28}$$

Let $x, y \in \mathscr{C}^\alpha$; then we have

$$\begin{aligned}
\Gamma(y)(t) - \Gamma(x)(t) &= S\left(\frac{t^\alpha}{\alpha}\right)[g(y) - g(x)] + \int_0^t s^{\alpha-1} S\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\left[f\left(s, y(s), \frac{d^\gamma y(s)}{dt^\gamma}\right) - f\left(s, x(s), \frac{d^\gamma x(s)}{dt^\gamma}\right)\right] ds, \\
\frac{d^\alpha}{dt^\alpha}[\Gamma(y)(t) - \Gamma(x)(t)] &= C\left(\frac{t^\alpha}{\alpha}\right)[g(y) - g(x)] + \int_0^t s^{\alpha-1} C\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\left[f\left(s, y(s), \frac{d^\gamma y(s)}{dt^\gamma}\right) - f\left(s, x(s), \frac{d^\gamma x(s)}{dt^\gamma}\right)\right] ds.
\end{aligned} \tag{29}$$

Accordingly, we obtain

$$\|\Gamma(y)(t)-\Gamma(x)(t)\|+\left\|\frac{d^{\alpha}}{dt^{\alpha}}[\Gamma(y)(t)-\Gamma(x)(t)]\right\| \leq 2\left[\varepsilon+\max\left(\frac{a\tau^{\alpha}}{\alpha},\frac{b\tau^{2\alpha-\gamma}}{2\alpha-\gamma}\right)\right]|y-x|. \tag{30}$$

Then, we get

$$|\Gamma(y)-\Gamma(x)| \leq 2\left[\varepsilon+\max\left(\frac{a\tau^{\alpha}}{\alpha},\frac{b\tau^{2\alpha-\gamma}}{2\alpha-\gamma}\right)\right]|y-x|. \tag{31}$$

Finally, Γ has an unique fixed point x in $(\mathscr{C}^{\alpha},|.|)$, which is the integral solution of the equation (22). □

Now, we give a result that is better than the previous one.

Theorem 3. *The Cauchy problem (22) has a unique integral solution, provided that*

$$\varepsilon<\frac{1}{2}. \tag{32}$$

Proof. Define the operator $\Gamma:\mathscr{C}^{\alpha}\longrightarrow\mathscr{C}^{\alpha}$ by

$$\Gamma(x)(t) = C\left(\frac{t^{\alpha}}{\alpha}\right)x_0+S\left(\frac{t^{\alpha}}{\alpha}\right)[x_1+g(x)] + \int_0^t s^{\alpha-1}S\left(\frac{t^{\alpha}-s^{\alpha}}{\alpha}\right)f\left(s,x(s),\frac{d^{\gamma}x(s)}{dt^{\gamma}}\right)ds, \quad t\in[0,\tau]. \tag{33}$$

Next, we define a new norm $|.|_{\alpha}$ in $\mathscr{C}^{\alpha}$ by

$$|x|_{\alpha}=\left|\exp\left(\frac{-\theta(.)^{\alpha}}{\alpha}\right)x\right|, \tag{34}$$

where

$$\theta=\frac{3\max(a,b\tau^{\alpha-\gamma})}{1-2\varepsilon}. \tag{35}$$

For $x,y\in\mathscr{C}^{\alpha}$ and $t\in[0,\tau]$, we have

$$\Gamma(y)(t)-\Gamma(x)(t)=S\left(\frac{t^{\alpha}}{\alpha}\right)[g(y)-g(x)] + \int_0^t s^{\alpha-1}S\left(\frac{t^{\alpha}-s^{\alpha}}{\alpha}\right)\left[f\left(s,y(s),\frac{d^{\gamma}y(s)}{dt^{\gamma}}\right) - f\left(s,x(s),\frac{d^{\gamma}x(s)}{dt^{\gamma}}\right)\right]ds. \tag{36}$$

Therefore, we obtain

$$\begin{aligned}\|\Gamma(y)(\mathrm{t})-\Gamma(x)(t)\| &\leq \left[\varepsilon\exp\left(\frac{\theta t^{\alpha}}{\alpha}\right) + \max(a,b\tau^{\alpha-\gamma})\int_0^t s^{\alpha-1}\exp\left(\frac{\theta s^{\alpha}}{\alpha}\right)ds\right]|y-x|_{\alpha},\\ \left\|\frac{d^{\alpha}}{dt^{\alpha}}[\Gamma(y)(t)-\Gamma(x)(t)]\right\| &\leq \left[\varepsilon\exp\left(\frac{\theta t^{\alpha}}{\alpha}\right) + \max(a,b\tau^{\alpha-\gamma})\int_0^t s^{\alpha-1}\exp\left(\frac{\theta s^{\alpha}}{\alpha}\right)ds\right]|y-x|_{\alpha}.\end{aligned} \tag{37}$$

Accordingly, we get

$$|\Gamma(y)-\Gamma(x)|_{\alpha}\leq 2\left[\varepsilon+\frac{\max(a,b\tau^{\alpha-\gamma})}{\theta}\right]|y-x|_{\alpha}. \tag{38}$$

Hence, we conclude that

$$|\Gamma(y)-\Gamma(x)|_{\alpha}\leq\frac{2(\varepsilon+1)}{3}|y-x|_{\alpha}. \tag{39}$$

The fact $2(\varepsilon+1)/3<1$ proves that Γ has an unique fixed point x in $(\mathscr{C}^{\alpha},|.|_{\alpha})$, which is the integral solution of equation (22). □

3.2. Stability of the Integral Solution. Here, we give a result concerning the nonlocal-condition effect on the stability of the integral solution.

Theorem 4. *Let x and y be solutions associated with (x_0,x_1) and (x_0,y_1), respectively. Then, we have the following estimate:*

$$|y-x| \leq \frac{2}{1-2(\varepsilon+\max(a\tau^{\alpha}/\alpha,b\tau^{2\alpha-\gamma}/(2\alpha-\gamma)))}\|y_1-x_1\|, \tag{40}$$

provided that

$$\varepsilon+\max\left(\frac{a\tau^{\alpha}}{\alpha},\frac{b\tau^{2\alpha-\gamma}}{2\alpha-\gamma}\right)<\frac{1}{2}. \tag{41}$$

Proof. We have

$$
\begin{aligned}
y(t)-x(t) &= S\left(\frac{t^\alpha}{\alpha}\right)\left[y_1-x_1+g(y)-g(x)\right] \\
&\quad + \int_0^t s^{\alpha-1} S\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)\left[f\left(s,y(s),\frac{d^\gamma y(s)}{dt^\gamma}\right)\right. \\
&\quad \left. - f\left(s,x(s),\frac{d^\gamma x(s)}{dt^\gamma}\right)\right] ds, \\
\frac{d^\alpha}{dt^\alpha}\left[y(t)-x(t)\right] &= C\left(\frac{t^\alpha}{\alpha}\right)\left[y_1-x_1+g(y)\right. \\
&\quad \left. - g(x)\right] + \int_0^t s^{\alpha-1} C\left(\frac{t^\alpha-s^\alpha}{\alpha}\right) \\
&\quad \cdot \left[f\left(s,y(s),\frac{d^\gamma y(s)}{dt^\gamma}\right)\right. \\
&\quad \left. - f\left(s,x(s),\frac{d^\gamma x(s)}{dt^\gamma}\right)\right] ds.
\end{aligned} \tag{42}
$$

Consequently, we get

$$
\begin{aligned}
\left|y-x\right| &\le 2\left(\varepsilon+\max\left(\frac{a\tau^\alpha}{\alpha},\frac{b\tau^{2\alpha-\gamma}}{2\alpha-\gamma}\right)\right)\left|y-x\right| \\
&\quad + 2\left\|y_1-x_1\right\|.
\end{aligned} \tag{43}
$$

Finally, we obtain the following estimation:

$$
\begin{aligned}
&\left|y-x\right| \\
&\le \frac{2}{1-2\left(\varepsilon+\max\left(a\tau^\alpha/\alpha, b\tau^{2\alpha-\gamma}/\left(2\alpha-\gamma\right)\right)\right)}\left\|y_1\right. \\
&\quad \left. - x_1\right\|.
\end{aligned} \tag{44}
$$

□

4. Implicit Fundamental Solution in the Case When $\gamma=\alpha$

Here, we give the implicit fundamental solution of (4) by using the separating variables method. To do so, let $u(t,\ell)=x(t)y(\ell)$. Then, (4) becomes as follows:

$$
\begin{aligned}
&ax(t)\,y(\ell)+by(\ell)\frac{d^\alpha x(t)}{dt^\alpha}+y(\ell)\frac{d^\alpha}{dt^\alpha}\frac{d^\alpha x(t)}{dt^\alpha} \\
&\quad = cx(t)\frac{d^2 y(\ell)}{d\ell^2}.
\end{aligned} \tag{45}
$$

Then, there exists a constant λ^2 such as

$$
\left(a+\lambda^2\right)x(t)+b\frac{d^\alpha x(t)}{dt^\alpha}+\frac{d^\alpha}{dt^\alpha}\frac{d^\alpha x(t)}{dt^\alpha}=0, \tag{46}
$$

$$
c\frac{d^2 y(\ell)}{d\ell^2}=-\lambda^2 y(\ell). \tag{47}
$$

According to (5) and (6), the solution of (47) is given by

$$
y_n(\ell)=\sin(n\ell). \tag{48}
$$

Moreover, the solution of (4) can written as

$$
u(t,\ell)=\sum_{n=0}^{+\infty} x_n(t)\,y_n(\ell), \tag{49}
$$

where $y_n(\ell)=\sin(n\ell)$ and $x_n(t)$ is the solution of the following ordinary differential equation:

$$
\left(a+cn^2\right)x_n(t)+b\frac{d^\alpha x_n(t)}{dt^\alpha}+\frac{d^\alpha}{dt^\alpha}\frac{d^\alpha x_n(t)}{dt^\alpha}=0, \tag{50}
$$

with the nonlocal initial conditions

$$
\begin{aligned}
x_n(0) &= \frac{2}{\pi}\int_0^\pi \sin(ns)\,u(0,s)\,ds, \\
\frac{d^\alpha x_n(0)}{dt^\alpha} &= \frac{2}{\pi}\int_0^\pi \sin(ns)\frac{\partial^\alpha u(0,s)}{\partial t^\alpha}ds.
\end{aligned} \tag{51}
$$

By using the fractional Laplace transform in (50), we get

$$
\begin{aligned}
x_n(t) &= \frac{2}{\pi}\exp\left(-\frac{bt^\alpha}{2\alpha}\right)\int_0^\pi \cos\left(\frac{\sqrt{4cn^2+4a-b^2}}{2\alpha}t^\alpha\right) \\
&\quad \cdot x_0\sin(ns)\,ds+\frac{2b}{\pi}\exp\left(-\frac{bt^\alpha}{2\alpha}\right) \\
&\quad \cdot \int_0^\pi \frac{\sin\left(\left(\sqrt{4cn^2+4a-b^2}/2\alpha\right)t^\alpha\right)}{\sqrt{4cn^2+4a-b^2}}x_0\sin(ns)\,ds \\
&\quad + \frac{4}{\pi}\exp\left(-\frac{bt^\alpha}{2\alpha}\right) \\
&\quad \cdot \sum_{i=1}^{p}\int_0^\pi \frac{\sin\left(\left(\sqrt{4cn^2+4a-b^2}/2\alpha\right)t^\alpha\right)}{\sqrt{4cn^2+4a-b^2}}\left[\frac{x_1}{p}\right. \\
&\quad \left. + \varepsilon_i u(t_i,s)\right]\sin(ns)\,ds.
\end{aligned} \tag{52}
$$

Finally, replacing $x_n(t)$ in (49), we get

$$
\begin{aligned}
u(t,\ell) &= \frac{2}{\pi}\exp\left(-\frac{bt^\alpha}{2\alpha}\right)\sum_{n=0}^{+\infty}\int_0^\pi \cos\left(\frac{\sqrt{4cn^2+4a-b^2}}{2\alpha}\right. \\
&\quad \left. \cdot t^\alpha\right)x_0\sin(n\ell)\sin(ns)\,ds+\frac{2b}{\pi}\exp\left(-\frac{bt^\alpha}{2\alpha}\right) \\
&\quad \cdot \sum_{n=0}^{+\infty}\int_0^\pi \frac{\sin\left(\left(\sqrt{4cn^2+4a-b^2}/2\alpha\right)t^\alpha\right)}{\sqrt{4cn^2+4a-b^2}}x_0\sin(n\ell) \\
&\quad \cdot \sin(ns)\,ds+\frac{4}{\pi}\exp\left(-\frac{bt^\alpha}{2\alpha}\right) \\
&\quad \cdot \sum_{n=0}^{+\infty}\sum_{i=1}^{p}\int_0^\pi \frac{\sin\left(\left(\sqrt{4cn^2+4a-b^2}/2\alpha\right)t^\alpha\right)}{\sqrt{4cn^2+4a-b^2}}\left[\frac{x_1}{p}\right. \\
&\quad \left. + \varepsilon_i u(t_i,s)\right]\sin(n\ell)\sin(ns)\,ds.
\end{aligned} \tag{53}
$$

Proposition 5. *For $\alpha = 1$ and $\varepsilon = 0$ the formula (53) coincides with the fundamental solution of the classical telegraph equation (1).*

Proof. Based on the separating variables method in (1), we obtain

$$u(t,\ell)=\sum_{n=0}^{+\infty}x_n(t)\sin(n\ell), \tag{54}$$

where $x_n(t)$ is the solution of the following ordinary differential equation:

$$\left(a+cn^2\right)x_n(t)+b\frac{\mathrm{d}x_n(t)}{dt}+\frac{d^2x_n(t)}{dt^2}=0, \tag{55}$$

with the initial conditions

$$\begin{aligned} x_n(0)&=\frac{2}{\pi}\int_0^\pi \sin(ns)\,u(0,s)\,ds,\\ \frac{dx_n(0)}{dt}&=\frac{2}{\pi}\int_0^\pi \sin(ns)\,\frac{\partial u(0,s)}{\partial t}ds. \end{aligned} \tag{56}$$

By using the classical Laplace transform in (55), we get

$$\begin{aligned} x_n(t)&=\frac{2}{\pi}\exp\left(-\frac{b}{2}t\right)\\ &\cdot\int_0^\pi\cos\left(\frac{\sqrt{4cn^2+4a-b^2}}{2}t\right)x_0\sin(ns)\,ds+\frac{2b}{\pi}\\ &\cdot\exp\left(-\frac{b}{2}t\right)\\ &\cdot\int_0^\pi\frac{\sin\left(\left(\sqrt{4cn^2+4a-b^2}/2\right)t\right)}{\sqrt{4cn^2+4a-b^2}}x_0\sin(ns)\,ds\\ &+\frac{4}{\pi}\exp\left(-\frac{b}{2}t\right)\\ &\cdot\int_0^\pi\frac{\sin\left(\left(\sqrt{4cn^2+4a-b^2}/2\right)t\right)}{\sqrt{4cn^2+4a-b^2}}x_1\sin(ns)\,ds. \end{aligned} \tag{57}$$

Replacing $x_n(t)$ in (54), we find the fundamental solution of the classical telegraph equation (1) as follows:

$$\begin{aligned} u(t,\ell)&=\frac{2}{\pi}\exp\left(-\frac{b}{2}t\right)\\ &\cdot\sum_{n=0}^{+\infty}\int_0^\pi\cos\left(\frac{\sqrt{4cn^2+4a-b^2}}{2}t\right)x_0\sin(n\ell)\\ &\cdot\sin(ns)\,ds+\frac{2b}{\pi}\exp\left(-\frac{b}{2}t\right)\\ &\cdot\sum_{n=0}^{+\infty}\int_0^\pi\frac{\sin\left(\left(\sqrt{4cn^2+4a-b^2}/2\right)t\right)}{\sqrt{4cn^2+4a-b^2}}x_0\sin(n\ell)\\ &\cdot\sin(ns)\,ds+\frac{4}{\pi}\exp\left(-\frac{b}{2}t\right)\\ &\cdot\sum_{n=0}^{+\infty}\int_0^\pi\frac{\sin\left(\left(\sqrt{4cn^2+4a-b^2}/2\right)t\right)}{\sqrt{4cn^2+4a-b^2}}x_1\sin(n\ell)\\ &\cdot\sin(ns)\,ds. \end{aligned} \tag{58}$$

□

Remark 6. If we consider the fractional derivative in Caputo's sense, the implicit fundamental solution of (4) can be written in terms of the Mittag-Leffler function. However, in our case we have found the implicit fundamental solution in terms of the classical trigonometric functions.

Remark 7. The integral solution does not impose any constraint concerning the choice of the derivation parameters. This provides more freedom concerning the choice of sensitive parameters α and γ in practice situations for modeling naturel phenomena.

5. Conclusion

We have studied a time-conformable fractional telegraph equation with nonlocal condition. In the case when $\alpha = \gamma$, we have given the implicit fundamental solution in terms of the classical trigonometric functions. In the general case, we have established the existence, uniqueness, and stability of the integral solution.

As for future work, we intend to give the fundamental solution for all values of α and γ.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] E. C. Eckstein, J. A. Goldstein, and M. Leggas, "The mathematics of suspensions: Kac walks and asymptotic analyticity," *Electronic Journal of Differential Equations*, vol. 3, pp. 39–50, 1999.

[2] R. C. Cascaval, E. C. Eckstein, C. L. Frota, and J. A. Goldstein, "Fractional telegraph equations," *Journal of Mathematical Analysis and Applications*, vol. 276, no. 1, pp. 145–159, 2002.

[3] J. Chen, F. Liu, and V. Anh, "Analytical solution for the time-fractional telegraph equation by the method of separating variables," *Journal of Mathematical Analysis and Applications*, vol. 338, no. 2, pp. 1364–1377, 2008.

[4] S. Das, K. Vishal, P. K. Gupta, and A. Yildirim, "An approximate analytical solution of time-fractional telegraph equation," *Applied Mathematics and Computation*, vol. 217, no. 18, pp. 7405–7411, 2011.

[5] V. R. Hosseini, W. Chen, and Z. Avazzadeh, "Numerical solution of fractional telegraph equation by using radial basis functions,"

Engineering Analysis with Boundary Elements, vol. 38, no. 12, pp. 31–39, 2014.

[6] S. Kumar, "A new analytical modelling for fractional telegraph equation via Laplace transform," *Applied Mathematical Modelling*, vol. 38, no. 13, pp. 3154–3163, 2014.

[7] S. Momani, "Analytic and approximate solutions of the space- and time-fractional telegraph equations," *Applied Mathematics and Computation*, vol. 170, no. 2, pp. 1126–1134, 2005.

[8] V. K. Srivastava, M. K. Awasthi, and S. Kumar, "Analytical approximations of two and three dimensional time-fractional telegraphic equation by reduced differential transform method," *Egyptian Journal of Basic and Applied Sciences*, vol. 1, no. 1, pp. 60–66, 2014.

[9] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North Holland Mathematics Studies 204, Elsevier, New York, NY, USA, 2006.

[10] K. S. Miller, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, 1993.

[11] K. B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, NY, USA, 1974.

[12] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.

[13] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Yverdon, Switzerland, 1993.

[14] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, "A new definition of fractional derivative," *Journal of Computational and Applied Mathematics*, vol. 264, pp. 65–70, 2014.

[15] W. S. Chung, "Fractional Newton mechanics with conformable fractional derivative," *Journal of Computational and Applied Mathematics*, vol. 290, pp. 150–158, 2015.

[16] L. Martínez, J. Rosales, C. Carreño, and J. Lozano, "Electrical circuits described by fractional conformable derivative," *International Journal of Circuit Theory and Applications*, vol. 46, no. 5, pp. 1091–1100, 2018.

[17] H. W. Zhou, S. Yang, and S. Q. Zhang, "Conformable derivative approach to anomalous diffusion," *Physica A: Statistical Mechanics and its Applications*, vol. 491, pp. 1001–1013, 2018.

[18] L. Byszewski, "Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem," *Journal of Mathematical Analysis and Applications*, vol. 162, no. 2, pp. 494–505, 1991.

[19] K. Deng, "Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions," *Journal of Mathematical Analysis and Applications*, vol. 179, no. 2, pp. 630–637, 1993.

[20] W. E. Olmstead and C. A. Roberts, "The one-dimensional heat equation with a nonlocal initial condition," *Applied Mathematics Letters*, vol. 10, no. 3, pp. 89–94, 1997.

[21] T. Abdeljawad, "On conformable fractional calculus," *Journal of Computational and Applied Mathematics*, vol. 279, pp. 57–66, 2015.

[22] C. C. Travis and G. F. Webb, "Cosine families and abstract nonlinear second order differential equations," *Acta Mathematica Hungarica*, vol. 32, no. 1-2, pp. 75–96, 1978.

[23] M. E. Hernández, "Existence of solutions to a second order partial differential equation with nonlocal conditions," *Electronic Journal of Differential Equations*, vol. 2003, pp. 1–10, 2003.