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Research Article

The Equations and Characteristics of the Magnetic Curves in the Sphere Space

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We investigate some geometrical properties of magnetic curves in $\mathbb{S}^3$ under the action of the Killing magnetic field $V = a\partial_x + b\partial_y + c\partial_z$. The other main result is provided about the classification of the equations of the geodesics in $\mathbb{S}^3$. Moreover, some most relevant graphs of the main results were drawn in this paper.

1. Introduction

The study of magnetic fields and their corresponding magnetic curves on different manifolds is one of the important research topics between differential geometry and physics. The magnetic curves on the Riemannian manifolds are trajectories of charged particles moving on M under the magnetic field. Meanwhile, the different magnetic fields were extended to different ambient spaces [1–12]. Corresponding to parallel Lorentz forces, the magnetic trajectories are obtained on some 2-dimensional space [1, 2]. In [3, 4], the authors had researched the magnetic fields in complex space, which are called Kähler form, and in Sasakian 3-manifold. The classification of the magnetic curves in 3-dimensional Minkowski space with Killing magnetic field and in three-dimensional almost paracontact manifolds was given in [5, 6]. The authors obtained the magnetic trajectories as solutions of a variational problem that neither involves any local potential nor constraints the topology given a magnetic field in 3D [7]. And the classification for the Killing magnetic trajectories in two special 3-dimensional manifolds, namely, $\mathbb{E}^3$ and $\mathbb{S}^2 \times \mathbb{R}$, was studied in [8, 11], respectively.

However, if we want to extend this concept to other ambient spaces, it is necessary to distinguish between the manifolds and the tangent vector spaces. In this regard, the sphere spaces play an important role among these manifolds, for their normal vectors direct to the original, and there is a smooth deformation through constant mean curvature surfaces with the same topology, which can be expressed in terms of changing radii. The authors extend the rectifying theory and the relative results in the 3-dimensional sphere [13].

Looking over all these results obtained in classification of magnetic trajectories corresponding to magnetic fields in different ambient spaces, until recently, and to the best of our knowledge, there has been little information available about the magnetic curves in the 3-dimensional sphere. In the present paper, we give some geometrical properties of magnetic curves in $\mathbb{S}^3$, especially, the magnetic curves corresponding to $V = a\partial_x + b\partial_y + c\partial_z$.

The outlines of this work are as follows: we introduce the magnetic curves in $\mathbb{S}^3$ in Section 2. In Section 3, for particular geodesics, we adopt the first approach to classify the equation of the geodesics in $\mathbb{S}^3$ (Theorem 1). Then, we deal with the magnetic vector field $V = a\partial_x + b\partial_y + c\partial_z$ tangent to the $\mathbb{R}$ factor, which generates the magnetic trajectories described in Theorem 2. In Section 4, as an application, we give some examples and graphs to certify our conclusions. And then, we investigate the trajectories of the magnetic fields called N-magnetic curves. Moreover, we obtain some solutions of the Lorentz force equation and give an example of this curve by drawing their pictures using Mathematica.

2. Preliminaries

Let M be a n($n \geq 2$)-dimensional oriented Riemannian manifold. A *magnetic curve* represents the trajectory of a charged particle moving in the manifold under the action of a magnetic field. A *magnetic field* in (M, g) is a closed two-form F. The corresponding *Lorentz force* of a magnetic field F on (M, g) is a skew symmetric $(1,1)$−tensor field ϕ defined by

$$g(\phi(X), Y) = F(X, Y), \tag{1}$$

for any $X, Y \in \chi(M)$.

The *magnetic trajectories* of F are curves γ on M which satisfy the *Lorentz equation*

$$\nabla_{\gamma'}\gamma' = \phi(\gamma'). \tag{2}$$

The curve $\gamma(s)$ is also known as the *flowline* of the dynamical system associated with the magnetic field F. When the magnetic curve $\gamma(s)$ is arc length parametrized $v_0 = 1$, it is called a *normal magnetic curve*.

Let $\gamma(s)$ be a magnetic curve; if the curve satisfies the equation $\nabla_{\gamma'}\gamma' = 0$, we call the curve *geodesic*. Therefore, from the point of view of dynamical systems, a geodesic corresponds to a trajectory of a particle when $F = 0$.

A field vector field V on M is Killing if and only if it satisfies the Killing equation:

$$g(\nabla_Y V, Z) + g(\nabla_Z V, Y) = 0 \tag{3}$$

for any vector fields Y, Z on M, where ∇ is the Levi-Cicita connection on M.

Let V be a Killing vector field on M and $F_V = \iota_V dv_g$ the corresponding Killing magnetic field, where the inner product is denoted by ι. Then, the Lorentz force of the F_V is given by [10]

$$\phi(X) = V \times X. \tag{4}$$

Consequently, the Lorentz force equation may be written as

$$\nabla_{\gamma'}\gamma' = V \times \gamma'. \tag{5}$$

In this paper, we will introduce some characteristics of magnetic curves in the three-dimensional sphere.

Let $\mathbb{S}^3$ denote the three-dimensional unit sphere in $\mathbb{R}^4$ centered at the origin and defined by

$$\mathbb{S}^3 = \left\{\boldsymbol{x} = (x_1, x_2, x_3, x_4) \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\right\}. \tag{6}$$

Let $\overline{\nabla}$ and ∇^0 denote the Levi-Civita connections in $\mathbb{S}^3$ and $\mathbb{R}^4$, respectively. If X and Y are vector fields tangent to $\mathbb{S}^3$, then $\overline{\nabla}$ and ∇^0 are related by the Gauss formula as follows:

$$\nabla^0_X Y = \overline{\nabla}_X Y - h(X, Y)G, \tag{7}$$

where $G : \mathbb{S}^3 \longrightarrow \mathbb{R}^4$ denotes the position vector.

In three-dimensional manifold (M^3, g), the mixed product of the vector fields $X, Y, Z \in \chi(M^3)$ is defined by $g(X \wedge Y, Z) = \Omega_3(X, Y, Z)$. In the sphere space $\mathbb{S}^3$, we can define a cross product as follows. Consider a point $q \in \mathbb{S}^3$ and take two tangent vectors $v_1, v_2 \in T_q(\mathbb{S}^3)$; the cross product of the v_1 and v_2 is the unique tangent vector $v_1 \times v_2$ in $T_q(\mathbb{S}^3)$ such that

$$\langle v_1 \times v_2, \omega\rangle = \det(v_1, v_2, \omega, q), \quad \forall \omega \in T_q(\mathbb{S}^3), \tag{8}$$

where $\langle , \rangle$ denotes the induced metric in $\mathbb{S}^3$ and the vectors are considered as column vectors in $\mathbb{R}^4$ [13].

Consider a unit speed curve $\gamma : I \longrightarrow \mathbb{S}^3$, where I is a real open interval and assume that γ is not a geodesic curve. Let $\boldsymbol{T}(s) = \gamma'(s)$, and then, there is a unique vector field $\mathbf{N}(s)$ and a positive function $\kappa(s)$ so that $\overline{\nabla}_{\boldsymbol{T}(s)}\boldsymbol{T}(s) = \kappa(s)\boldsymbol{N}(s)$. Here, $\overline{\nabla}_{\boldsymbol{T}(s)}\boldsymbol{T}(s)$ denotes the covariant derivative of $\gamma(s)$ in $\mathbb{S}^3$, and $\boldsymbol{N}(s)$ is called the principal normal vector fields and $\kappa(s)$, the curvature of the given curve. Given a unit speed curve in $\mathbb{S}^3$, the binormal vector field of the curve $\gamma(s)$ is defined by $\boldsymbol{B}(s) = \boldsymbol{T}(s) \times \boldsymbol{N}(s)$, which is a unit vector field orthogonal to both $\boldsymbol{T}(s)$ and $\boldsymbol{N}(s)$. Since $\overline{\nabla}_{\boldsymbol{T}(s)}\boldsymbol{B}(s)$ is collinear with $\boldsymbol{N}(s)$, we can write $\overline{\nabla}_{\boldsymbol{T}(s)}\boldsymbol{B}(s) = -\tau(s)\boldsymbol{N}(s)$, and the differentiable function $\tau(s)$ is called the torsion of $\gamma(s)$. There exists a *Frenet frame* $\{\boldsymbol{T}(s), \boldsymbol{N}(s), \boldsymbol{B}(s)\}$ satisfying [13]

$$\begin{aligned}
\overline{\nabla}_{\boldsymbol{T}(s)}\boldsymbol{T}(s) &= \kappa(s)\boldsymbol{N}(s)\\
\overline{\nabla}_{\boldsymbol{T}(s)}\boldsymbol{N}(s) &= -\kappa(s)\boldsymbol{T}(s) + \tau(s)\boldsymbol{B}(s)\\
\overline{\nabla}_{\boldsymbol{T}(s)}\boldsymbol{B}(s) &= -\tau(s)\boldsymbol{N}(s).
\end{aligned} \tag{9}$$

By using the Gauss formula, we can rewrite the equations in the Euclidean connection ∇^0 as follows:

$$\begin{aligned}
\nabla^0_{\boldsymbol{T}(s)}\boldsymbol{T}(s) &= \kappa(s)\boldsymbol{N}(s) - \gamma(s)\\
\nabla^0_{\boldsymbol{T}(s)}\boldsymbol{N}(s) &= -\kappa(s)\boldsymbol{T}(s) + \tau(s)\boldsymbol{B}(s)\\
\nabla^0_{\boldsymbol{T}(s)}\boldsymbol{B}(s) &= -\tau(s)\boldsymbol{N}(s).
\end{aligned} \tag{10}$$

3. Some Characteristics of the Magnetic Curves in 3D Sphere

By the Gauss formula $\nabla^0_X Y = \overline{\nabla}_X Y - h(X, Y)G$, where $\overline{\nabla}$ and ∇^0 denote the Levi-Civita connections in $\mathbb{S}^3$ and $\mathbb{R}^4$, respectively. Suppose the curve $\gamma(s)$ is the geodesics in three-dimensional sphere, and $\gamma(s) = (x(s), y(s), z(s), t(s))$, the tangent vector $\boldsymbol{T}(s) = \gamma'(s) = \{\dot{x}(s), \dot{y}(s), \dot{z}(s), \dot{t}(s)\}$. We construct a mapping $h : \mathbb{S}^3 \longrightarrow \mathbb{R}^4$, which is the second fundamental form of $\mathbb{S}^3$ in $\mathbb{R}^4$, and the position vector $G = \{x, y, z, 0\}$.

Let $\gamma : I \subset \mathbb{R} \longrightarrow \mathbb{S}^3 \subset \mathbb{R}^4$ be a smooth curve in $\mathbb{S}^3$. The metric in $\mathbb{S}^3$ is given by the restriction of the usual scalar product $\langle , \rangle$ in $\mathbb{R}^4$ and the Levi-Civita connections ∇^0 and $\overline{\nabla}$ in $\mathbb{S}^3$ and $\mathbb{R}^4$ respectively. We can consider the natural projection

$$\begin{aligned}
\pi : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}^3 \subset \mathbb{R}^4 : \pi(v_1, v_2, v_3, v_4)\\
= (v_1, v_2, v_3, 0).
\end{aligned} \tag{11}$$

The coordinates for the arc length parameter curve $\gamma(s)$ in $\mathbb{S}^3$ are parameterized by $\gamma(s) = (x(s), y(s), z(s), t(s))$ such that $x^2(s)+y^2(s)+z^2(s)+t^2(s) = 1$ and $\dot{x}^2(s)+\dot{y}^2(s)+\dot{z}^2(s)+\dot{t}^2(s) = 1$, satisfying the conditions

$$\begin{aligned} x(0) &= x_0, \\ y(0) &= y_0, \\ z(0) &= z_0, \\ t(0) &= t_0, \\ \dot{x}(0) &= u_0, \\ \dot{y}(0) &= v_0, \\ \dot{z}(0) &= \omega_0, \\ \dot{t}(0) &= \xi_0, \end{aligned} \tag{12}$$

with $x_0^2+y_0^2+z_0^2+t_0^2 = u_0^2+v_0^2+\omega_0^2+\xi_0^2 = 1$ and $x_0u_0+y_0v_0+z_0\omega_0+t_0\xi_0 = 0$.

When the Lorentz force vanished, the geodesics may be considered as particular magnetic trajectories. Hence, at first, we consider the classification of the geodesics in the $\mathbb{S}^3$.

Theorem 1. *The expression form of the geodesics in the manifold $\mathbb{S}^3$ is one of the following three cases:*

Case 1. $\gamma(s) = \gamma(s_0) = (x_0, y_0, z_0, t_0)$;

Case 2. $\gamma(s) = (x_0\cos s + \sqrt{u_0^2-x_0^2}\sin s, y_0\cos s + \sqrt{v_0^2-y_0^2}\sin s, z_0\cos s + \sqrt{\omega_0^2-z_0^2}\sin s, t_0)$.

Case 3

$$\begin{aligned} \gamma(s) = \Big(& m\sqrt{a^2-x_0^2}\sin\sqrt{1-\xi_0^2}s \\ & + x_0\cos\sqrt{1-\xi_0^2}s, m\sqrt{a^2-y_0^2}\sin\sqrt{1-\xi_0^2}s \\ & + y_0\cos\sqrt{1-\xi_0^2}s, m\sqrt{a^2-z_0^2}\sin\sqrt{1-\xi_0^2}s \\ & + z_0\cos\sqrt{1-\xi_0^2}s, \xi_0 s + t_0\Big); \end{aligned} \tag{13}$$

where $m = ((1-\xi_0^2)y_0^2+v_0^2)/(1-\xi_0^2)$.

Proof. By Gauss formula (7), where $X, Y \in \chi(\mathbb{S}^3)$ and $h : \mathbb{S}^3 \longrightarrow \mathbb{R}^4$ are the second fundamental form of $\mathbb{S}^3$ in $\mathbb{R}^4$, and $\pi(\{x, y, z, t\}) = \{x, y, z, 0\}$ is the projection in $\mathbb{S}^3$, we obtain

$$\ddot{\gamma} = \overline{\nabla}_{\dot{\gamma}}\dot{\gamma} - \langle \pi(\dot{\gamma}), \pi(\dot{\gamma})\rangle \pi(\gamma), \tag{14}$$

where π is the natural projection. Substituting expression (7) in (14), we can get

$$\ddot{\gamma} = \overline{\nabla}_{\dot{\gamma}}\dot{\gamma} - \left(1-\dot{t}^2\right)(x, y, z, 0), \tag{15}$$

and if the curve $\gamma(s)$ is the geodesics in $\mathbb{S}^3$, we know $\overline{\nabla}_{\dot{\gamma}}\dot{\gamma} = 0$ [11] and

$$\ddot{\gamma} = -\left(1-\dot{t}^2\right)(x, y, z, 0). \tag{16}$$

Equivalently,

$$\begin{aligned} \ddot{x} &= -\left(1-\dot{t}^2\right)x, \\ \ddot{y} &= -\left(1-\dot{t}^2\right)y, \\ \ddot{z} &= -\left(1-\dot{t}^2\right)z, \\ \ddot{t} &= 0. \end{aligned} \tag{17}$$

At the initial conditions, we solve the fourth equation of this system, and we can obtain $t(s) = \xi_0 s + t_0$, where $\xi_0 \in [-1, 1]$.

Let us consider the following cases:

$$\begin{aligned} \ddot{x} &= -\left(1-\xi_0^2\right)x, \\ \ddot{y} &= -\left(1-\xi_0^2\right)y, \\ \ddot{z} &= -\left(1-\xi_0^2\right)z, \\ \ddot{t} &= 0. \end{aligned} \tag{18}$$

Case 1. When $\xi_0 = \pm 1$, we can find $\ddot{x} = \ddot{y} = \ddot{z} = 0$, and

$$\begin{aligned} \gamma(s) &= (x, y, z, t) \\ &= (u_0 s + x_0, v_0 s + y_0, \omega_0 s + z_0, \xi_0 s + t_0). \end{aligned} \tag{19}$$

And $(u_0 s + x_0)^2 + (v_0 s + y_0)^2 + (\omega_0 s + z_0)^2 + (\xi_0 s + t_0)^2 = 1$. Also, there are two equal solutions $s_1 = s_2 = 0$, which stand for one fixed point γ_0 in $\mathbb{S}^3$.

Case 2. When $\xi_0 = 0$, we can find

$$\begin{aligned} \ddot{x} &= -x, \\ \ddot{y} &= -y, \\ \ddot{z} &= -z, \end{aligned} \tag{20}$$

and $t = t_0$, and solving the equations above, we obtain

$$\begin{aligned} x &= u_0 \sin\left(\pm s + \arcsin\frac{x_0}{u_0}\right), \\ y &= v_0 \sin\left(\pm s + \arcsin\frac{y_0}{v_0}\right), \\ z &= \omega_0 \sin\left(\pm s + \arcsin\frac{z_0}{\omega_0}\right), \end{aligned} \tag{21}$$

and if $x_0/u_0 = y_0/v_0 = z_0/\omega_0$, we can find the $\gamma(s)$ is the great circle in $\mathbb{S}^3$.

If $x_0/u_0 \neq y_0/v_0 \neq z_0/\omega_0$, $\gamma(s)$ is included in a plane passing through the origin and parameterized as

$$\begin{aligned} \gamma(s) = \Big(& x_0\cos s + \sqrt{u_0^2-x_0^2}\sin s, y_0\cos s \\ & + \sqrt{v_0^2-y_0^2}\sin s, z_0\cos s + \sqrt{\omega_0^2-z_0^2}\sin s, t_0\Big). \end{aligned} \tag{22}$$

Case 3. When $\xi_0 \in (-1, 0) \cup (0, 1)$, the general case, we can find the $t(s) = \xi_0 s + t_0$, and $\gamma(s)$ is a circle passing through the original point and parameterized by

$$x = m\sqrt{a^2-x_0^2}\sin\sqrt{1-\xi_0^2}s + x_0\cos\sqrt{1-\xi_0^2}s,$$

$$\begin{aligned} y &= m\sqrt{a^2 - y_0^2}\sin\sqrt{1-\xi_0^2}s + y_0\cos\sqrt{1-\xi_0^2}s, \\ z &= m\sqrt{a^2 - z_0^2}\sin\sqrt{1-\xi_0^2}s + z_0\cos\sqrt{1-\xi_0^2}s, \\ t &= \xi_0 s + t_0, \end{aligned} \tag{23}$$

where $m = ((1-\xi_0^2)y_0^2 + v_0^2)/(1-\xi_0^2)$, and

$$\begin{aligned} \gamma(s) = \Big(& m\sqrt{a^2 - x_0^2}\sin\sqrt{1-\xi_0^2}s \\ & + x_0\cos\sqrt{1-\xi_0^2}s, m\sqrt{a^2 - y_0^2}\sin\sqrt{1-\xi_0^2}s \\ & + y_0\cos\sqrt{1-\xi_0^2}s, m\sqrt{a^2 - z_0^2}\sin\sqrt{1-\xi_0^2}s \\ & + z_0\cos\sqrt{1-\xi_0^2}s, \xi_0 s + t_0\Big). \end{aligned} \tag{24}$$

We conclude the proof noticing that indeed the geodesics in $\mathbb{S}^3$ are parameterized by (20)-(23). □

Motivated by the fact that the equations of the geodesics are particular Lorentz equations, when the Lorentz force vanishes identically, the geodesics may be regarded as particular magnetic trajectories, the magnetic field in $\mathbb{S}^3$ is a closed 2-form F, and the Lorentz force corresponding to F is a (1,1)-type tensor field ϕ as (1).

A toy example for a Killing vector field is on the upper half plane $\mathbb{R}^2$, $y \geq 0$ equipped matric $g = y^{-2}(dx^2 + dy^2)$. The pair (M, g) is called the hyperbolic plane and has Killing vector field ∂_x. This should be clear since the covariant derivative $\nabla_{\partial_z} g$ transports the metric along an integral curve genenrated by the vector field. In this paper, we mention the basis for vector fields $V = a\partial_x + b\partial_y + c\partial_z$, for any $a, b, c \in \mathbb{R}$, and the Killing magnetic field determined by V is $F_V = adx \wedge dy + bdy \wedge dz + cdz \wedge dx$.

Theorem 2. *Let $\gamma(s)$ be a magnetic trajectory corresponding to the Killing vector field $V = a\partial_x + b\partial_y + c\partial_z$ in $\mathbb{S}^3$, and then $\gamma(s)$ is*

$$\begin{aligned} \gamma(s) = \gamma(0) + \left\langle \gamma'(0), V\right\rangle sV + (1+\cos s)V\times\gamma'(0) \\ + \sin s\boldsymbol{W}, \end{aligned} \tag{25}$$

where $\boldsymbol{W} = \gamma'(0) - \langle\gamma'(0), V\rangle V$.

Proof. In this proof, we can choose a special parameter s, satisfying $\langle V, V\rangle = 1$. Then, we know V and $\gamma'(0)$ are linearly independent, and we can construct a new vector $\boldsymbol{W} = \gamma'(0) - \langle\gamma'(0), V\rangle V$. And

$$\begin{aligned} \left\langle V\times\gamma'(0), V\times\gamma'(0)\right\rangle = 1, \\ \left\langle V, V\times\gamma'(0)\right\rangle = 0, \end{aligned} \tag{26}$$

and meanwhile,

$$\begin{aligned} &\langle V, \boldsymbol{W}\rangle = \left\langle V, \gamma'(0) - \left\langle\gamma'(0), V\right\rangle V\right\rangle = 0, \\ &\left\langle V\times\gamma'(0), \boldsymbol{W}\right\rangle \\ &\quad = \left\langle V\times\gamma'(0), \gamma'(0) - \left\langle\gamma'(0), V\right\rangle V\right\rangle = 0, \\ &\langle\boldsymbol{W}, \boldsymbol{W}\rangle \\ &\quad = \left\langle\gamma'(0), \gamma'(0)\right\rangle - 2\left\langle V, \gamma'(0)\right\rangle^2 - \left\langle V, \gamma'(0)\right\rangle^2 \\ &\qquad + \left\langle V, \gamma'(0)\right\rangle^2\left\langle\gamma'(0), \gamma'(0)\right\rangle. \end{aligned} \tag{27}$$

Case 1. When $\langle\gamma'(0), \gamma'(0)\rangle = \langle V, \gamma'(0)\rangle^2$, we know

$$\langle\boldsymbol{W}, \boldsymbol{W}\rangle = \left\langle V\times\gamma'(0), V\times\gamma'(0)\right\rangle = 0. \tag{28}$$

Hence, $\boldsymbol{W} = V\times\gamma'(0) = \boldsymbol{0}$, which is lineally dependent between V and $\gamma'(0)$.

Case 2. When $\langle\gamma'(0), \gamma'(0)\rangle \neq \langle V, \gamma'(0)\rangle^2$, $\langle V, \gamma'(0)\rangle^2 \neq 0$, we know $V, V\times\gamma'(0), \boldsymbol{W}$ are linearly independent, supposing $V, V\times\gamma'(0), \boldsymbol{W}\}$ is the frame in $\mathbb{S}^3$.Thus, in the following, we only consider the case V and $\gamma'(0)$ are lineally independent. Let the curve

$$\gamma(s) = \gamma(0) + A(s)V + B(s)V\times\gamma'(0) + C(s)\boldsymbol{W}, \tag{29}$$

where $A(s), B(s), C(s)$ are functions satisfying the original conditions $A(0) = B(0) = C(0) = 0$, $A'(0) = \langle\gamma'(0), V\rangle$, $B'(0) = 0, C'(0) = 1$.

Then, the Lorentz equation is equivalent to $\gamma''(s) = V\times\gamma'(s)$ which can be written as follows:

$$\begin{aligned} \gamma''(s) &= A''(s)V + B''(s)V\times\gamma'(0) + C''(s)\boldsymbol{W} \\ &= V\times\left(A'(s)V + B'(s)V\times\gamma'(0) + C'(s)\boldsymbol{W}\right) \\ &= B'(s)V\times\left(V\times\gamma'(0)\right) + C'(s)\boldsymbol{W} \\ &= C'(s)V\times\gamma'(0) \\ &\quad + B'(s)\left[-\gamma'(0) + \left\langle V, \gamma'(0)\right\rangle V\right] \\ &= C'(s)V\times\gamma'(0) - B'(s)\boldsymbol{W}. \end{aligned} \tag{30}$$

And we know

$$\begin{aligned} A''(s) &= 0, \\ B''(s) &= C'(s), \\ C''(s) &= -B'(s), \end{aligned} \tag{31}$$

and for the original conditions, we know $A(s) = \langle V, \gamma\prime(0)\rangle s$. By solving

$$\begin{aligned} B''(s) &= C'(s), \\ C''(s) &= -B'(s), \end{aligned} \tag{32}$$

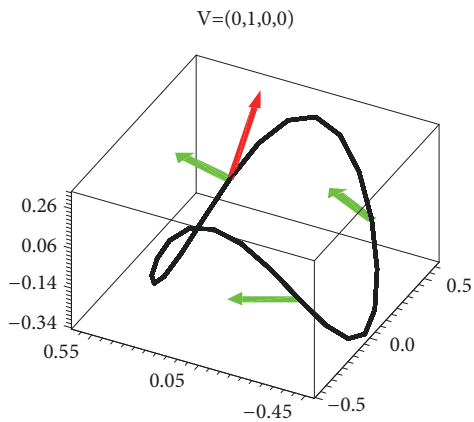

FIGURE 1: Red vector is $\gamma'(0) = \{1, -1, 0, -2\}$, and green vectors are magnetic fields $V = \{0, 1, 0, 0\}$.

we know

$$B(s) = \cos s + 1,$$
$$C(s) = \sin s. \tag{33}$$

Hence, we can get the equation of the curve

$$\gamma(s) = \gamma(0) + \langle \gamma'(0), V \rangle sV + (1 + \cos s) V \times \gamma'(0) + \sin s\boldsymbol{W}. \tag{34}$$

Remark. When $\langle \gamma'(0), \gamma'(0) \rangle \neq \langle V, \gamma'(0) \rangle^2$, $\langle V, \gamma'(0) \rangle^2 = 0$, the equation of the curve is

$$\gamma(s) = \gamma(0) + (1 + \cos s) V \times \gamma'(0) + \sin s\gamma'(0), \tag{35}$$

which is a special case of above equation. □

4. Some Examples

We will give some projected graphs to appear in the proof of the Theorem 2. For the dimension, we only give the projection of the curve and vectors to three dimensions.

Example 1. The expression of curve is

$$\gamma(s) = \left\{ \frac{\sqrt{3}}{3}(\cos s + \sin s), \frac{\sqrt{3}}{3}(\cos s - \sin s), \frac{\sqrt{3}}{3} \cdot \cos 2s, \frac{\sqrt{3}}{3} \sin 2s \right\}. \tag{36}$$

Case 1. When the magnetic fields are $V = \{0, 1, 0, 0\}$ and $V = (\sqrt{3}/3)\{1, 0, 1, 1\}$ satisfying

$$\langle \gamma'(0), \gamma'(0) \rangle \neq \langle V, \gamma'(0) \rangle^2 \neq 0, \tag{37}$$

the figures are Figures 1 and 2, respectively.

Case 2. When the magnetic field is $V = (1/2)\{1, 1, 1, 1\}$ satisfying $\langle \gamma'(0), \gamma'(0) \rangle = \langle V, \gamma'(0) \rangle^2$, the figure is Figure 3.

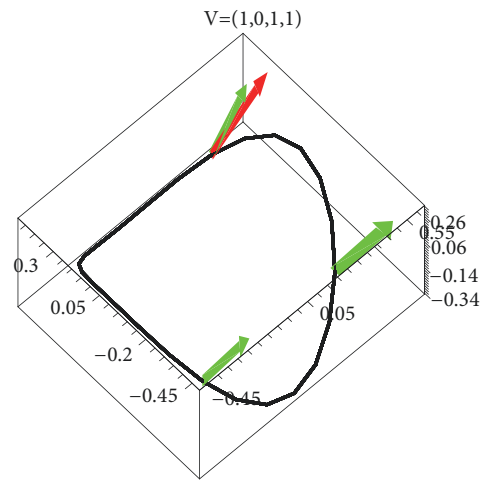

FIGURE 2: Red vector is $\gamma'(0) = \{1, -1, 0, -2\}$, and green vectors are magnetic fields $V = (\sqrt{3}/3)\{1, 0, 1, 1\}$.

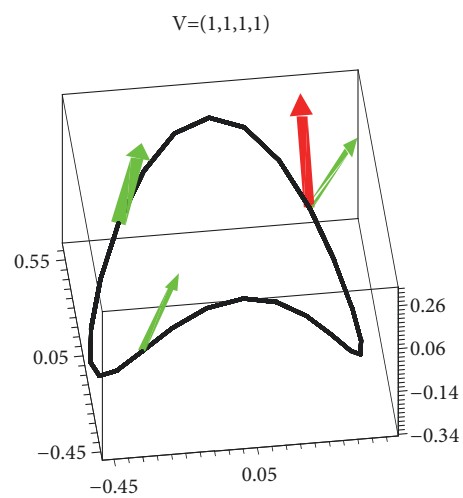

FIGURE 3: Red vector is $\gamma'(0) = \{1, -1, 0, -2\}$, and green vectors are magnetic fields $V = (1/2)\{1, 1, 1, 1\}$.

Example 2. The expression of curve is

$$\gamma(s) = \left\{ \frac{\sqrt{2}}{2} \sin 2s, \frac{\sqrt{2}}{2} \cos 2s, \frac{\sqrt{2}}{2} \sin s, \frac{\sqrt{2}}{2} \cos s \right\}. \tag{38}$$

Case 1. When the magnetic field is $V = (1/2)\{1, 1, 1, 1\}$ satisfying $\langle \gamma'(0), \gamma'(0) \rangle \neq \langle V, \gamma'(0) \rangle^2 \neq 0$, the figure is Figure 4.

Case 2. When the magnetic field is $V = (\sqrt{2}/2)\{0, 1, 0, 1\}$ satisfying $\langle \gamma'(0), \gamma'(0) \rangle \neq \langle V, \gamma'(0) \rangle^2 = 0$, the figure is Figure 5.

Case 3. When the magnetic field is $V = (\sqrt{42}/21)\{\sqrt{2}/2, 1, 2\sqrt{2}, 1\}$ satisfying $\langle \gamma'(0), \gamma'(0) \rangle = \langle V, \gamma'(0) \rangle^2$, the figure is Figure 6.

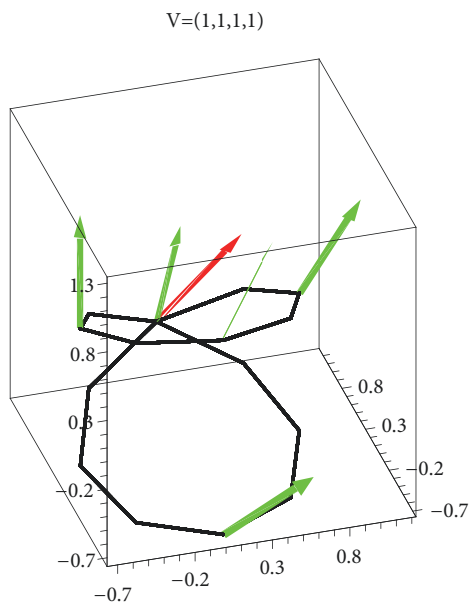

Figure 4: Red vector is $\gamma'(0) = \{2, 0, 1, 0\}$, and green vectors are magnetic fields $V = (1/2)\{1, 1, 1, 1\}$.

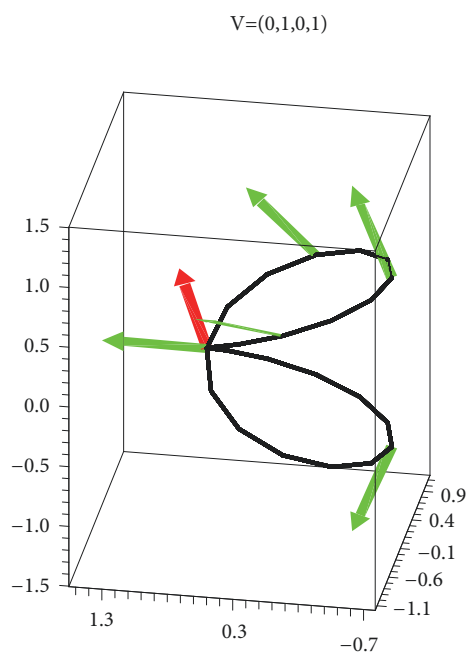

Figure 5: Red vector is $\gamma'(0) = \{2, 0, 1, 0\}$, and green vectors are magnetic fields $V = (\sqrt{2}/2)\{0, 1, 0, 1\}$.

5. A New Kind of Magnetic Curves in Three-Dimensional Sphere

As we all know, the Lorentz force is always perpendicular to both the velocity of the particle and the magnetic field that created it. When a charged particle moves in a static magnetic field, it traces a helical path in which the helix axis is parallel to the magnetic field. Also, if the charged particle moves parallel to magnetic field, the Lorentz force acts to be zero. Two vectors are perpendicular to the Lorentz force at the largest value. But we know that when a charged particle moves along a curve in a magnetic field V besides the velocity vector, the normal vector is also expressed according to the magnetic field V. Hence, the trajectories of the charged particle are changed. For example, when a charged particle moves in a static magnetic field in $\mathbb{R}^3$ and the normal vector is exposed to this field, it traces a slant helical path in which the slant-helix axis is parallel to the magnetic field V[14]. We give an example of the charged particle whose trajectories are N-magnetic curve in a magnetic field V.

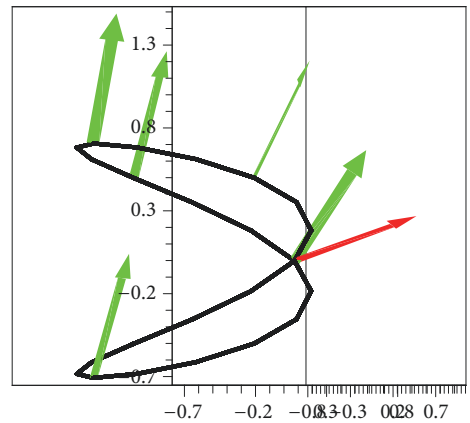

Figure 6: Red vector is $\gamma'(0) = \{2, 0, 1, 0\}$, and green vectors are magnetic fields $V = (\sqrt{42}/21)\{\sqrt{2}/2, 1, 2\sqrt{2}, 1\}$.

Definition 3. Let $\gamma : I \longrightarrow \mathbb{S}^3$ be a curve in $\mathbb{S}^3$ and V a magnetic field. We call the curve $\gamma(s)N$-magnetic curves if the normal vector field of the curve satisfies the Lorentz force equation; that is,

$$\overline{\nabla}_{\boldsymbol{T}}\boldsymbol{N} = \phi(\boldsymbol{N}) = V \times \boldsymbol{N}. \tag{39}$$

In three-dimensional Euclidean space $(\mathbb{R}^3, g)$ [8], a unit speed curve γ is a magnetic trajectory of a magnetic field V if and only if V can be written along γ as

$$V = \omega(s)\boldsymbol{T}(s) + \kappa(s)\boldsymbol{B}(s), \tag{40}$$

where the function $\omega(s)$ associated with each magnetic curve will be called its quasislope measured with respect to the magnetic field V, and $\boldsymbol{T}(s)$ is the tangent vector, and $\boldsymbol{B}(s)$ is the binormal vector of the curve [10, 14]. In this section, we give a new kind of magnetic curve called N-magnetic curve in three-dimensional sphere. Moreover, we obtain some characterizations and an example of this kind of curve.

Theorem 4. *Let $\gamma(s)$ be a unit speed N-magnetic curve in $\mathbb{S}^3$ with the Lorentz force in the Frenet frame being written as*

$$\begin{aligned} \phi(\boldsymbol{T}) &= \kappa(s)\boldsymbol{N}(s) \\ \phi(\boldsymbol{N}) &= -\kappa(s)\boldsymbol{T}(s) + \tau(s)\boldsymbol{B}(s) \\ \phi(\boldsymbol{B}) &= \omega_1(s)\boldsymbol{T}(s) - \omega_2(s)\boldsymbol{N}(s) \end{aligned} \tag{41}$$

and the magnetic field is

$$V = \tau(s)\boldsymbol{T} + \omega_1(s)\boldsymbol{N} - \kappa(s)\boldsymbol{B}, \tag{42}$$

where $\omega_1(s) = \langle V \times \boldsymbol{B}, \boldsymbol{T}\rangle$, $\omega_2(s) = \langle V \times \boldsymbol{B}, \boldsymbol{N}\rangle$.

Proof. Since $\gamma(s)$ is a unit speed N-magnetic curve on three-dimensional sphere and there exist three functions $\lambda_1(s), \eta_1(s), \zeta_1(s)$, we have $\phi(\boldsymbol{T}) = \lambda_1(s)\boldsymbol{T} + \eta_1(s)\boldsymbol{N} + \zeta_1(s)\boldsymbol{B}$ and $\lambda_1 = \langle \phi(\boldsymbol{T}), \boldsymbol{T}\rangle = 0$.

Meanwhile,

$$\eta_1(s) = \langle \phi(\boldsymbol{T}), \boldsymbol{N}\rangle = -\langle -\kappa(s)\boldsymbol{T}(s), \boldsymbol{T}(s)\rangle = \kappa(s), \tag{43}$$

and

$$\zeta_1(s) = \langle \phi(\boldsymbol{T}), \boldsymbol{B}\rangle = \langle V \times \boldsymbol{T}, \boldsymbol{B}\rangle = 0. \tag{44}$$

Hence, $\phi(\boldsymbol{T}) = \kappa(s)\boldsymbol{N}(s)$.

As the same methods, we suppose $\phi(\boldsymbol{N}) = \lambda_2(s)\boldsymbol{T} + \eta_2(s)\boldsymbol{N} + \zeta_2(s)\boldsymbol{B}$ and can obtain $\eta_2 = 0$,

$$\begin{aligned} \lambda_2(s) &= \langle \phi(\boldsymbol{N}), \boldsymbol{T}\rangle = \langle V \times \boldsymbol{N}, \boldsymbol{T}\rangle = -\kappa(s), \\ \zeta_2(s) &= \tau(s). \end{aligned} \tag{45}$$

Hence, $\phi(\boldsymbol{N}) = -\kappa(s)\boldsymbol{T}(s) + \tau(s)\boldsymbol{B}(s)$.

We suppose $\phi(\boldsymbol{B}) = \lambda_3(s)\boldsymbol{T} + \eta_3(s)\boldsymbol{N} + \zeta_3(s)\boldsymbol{B}$ and can obtain $\zeta_3 = 0$,

$$\begin{aligned} \lambda_3(s) &= \langle \phi(\boldsymbol{B}), \boldsymbol{T}\rangle = \langle V \times \boldsymbol{B}, \boldsymbol{T}\rangle = \omega_1(s), \\ \zeta_3(s) &= \langle \phi(\boldsymbol{B}), \boldsymbol{N}\rangle = \langle V \times \boldsymbol{B}, \boldsymbol{N}\rangle = \omega_2(s). \end{aligned} \tag{46}$$

And we can obtain

$$\phi(\boldsymbol{B}) = \omega_1\boldsymbol{T}(s) + \omega_2\boldsymbol{N}(s). \tag{47}$$

Let $\gamma(s)$ be a unit speed N-magnetic curve, and we suppose the magnetic field $V = \lambda\boldsymbol{T} + \eta\boldsymbol{N} + \zeta\boldsymbol{B}$, by the definition of the N-magnetic curve, $\phi(\boldsymbol{T}) = \kappa(s)\boldsymbol{N}(s) = V \times \boldsymbol{T}$. We know $\zeta = -\kappa(s)$, the same as $\lambda = \tau(s)$, $\eta = \omega_1$. Hence,

$$V = \tau(s)\boldsymbol{T} + \omega_1(s)\boldsymbol{N} - \kappa(s)\boldsymbol{B}. \tag{48}$$

□

Example 3. Let curve $\gamma(s)$ be a N-magnetic curve as

$$\begin{aligned} &\gamma(s) \\ &\quad = \{\cos s\cos 2s, \cos s\sin 2s, \sin s\cos 2s, \sin s\sin 3s\}, \end{aligned} \tag{49}$$

and the Frenet frames $\{\boldsymbol{T}(s), \boldsymbol{B}(s), \boldsymbol{N}(s), \kappa(s), \tau(s)\}$ as follows:

$$\begin{aligned} \boldsymbol{T}(s) &= \{-\cos 2s\sin s - 2\cos s\sin 2s, -\sin s\sin 2s \\ &\quad + 2\cos s\cos 2s, -\cos s\cos 3s \\ &\quad - 3\sin s\cos 3s, \cos s\sin 3s + 3\cos 3s\sin s\}, \\ \boldsymbol{N}(s) &= \{5\cos s\sin 2s + 4\cos 2s\sin s, -5\cos s\cos 2s, \\ &\quad - 10\sin s\cos 3s, 6\cos s\cos 3s - 10\sin s\sin 3s\}, \\ \boldsymbol{B}(s) &= \{17\cos s\cos 3s + \cos 2s\sin s, 8\cos 3s\sin s, \\ &\quad - 15\sin 2s\cos s + 3\cos s\sin 2s, 15\cos 2s\cos s\}, \end{aligned} \tag{50}$$

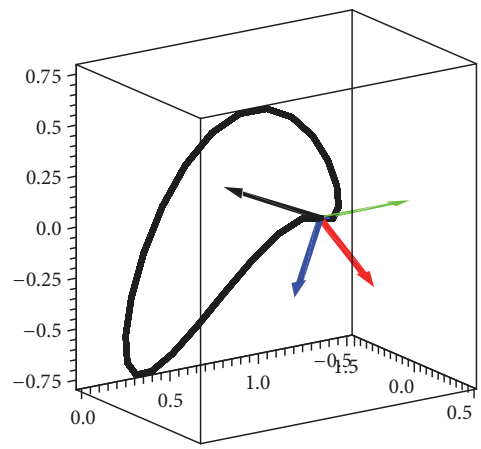

Figure 7: Red vector is $\boldsymbol{T}(0)$, blue vector is $\boldsymbol{N}(0)$, green vector is $\boldsymbol{B}(0)$, and black vector is V.

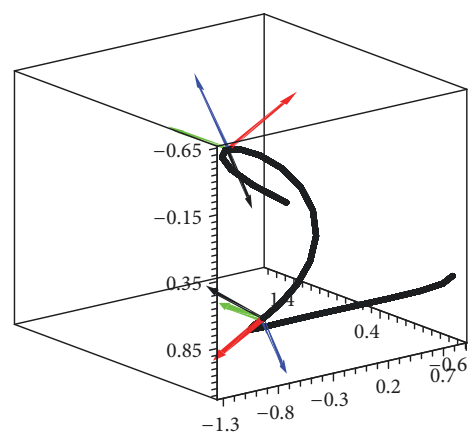

Figure 8: Red vector is $\boldsymbol{T}$, blue vector is $\boldsymbol{N}$, green vector is $\boldsymbol{B}$, and black vector is V.

and $\kappa(s) = \sin s + \cos s$, $\tau(s) = \cos s(\sin 2s\cos s - \cos 3s)$. When $\omega_1(s) = 1$, at the original point, we can draw the projection figure of the curve $\gamma(s)$ and the vector V in Figure 7. When $\omega_1(s) = 0$, at the original and $s = \pi/3$ points, we can draw the projection figure of the curve $\gamma(s)$ and the vector V in Figure 8.

Data Availability

The data supporting the conclusions of this manuscript are some open-access articles that have been properly cited, and the readers can easily obtain these articles to verify the conclusions, replicate the analysis, and conduct secondary analysis. Therefore, a publicly available data repository was not created.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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