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Research Article

On a New Characterization of Some Class Nonlinear Eigenvalue Problem

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A normal mode analysis of a vibrating mechanical or electrical system gives rise to an eigenvalue problem. Faber made a fairly complete study of the existence and asymptotic behavior of eigenvalues and eigenfunctions, Green's function, and expansion properties. We will investigate a new characterization of some class nonlinear eigenvalue problem.

1. Introduction

In this paper, we derive a new boundedness and compactness result for the Hardy operator in variable exponent Lebesgue spaces (VELS) $L^{p(.)}(0,l)$. A maximally weak condition is assumed on the exponent function. The last time such a study was carried out was in [1–12]. For a study of the Dirichlet problem of some class nonlinear eigenvalue problem with nonstandard growth condition the obtained results are applied. Such equations arise in the studies of the so-called Winslow effect physical phenomena [13] in the smart materials. In this connection, we mention recent studies for the multidimensional cases with application of Ambrosetti-Rabinowitz's Mountain Pass theorem approaches (see, e.g., [1, 14, 15]).

Theorem A. *Let $q,p:(0,l)\longrightarrow(1,\infty)$ be measurable functions with $q(x)\geq p(x)$ on $(0,l)$. Assume p is monotonically increasing and the function $x^{-1/p'(x)+\delta}$ is almost decreasing on $(0,l)$.. Then operator H boundedly acts the space $L^{p}(0,l)$ into $L^{q(.),-1/p'-1/q(.)}(0,l)$. Moreover, the norm of mapping depends on p^-,p^+,δ,β.*

Theorem B. *Let $q,p:(0,l)\longrightarrow(1,\infty)$ be measurable functions such that $\infty>q^+\geq q(x)\geq p(x)\geq p^->1$ for all $x\in(0,l)$. Assume that p is monotonically increasing and $x^{-1/p'+\varepsilon}$ is almost decreasing. Then the identity operator maps boundedly the space $W^{1.}_{p(.)}(0,l)$ into $L^{q(.),-1/p'-1/q(.)}(0,l)$. Moreover, the norm of mapping is estimated by a constant depending on $p^-,p^+,q,\varepsilon,\beta$.*

Notice that Theorem B states the inequality

$$\left\|yx^{-1/p'-1/q(.)}\right\|_{L^{q(.)}(0,l)}\leq\left\|y'\right\|_{L^{p(.)}(0,l)}\tag{1}$$

for any absolutely continuous function $y:(0,l)\longrightarrow\mathbb{R}$ with $y(0)=0$.

In the given assertions, $L^{p,\alpha}(0,l)$ denotes the space of measurable functions with finite norm $\|yx^{\alpha}\|_{L^{p(.)}(0,l)}$, while $W^{1.}_{p(.),\alpha}(0,l)$ stands for the space of absolutely continuous functions y with $y(0)=0$ and finite norm

$$\|y\|_{W^{1.}_{p(.)}}=\left\|y'\right\|_{L^{p(.)}}.\tag{2}$$

We say that the function $\alpha:(0,l)\longrightarrow(0,\infty)$ is almost increasing (decreasing) if there exists a constant $C>0$ such that for any $0<t_1<t_2<l$ it holds $\alpha(t_1)\leq C\alpha(t_2)$ $(\alpha(t_1)\geq C\alpha(t_2))$.

We need the following assertion.

Lemma 1. *Let $p(x)$ be increasing for $x\in(0,l)$. Let $t\in A_n(x)=(2^{-n-1}x,2^{-n}x]$, where n is the natural number. Then it holds*

$$t^{-1/p'(t)} \le C t^{-1/(p^-_{x,n})'}, \tag{3}$$

where $p_{x,n} = \inf_{t \in A_n(x)} p(t)$.

We will be inspired by [6–8] while proving Lemma 1.

Proof. Let $y \in A_n(x)$ be a point with $t^{-1/p'(y)} \le 2t^{-1/(p^-_{x,n})'}$. Let $y < t$ and both lie in $A_n(x)$. Then using the almost decrease of $x^{-1/p'+\varepsilon}$ it follows that

$$t^{-1/p'(t)+\varepsilon} \le c y^{-1/p'(y)+\varepsilon} \tag{4}$$

Using $t, y \in A_n(x)$, $(p^-_{x,n})' > 1$ it follows that

$$t^{-1/p'(t)} \le 2^{+\varepsilon} C y^{-1/p'(y)} \le 2^{+2+\varepsilon} C t^{-1/(p^-_{x,n})'} \tag{5}$$

Now let $y > t$; then, using the increase of p, $1/p'$ also will be increasing. Since $1/p'(t) < 1/p'(y)$, it follows that

$$\left(\frac{1}{t}\right)^{1/p'(t)} \le C \left(\frac{1}{t}\right)^{1/p'(y)} \le 2Ct^{-1/(p^-_{x,n})'}, \tag{6}$$

where $c = l^{1/p^{-\prime}} + l^{1/p^{+\prime}}$.

Lemma 1 has been proved. □

In the light of the information given above, we can give proof of Theorem A.

Proof of Theorem A. Let $f : (0, l) \longrightarrow (0, \infty)$ be a positive measurable function. It holds the identity

$$Hf(x) = \sum_{n=1}^{\infty} \int_{2^{-n-1}x}^{2^{-n}x} f(t)\,dt \tag{7}$$

Assume $\|f\|_p = 1$. Using the triangle property of $p(.)-$ norms

$$\|x^\alpha Hf\|_{q(.)} \le \sum_{n=1}^{\infty} \left\| x^\alpha \int_{A_n(x)} f(t)\,dt \right\|_{q(.)}, \tag{8}$$

with $\alpha(x) = -1/p'(x) - 1/q(x)$ (recall $A_n(x) = (2^{-n-1}x, 2^{-n}x]$).

Derive estimation for every summand in (8). In this purpose get estimation for the proper modular

$$\begin{aligned} &I_{q(.)}\left(x^{\alpha(.)} \int_{A_n(x)} f(t)\,dt\right) \\ &\quad := \int_0^l \left(x^{\alpha(.)} \int_{A_n(x)} f(t)\,dt\right)^{q(x)} dx. \end{aligned} \tag{9}$$

Applying the assumption on p (decreasing of $x^{-1/p'+\varepsilon}$), and using the expression for $q(x) = 1/(-\alpha - 1/p'(x))$, we have

$$\begin{aligned} &I_q\left(x^{-1/p'-1/q} \int_{A_n(x)} f(t)\,dt\right) \\ &\quad = \int_0^l \left(x^{-1/p'+\varepsilon} \int_{A_n(x)} f(t)\,dt\right)^{q(x)} \frac{dx}{x^{1+\varepsilon q(x)}} \\ &\quad \le C^{q^+} 2^{-n\varepsilon q^-} \cdot \\ &\quad \cdot \int_0^l \frac{dx}{x} \left(\int_{A_n(x)} f(t)\, t^{-1/p'(t)} dt\right)^{q(x)} \end{aligned} \tag{10}$$

Notice that we have used $x^{-1/p'(x)+\varepsilon} \le ct^{-1/p'(t)+\varepsilon}$ for any $0 < x < l$ and $2^{-n-1}x < t \le 2^{-n}x$ by using the almost decrease of $x^{-1/p'(x)+\varepsilon}$.

Therefore, from (8), using Hölder's inequality, it follows that

$$\begin{aligned} &I_q\left(x^{\alpha(.)} \int_{A_n(x)} f(t)\,dt\right) \le C^{q^+} 2^{-n\varepsilon q^-} \int_0^l \frac{dx}{x} \\ &\quad \cdot \left(\int_{A_n(x)} (f(t))^{p^-_{x,n}} dt\right)^{q(x)/p^-_{x,n}} \\ &\quad \cdot \left(\int_{A_n(x)} t^{-(p^-_{x,n})'/p'(t)} dt\right)^{q(x)/(p^-_{x,n})} \end{aligned} \tag{11}$$

Applying Lemma 1 and estimate (3) it follows from (11) that

$$\begin{aligned} &I_q\left(x^{\alpha(.)} \int_{A_{n(x)}} f(t)\,dt\right) \le (c \ln 2)^{q^+/p^-} \\ &\quad \cdot 2^{-n\varepsilon q^-} C^{q^+} \int_0^l \frac{dx}{x} \left(\int_{A_n(x)} (f(t))^{p^-_{x,n}} dt\right)^{q(x)/p^-_{x,n}} \end{aligned} \tag{12}$$

Since

$$\begin{aligned} \int_{A_n(x)} (f(t))^{p^-_{x,n}} dt &\le \int_{A_n(x)} (f(t))^{p(t)} dt + \int_{A_n(x)} dt \\ &\le 1 + 2^{-n}x \le 1 + 2^{-n}l \le l + 1, \end{aligned} \tag{13}$$

It follows that

$$\begin{aligned} &I_q\left(x^\alpha \int_{A_n(x)} f(t)\,dt\right) \le (c \ln 2)^{q^+} 2^{-n\varepsilon q^-} \int_0^l \frac{dx}{x} \\ &\quad \cdot \left(\frac{1}{l+1} \int_{A_n(x)} (f(t))^{p^-_{x,n}} dt\right)^{q(x)/p^-_{x,n}} (l+1)^{q^+} \\ &\quad \le (c \ln 2\,(l+1))^{q^+} \\ &\quad \cdot \int_0^l \left(\frac{1}{l+1} \int_{A_n(x)} \left[\left(f(t)^{p(t)} + 1\right)\right] dt\right)^{p(x)/p^-_{x,n}} \frac{dx}{x} \\ &\quad \le 2^{-n\varepsilon q^-} C^{q^+} (c \ln 2)^{q^+} (l+1)^{q^+-1} \int_0^l \frac{dx}{x} \\ &\quad \cdot \left(\int_{A_n(x)} \left[\left(f(t)^{p(t)} + 1\right)\right] dt\right). \end{aligned} \tag{14}$$

Hence,

$$\begin{aligned} I_q&\left(x^{\alpha(.)}\int_{A_{n(x)}} f(t)\,dt\right)\\ &\le c_3 2^{-n\varepsilon q^-} C^{q^+}\int_0^l\left(\int_{A_n(x)}\left[\left(f(t)\right)^{p(t)}+1\right]dt\right)\frac{dx}{x}\\ &\le c_3\int_0^{2^{-n}l}\left[\left(f(t)^{p(t)}+1\right)\right]\int_{2^n t}^{2^{n+1}t}\frac{dx}{x}\\ &= C^{q^+}c_3 2^{-n\varepsilon q^-}\ln 2\int_0^{2^{-n}l}\left[\left(f(t)^{p(t)}+1\right)\right]dt\\ &\le c^{q^+}c_3 2^{-n\varepsilon q^-}\ln 2\left(1+2^{-n}l\right)=c_4 2^{-n\varepsilon q^-}. \end{aligned} \tag{15}$$

Therefore, it has been proved that

$$I_q\left(x^{-1/p'-1/q}\int_{A_{n(x)}} f(t)\,dt\right)\le c_4 2^{-n\varepsilon q^-}, \tag{16}$$

which implies

$$\left\|x^{-1/p'-1/q}\int_{A_{n(x)}} f(t)\,dt\right\|_{q(.);(0,l)}\le c_4^{1/q^+}2^{-n\varepsilon(q^-/q^+)} \tag{17}$$

Inserting (17) into (8), we get

$$\left\|x^{-1/p'-1/q}Hf\right\|_{q(.);(0,l)}\le c_4^{1/q^+}\sum_{n=1}^{\infty}2^{-n\varepsilon(q^-/q^+)}=c_5 \tag{18}$$

Theorem A has been proved. □

Theorem C. *Let $q,p\ :\ (0,l)\longrightarrow(1,\infty)$ be measurable functions such that $\infty>q^+\ge q(x)\ge q^->p^+\ge p(x)\ge p^->1$.*

Assume that the function p increases on $(0,l)$ and $x^{-1/p'+\varepsilon}$ is almost decreasing in $(0,l)$. Then operator H acts compactly the space $L^{p,\beta}(0,l)$ into $L^{q,-1/p'-(1-\delta)/q}(0,l)$ for any $\delta\in(0,1)$.

Proof. In order to proof Theorem C, we may apply the approaches from [3–5]. In this way, insert the operators

$$\begin{aligned} P_1 f(x)&=X_{(0,a)}(x)\,x^{-1/p'-(1-\delta)/q}\int_0^x f(t)\,dt;\\ P_2 f(x)&=X_{(a,l)}(x)\,x^{-1/p'-(1-\delta)/q}\int_0^a f(t)\,dt;\\ P_3 f(x)&=X_{(a,l)}(x)\,x^{-1/p'-(1-\delta)/q}\int_a^x f(t)\,dt; \end{aligned} \tag{19}$$

As it was stated in [3], P_3 is a limit of finite rank operators, while P_2 is a finite rank operator. From the condition $\lim_{t\longrightarrow 0}B(t)=0$ it follows that

$$\begin{aligned} \left\|Hf-P_2 f-P_3 f\right\|_{L^{q(.)}(0,l)}&\le\left\|P_1 f\right\|_{L^{q(.)}(0,l)}\\ &\le ca^{\delta/p^+}\left\|f\right\|_p \end{aligned} \tag{20}$$

or

$$\begin{aligned} \left\|H-P_2-P_3\right\|_{L^p\longrightarrow L^{q,-1/p'-(1-\delta)/q}}&\le\left\|P_1\right\|_{L^{p,\beta}\longrightarrow L^{q,-1/p'-(1-\delta)/q}}\\ &\le ca^{\delta/p}\longrightarrow 0 \end{aligned} \tag{8}$$

as $a\longrightarrow 0$. To show the last estimation we shall use the arguments of Theorem A. Repeating all constructions there, we get the following estimates:

$$\begin{aligned} I_q&\left(x^{-1/p'-(1-\delta)/q}\int_{A_{n(x)}} f(t)\,dt\right)\\ &=\int_0^l\frac{dx}{x^{1-\delta+\varepsilon q}}\left(x^{-1/p'+\varepsilon}\int_{A_{n(x)}} f(t)\,dt\right)^{q(x)}\\ &\le C^{q^+}2^{-n\varepsilon q^-}\int_0^l\frac{dx}{x^{1-\delta}}\left(\int_{A_{n(x)}} t^{-1/p'(t)}f(t)\,dt\right)^{q(x)}. \end{aligned} \tag{21}$$

Notice that we have used $x^{-1/p'+\varepsilon}\le ct^{-1/p'(t)+\varepsilon}$ for any $t\in A_n(x)$, where n belongs to the natural number.

Therefore, using Hölder's inequality,

$$\begin{aligned} I_q&\left(x^{-1/p'-(1-\delta)/q}\int_{A_{n(x)}} f(t)\,dt\right)\le C^{q^+}2^{-n\varepsilon q^-}\int_0^l\frac{dx}{x^{1-\delta}}\\ &\cdot\left(\int_{A_{n(x)}}\left(f(t)\right)^{p^-_{x,n}}dt\right)^{q(x)/p^-_{x,n}}\\ &\cdot\left(\int_{A_{n(x)}} t^{-(p^-_{x,n})'/p'(t)}dt\right)^{q(x)/(p^-_{x,n})'}. \end{aligned} \tag{22}$$

Applying Lemma 1 and the arguments above, we attain the estimates

$$\begin{aligned} I_q&\left(x^{-1/p'-(1-\delta)/q}\int_{A_{n(x)}} f(t)\,dt\right)\le C^{q^+}2^{-n\varepsilon q^-}(c\\ &\cdot\ln 2)^{q^+}(l+1)^{q^+-1}\\ &\cdot\int_0^l\frac{dx}{x^{1-\delta}}\left(\int_{A_n(x)}\left[\left(f(t)\right)^{p(t)}+1\right]dt\right)\\ &\le C_3 2^{-n\varepsilon q^-}\int_0^{-2^{-n}}\left[\left(f(t)\right)^{p(t)}+1\right]\\ &\cdot\left(\int_{2^n t}^{2^{n+1}t}\frac{dx}{x^{1-\delta}}\right)dt\le 2^{-n\varepsilon q^-}\left(1+2^{-n}l\right)C_3 C^{q^+}l^{\delta} \end{aligned} \tag{23}$$

Therefore, it has been shown that

$$I_q\left(x^{-1/p'-(1-\delta)/q}\int_{A_{n(x)}} f(t)\,dt\right)\le cl^{\delta}2^{-n\varepsilon q^-} \tag{24}$$

if $\|f\|_p\le 1$. This implies

$$\begin{aligned} &\left\|x^{-1/p'-(1-\delta)/q}\int_{A_{n(x)}} f(t)\,dt\right\|_{q(.);(0,l)}\\ &\quad\le C^{1/q^+}l^{\delta/q^+}2^{-n\varepsilon q^-/q^+} \end{aligned} \tag{25}$$

Inserting these estimates over $n = 1, 2, \ldots$ in the expression

$$\left\| x^{-1/p'-(1-\delta)/q} Hf \right\|_{q(.);(0,l)} \leq c^{1/q^+} l^{\delta/q^+} \sum_{n=1}^{\infty} 2^{-n\varepsilon q^-/q^+} = c_5 l^{\delta/q^+} \tag{26}$$

The last estimate is a needed estimation which completes the proof of Theorem C.

Consider the problem

$$\frac{d}{dx}\left(\left|\frac{dy}{dx}\right|^{p(x)-2}\frac{dy}{dx}\right) = \lambda y^{q(x)-1}\left(lx - x^2\right)^{q(-1/p'-(1-\delta)/q)}$$
$$y(0) = y(l) = 0, \tag{27}$$
$$y(x) > 0$$

where $\delta \in (0,1)$, $x \in (0,l)$. □

Theorem D. *Let $1 < p^- < p(x) \leq p^+ < q^- \leq q(x) \leq q^+ < \infty$ be measurable functions on $(0,l)$ such that $p(x)$ is increasing and the function $x^{-1/p'+\varepsilon}$ is almost decreasing on $(0,l)$. Then for any fixed $\lambda > 0$ there exists a nontrivial solution of the problem (27).*

Proof. To prove this assertion, we shall use the well-known Mountain Pass theorem approaches. Define the functional

$$I_\lambda(y) = \int_0^l \frac{1}{p(x)}\left|y'\right|^{p(x)} dx - \lambda \int_0^l \frac{1}{q(x)}\left[\left(xl - x^2\right)^{-1/p'-(1-\delta)/q} y_+\right]^{q(x)} dx \tag{28}$$

Define the space $E = \overline{W}_p^1(0,l)$ a closure of absolutely continuous functions on $(0,l)$, such that $y(0) = y(l) = 0$ with a norm

$$\|y\|_{\overline{W}_p^1(0,l)} = \left\|\frac{dy}{dx}\right\|_{p(.);(0,l)}, \tag{29}$$

Define also the space $\overline{L}^{p,\beta}(0,l)$ as a space of measurable functions with finite norm

$$\|f\|_{\overline{L}^p(0,l)} = \inf\left\{\lambda > 0 : \int_0^l \left|\frac{f}{\lambda}\right|^{p(x)} dx \leq 1\right\}. \tag{30}$$

Applying the standard approaches (see, e.g., [13]), it is not difficult to see that the functional $I_\lambda \in C^1(E,R)$. Further, $I'_\lambda \in E^*$ and

$$\left\langle I'_\lambda(y), v\right\rangle = \int_0^l \left|y'\right|^{p(x)} - \lambda \int_0^l \left(xl - x^2\right)^{(-1/p'-(1-\delta)/q)q(x)} y_+^{q(x)} dx \tag{31}$$

By applying Theorem B, we get the implication $\|f\|_E = 0 \longrightarrow f = 0$. Show that Palaisce-Smale (PS) condition is satisfied for the problem (27). Let $\{y_n\} \in E$ be a sequence such that it is satisfied by the following two conditions:

(1) $|I_\lambda(y_n)| \leq M$;
(2) $\|I'(y_n)\|_{E^*} \longrightarrow 0$ as $n \longrightarrow \infty$.

To prove the PS condition, we must prove that such a sequence is compact; that is, it contains a subsequence $\{y_{n_k}\}$ converging in E to a function $\in E$. In order to show it, establish the boundedness of $\{y_n\}$.

From (1) it follows that

$$\int_0^l \frac{\left|y'_n\right|^{p(x)}}{p(x)} dx - \lambda \int_0^l \frac{\lambda}{q(x)}\left[\left(xl - x^2\right)^{-1/p'-(1-\delta)/q} (y_n)_+\right]^{q(x)} dx \leq M \tag{32}$$

Then

$$\frac{1}{p^+}\int_0^l \left|y'_n\right|^{p(x)} dx \leq \frac{1}{q^-}\int_0^l \left(\left(xl - x^2\right)^{-1/p'-(1-\delta)/q} (y_n)_+\right)^{q(x)} dx + M \tag{33}$$

Using condition (2) $|I'_\lambda(y_n)| = 0(1)$ for $n \longrightarrow \infty$, it follows that $\langle I'_\lambda(y_n), y_n\rangle = O(1)\|y_n\|$; that is,

$$\lambda \int_0^l \left(\left(xl - x^2\right)^{-1/p'-(1-\delta)/q} (y_n)_+\right)^{q(x)} dx = O(1)\|y_n\| + \int_0^l \left|y'_n\right|^{p(x)} dx \tag{34}$$

Inserting this into (27), it follows that

$$\left(\frac{1}{p^+} - \frac{1}{q^-}\right)\int_0^l \left|y'_n\right|^{p(x)} dx \leq M + O(1)\|y_n\|. \tag{35}$$

From this, since $q^- > p^+$, it follows that

$$\int_0^l \left|y'_n\right|^{p(x)} dx \leq \frac{2Mp^+q^-}{q^- - p^+} + 0(1)\|y_n\| \tag{36}$$

or

$$\|y_n\|^{p^-} \leq O(1) + O(1)\|y_n\|. \tag{37}$$

Using Young's inequality from here it follows that

$$\left\|y'_n\right\|_{p(.)} \leq C(M). \tag{38}$$

This completes the boundedness of $\{y_n\}$ in E. Applying well-known fact, there exists a weak convergent subsequence

$y_{n_k} \longrightarrow y$ in E. Denote it again as y_n•It follows from the compact embedding

Theorem C that a strong convergence $y_n \longrightarrow y$ in $L^{q,-1/p'-(1-\delta)/q}(0,l)$ holds; that is,

$$\|y_n - y\|_{L^{q,-1/p'-(1-\delta)/q}(0,l)} \longrightarrow 0 \tag{39}$$

Now, we are ready to show the strong convergence $y_n \longrightarrow y$ in E. For this purpose, insert $v = y_n - y$ into the equality

$$\left\langle I'_\lambda (y_n), v\right\rangle = O(1)\|v\|. \tag{40}$$

Then

$$\begin{aligned}&\int_0^l |y'_n|^{p(x)-2} y'_n (y'_n - y') \\ &\quad - \lambda \int_0^l (xl - x^2)^{(-1/p'-(1-\delta)/q)q(x)} (y_n)_+^{q(x)-1} (y_n - y) \\ &\quad = O(1)\|y_n - y\|.\end{aligned} \tag{41}$$

From this, since $y_n \longrightarrow y$ in $L^{q,-1/p'-(1-\delta)/q}(0,l)$ and using Holder's inequality, it follows that

$$\begin{aligned}&\left|\int_0^l (lx - x^2)^{(-1/p'-(1-\delta)/q)q(x)} (y_n)_+^{q(x)-1} (y_n - y)\right| \\ &\quad \le \|y_n - y\|_{L^{q,-1/p'-(1-\delta)/q}(0,l)} \\ &\quad \cdot \left\|\left((y_n)_+ (lx - x^2)^{-1/p'-(1-\delta)/q}\right)^{q(x)-1}\right\|_{L^{q'}} = O(1) \\ &\quad \cdot \left\|y_n (lx - x^2)^{-1/p'-(1-\delta)/q}\right\|_q^{q^+-1} \longrightarrow 0,\end{aligned} \tag{42}$$

since $\{y_n\}$ is bounded in $L^{q,-1/p'-(1-\delta)/q}(0,l)$.

Therefore,

$$\int_0^l |y'_n|^{p(x)-2} y'_n (y'_n - y') = O(1) + O(1)\|y_n - y\|. \tag{43}$$

From this we infer that

$$\begin{aligned}&\int_0^l \left(|y'_n|^{p(x)-2} y'_n - |y'|^{p(x)-2} y'\right)(y'_n - y')\,dx \\ &\quad + \int_0^l |y'|^{p(x)-2} y' (y'_n - y')\,dx \\ &\quad = 0(1) + 0(1)\|y_n - y\|\end{aligned} \tag{44}$$

Since $y_n \longrightarrow y$ weakly in E it holds that

$$\int_0^l |y'|^{p(x)-2} y' (y'_n - y')\,dx \longrightarrow 0. \tag{45}$$

This ensures that

$$\begin{aligned}&\int_0^l \left(|y'_n|^{p(x)-2} y'_n - |y'|^{p(x)-2} y'\right)(y'_n - y')\,dx \\ &\quad = 0(1) + 0(1)\|y_n - y\|.\end{aligned} \tag{46}$$

We will apply the following two inequalities:

$$\begin{aligned}&\left(|y'_n|^{p(x)-2} y'_n - |y'|^{p(x)-2} y', y'_n - y'\right) \\ &\quad \ge \gamma_1(p)|y'_n - y'|^{p(x)}\end{aligned} \tag{47}$$

for $p(x) \ge 2$ and

$$\begin{aligned}&\left(|y'_n|^{p(x)-2} y'_n - |y'|^{p(x)-2} y', y'_n - y'\right) \\ &\quad \ge \gamma_2(p) \frac{|y'_n - y'|^2}{|y'_n|^{2-p} + |y'|^{2-p}}\end{aligned} \tag{48}$$

for $1 < p(x) < 2$. Applying (47), for the case $p(x) \ge 2$, we get

$$\int_0^l |y'_n - y'|^{p(x)}\,dx = O(1) + O(1)\|y_n - y\|. \tag{49}$$

As to the case $1 < p(x) < 2$, we have the inequality

$$\int_0^l \frac{(y'_n - y')^2}{|y'_n|^{2-p} + |y'|^{2-p}}dx \le 0(1) + 0(1)\|y_n - y\| \tag{50}$$

Using Young's inequality from here it follows that

$$\begin{aligned}\int_0^l |y'_n - y'|^p\,dx &\le \varepsilon \int_0^l (|y'_n| + |y'|)^{p(x)} x^{\beta p} dx \\ &\quad + C(\varepsilon) \int_0^l \frac{|y'_n - y'|^2}{(|y'_n| + |y'|)^{2-p}}dx \\ &\le M\varepsilon \\ &\quad + C(\varepsilon)(0(1) + 0(1)\|y_n - y\|), \\ &\qquad \forall \varepsilon > 0.\end{aligned} \tag{51}$$

Therefore,

$$\|y_n - y\|^{p^-} \le M\varepsilon + c(\varepsilon)(0(1) + 0(1)\|y_n - y\|) \tag{52}$$

where M does not depend on $n \in N$. This and the above inequality and Young's inequality give

$$\|y_n - y\|^{p^-} \le M\varepsilon + 0(1) \tag{53}$$

This inequality yields $y_n \longrightarrow y$ in E.

Now, we are ready to apply the Mountain Pass theorem. If $\|y_n - y\| \nrightarrow 0$ from preceding equality one gets

$$\|(y'_n - y')\|_{p(\bullet)}^{p^-} = O(1) + O(1)\|y'_n - y'\|_{p(\bullet)} \tag{54}$$

Therefore, using assumption $p^- > 1$ and Young's inequality we have

$$\|(y'_n - y')\|_{p(\bullet)} = O(1), \tag{55}$$

that is, $y_n \longrightarrow y$ in E strongly.

The proof of PS property has been completed. □

Now, apply the Mountain Pass theorem in order to show the existence of solution for the problem (27).

For $\|y\| \leq 1$ we have

$$I_\lambda(y) \geq \frac{1}{p^+}\int_0^l \left(\frac{|y'|}{\|y\|}\right)^{p(x)} \|y\|^{p^+} dx - \frac{\lambda}{q^-}\int_0^l \left(\frac{\left(xl-x^2\right)^{-1/p'-(1-\delta)/q} y_+}{\|y\|}\right)^{q(x)} \|y\|^{q^-} \tag{56}$$

By using Theorem A, it follows that

$$\|y\|_{L^{q,-1/p'-1/q}(0,l)} \leq c\|y\|_{L^p(0,l)} \tag{57}$$

Then (56) implies that

$$\begin{aligned} I_\lambda(y) &\geq \frac{1}{p^+}\|y\|^{p^+} - \frac{\lambda}{q^-}\int_0^l \left(\frac{\left(xl-x^2\right)^{-1/p'-1/q} c}{\|y\|_{L^{q,-1/p'-1/q}(0,l)}}\right)^{q(x)} \|y\|^{q^-} \\ &\geq \frac{1}{p^+}\|y\|^{p^+} - \frac{\lambda c^{q^+}}{q^-}\|y\|^{q^-} \end{aligned} \tag{58}$$

Hence for $\|y\| \leq 1$ it follows that

$$\begin{aligned} I_\lambda(y) &\geq \frac{1}{p^+}\|y\|^{p^+} - \frac{\lambda c^{q^+}}{q^-}\|y\|^{q^-} \\ &= \|y\|^{p^+}\left(\frac{1}{p^+} - \frac{\lambda c^{q^+} p^+}{q^-}\|y\|^{q^- - p^+}\right) \end{aligned} \tag{59}$$

Therefore,

$$I_\lambda(y) \geq \|y\|^{p^+} \frac{\lambda c^{q^+}}{q^-}\left(\frac{q^-}{\lambda c^{q^+} p^+} - \|y\|^{q^- - p^+}\right). \tag{60}$$

If we choose $\|y\| = \min\{1, (q^-/2\lambda c^{q^+} p^+)^{1/(q^- - p^+)}\}$, it follows that

$$I_\lambda(y) \geq \left(\frac{q^-}{2\lambda c^{q^+} p^+}\right)^{p^+/(q^- - p^+)} \frac{1}{2}\frac{q^-}{2\lambda c^{q^+} p^+} \tag{61}$$

Choose $R = (1/2\lambda c^{q^+} p^+)^{1/(q^- - p^+)}$ to apply the Mountain Pass theorem.

Now, what remains is to find a point $y_0 \in E$ where $I_\lambda(y_0) < 0$. To show this, apply the fibering method; for $y \in E$ to be fixed and sufficiently large $t > 1$ it holds that

$$\begin{aligned} I_\lambda(ty) &= \int_0^l \frac{t^{p(x)}}{p(x)}|y'|^{p(x)} dx - \int_0^l \frac{t^{q(x)}}{q(x)}\left(\left(xl-x^2\right)^{-1/p'-(1-\delta)/q} y_+\right)^{q(x)} dx \\ &\leq \frac{t^{p^+}}{p^-}\int_0^l |y'|^{p(x)} dx - \frac{t^{q^-}}{q^+}\int_0^l \left((x(l-x))^{\beta-1/p'-(1-\delta)/q} y_+\right)^{q(x)} dx \\ &< 0. \end{aligned} \tag{62}$$

Insert $y_0 = ty$ in order to get some point laid out of the ball $B(0,R)$ in E, such that $I_\lambda(y_0) < 0$. Applying Mountain Pass theorem, there exists a point $y_0 \in E$ with $I_\lambda(y_0) = c$ such that $I'_\lambda(y_0) = 0$, where

$$\begin{aligned} c = \inf_{\gamma(\mathrm{t})\in\Gamma} \sup &\left\{I_\lambda(\gamma(\mathrm{t})) : \text{with } \gamma : [0,1] \longrightarrow E,\ \gamma \right. \\ &\left. \in C^1\ [0,1;\mathrm{E}],\ \text{and } \gamma(0) = 0,\ \gamma(1) = 0\right\}. \end{aligned} \tag{63}$$

Therefore,

$I_\lambda(y_0) > 0$, $I'(y_0) = 0$. To show that y_0 is a positive solution of (27) insert in $\langle I'(y_0), v\rangle = 0$, $v = (y_0)_-$:

$$\int_0^l |y'|^{p(x)-2} y' y'_- - \lambda\int_0^l \left(xl-x^2\right)^{(-1/p'-(1-\delta)/q)q} y_+^{q(x)-1} y_- dx = 0 \tag{64}$$

Since the second integral is zero ($y_+^{q(x)-1} y_- \equiv 0$), we have

$$0 = \int_0^l |y'_-|^{p(x)-2}\left(y'_-\right)^2 dx \tag{65}$$

Therefore, $y'_- \equiv 0$; using imbedding Theorem B we infer $y_- \equiv 0$, which implies that $y_0(x) > 0$.

We have proved the existence of problem (27) for any $\lambda > 0$.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author declares that he has no conflicts of interest.

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