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Research Article

Extended Minimal Atomicity through Nondifferentiability: A Mathematical-Physical Approach

Gabriel Gavriluţ,[1] Alina Gavriluţ ,[2] and Maricel Agop [3,4]

[1] *Faculty of Physics, Alexandru Ioan Cuza University, Carol I Bd. 11, Iaşi, 700506, Romania*
[2] *Faculty of Mathematics, Alexandru Ioan Cuza University, Carol I Bd. 11, Iaşi, 700506, Romania*
[3] *Department of Physics, Gheorghe Asachi Technical University of Iaşi, Romania*
[4] *Academy of Romanian Scientists, Splaiul Independenţei 54, Sector 5, 050094, Bucureşti, Romania*

Correspondence should be addressed to Alina Gavriluţ; gavrilut@uaic.ro

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The mathematical concept of minimal atomicity is extended to fractal minimal atomicity, based on the nondifferentiability of the motion curves of physical system entities on a fractal manifold. For this purpose, firstly, different results concerning minimal atomicity from the mathematical procedure of the Quantum Measure Theory and also several physical implications are obtained. Further, an inverse method with respect to the common developments concerning the minimal atomicity concept has been used, showing that Quantum Mechanics is identified as a particular case of Fractal Mechanics at a given scale resolution. More precisely, for fractality through Markov type stochastic processes, i.e., fractalization through stochasticization, the standard Schrödinger equation is identified with the geodesics of a fractal space for motions of the physical system entities on nondifferentiable curves on fractal dimension two at Compton scale resolution. In the one-dimensional stationary case of the fractal Schrödinger type geodesics, a special symmetry induced by the homographic group in Barbilian's form "makes possible the synchronicity" of all entities of a given physical system. The integral and differential properties of this group under the restriction of defining a parallelism of directions in Levi-Civita's sense impose correspondences with the "dynamics" of the hyperbolic plane so that harmonic mappings between the ordinary flat space and the hyperbolic one generate (by means of a variational principle) *a priori* probabilities in Jaynes' sense. The explicitation of such situation specifies the fact that the hydrodynamical variant of a Fractal Mechanics is more easily approached and, from this, the fact that Quantum Measure Theory can be a particular case of a possible Fractal Measure Theory.

1. Introduction

Measure Theory concerns assigning a notion of size to sets. Recently, nonadditive measure theory was given an increasing interest due to its various applications in a wide range of areas. It is used to describe situations concerning conflicts or cooperations among intelligent rational players, giving an appropriate mathematical framework to predict the outcome of the process. Precisely, theories dealing with (pseudo)atoms and monotone measures are used in game theory, probabilities, statistics, and artificial intelligence.

(Non)atomic measures and purely atomic measures have been investigated in different variants (Chiţescu [1, 2], Cavaliere and Ventriglia [3], Gavriluţ and Agop [4], Gavriluţ and Croitoru [5, 6], Gavrilut [7, 8], Khare and Singh [9], Li *et al.* [10, 11], Pap [12–14], Pap *et al.* [15], Rao and Rao [16], Suzuki [17], and Wu and Bo [18]).

Thus, one important application of Measure Theory is in probability, where a measurable set is interpreted as an event and its measure as the probability that the event will occur. Since probability is an important notion in Quantum Mechanics, Measure Theory's techniques could be used to study quantum phenomena. Unfortunately, one of the foundational axioms of Measure Theory does not remain valid in its intuitive application to Quantum Mechanics.

Although classical measure theory imposes strict additivity conditions, a rich theory of nonadditive measures developed. Precisely, modifications of traditional Measure Theory (Pap [13, 14]) led to Quantum Measure Theory (Gudder [19–23], Salgado [24], Sorkin [25–27], and Surya and Waldlden

[28]). Practically, an extended notion of a measure has been introduced and its applications to the study of interference, probability, and spacetime histories in Quantum Mechanics have been discussed (Schweizer and Sklar [29]).

Introduced by Sorkin [25–27], quantum measures help us to describe Quantum Mechanics and its applications to Quantum Gravity and Cosmology (Hartle [30, 31], Phillips [32]). Quantum Measure Theory indicates a wide variety of applications, as its mathematical structure is used in the standard quantum formalism.

In [25–27], Sorkin proposed a history-based framework, which can accommodate both standard Quantum Mechanics and physical theories beyond the quantum formalism.

Recently, since Quantum Mechanics can be assimilated with a particular model of Fractal Mechanics at a given scale resolution in the form of Scale Relativity Theory in a constant fractal dimension and arbitrary (Mercheş and Agop [33]), fundamental concepts of Quantum Mechanics can be extended to similar concepts, but on fractal manifolds.

Such result is not a singular one: firstly, Feynman and Hibbs [34], from path integral formalism, then Laskin [35], and more recently Nottale [36] through Scale Relativity Theory (in fractal dimension two) proved that there exists an intrinsic relationship between the Fractal Theory (initiated by Mandelbrot [37]) and Quantum Mechanics.

The aim of this paper is to provide the mathematical-physical perspective that is necessary to extend some of these above-mentioned concepts. Precisely, we extend the concept of minimal atom to the fractal minimal atom. We also give characterizations from a mathematical viewpoint to these new concepts and we make explicit certain physical implications.

Firstly, we introduce the notion of a minimal atom from a mathematical perspective and we give some of its mathematical properties. In this framework, physical correspondences in the Quantum Mechanics (Phillips [32]) context are established. Secondly, taking into account that Quantum Mechanics is a particular case of Fractal Mechanics (for Peano curves at Compton scale resolution), situation when the Schrödinger equation is identified with a particular geodesics of a fractal manifold, we apply an inverse method. We make explicit the fundamental elements of Fractal Mechanics and starting from here we introduce the new concepts of a fractal minimal atom.

The present paper is organized as follows. After an Introductory part, Section 2 contains different results involving certain types of atoms, introduced from the Quantum Measure Theory mathematical perspective. Certain physical implications and interpretations are provided. Section 3 is devoted to a background of Quantum Measure Theory by means of Fractal Mechanics; fundaments of the model and its correspondence with Quantum Mechanics are highlighted. Sections 4 and 5 contain developments of the Fractal Mechanics either in the form of Barbilian's model of fractal differential geometry, as in Section 4, or in the form of the probabilities generated by means of harmonic mappings, as in Section 5. From such perspective, the background for extending the notions that are specific to atomicity, to those involving fractal atomicity, is provided, so that in Section 6 some properties of the fractal minimal atom are discussed.

2. Minimal Atoms

In this section, let T be an abstract nonvoid set, $\mathscr{C}$ a ring of subsets of T, X a Banach space, $\mathbb{P}_f(X)$ the family of all nonvoid closed subsets of X, and $\mu : \mathscr{C} \longrightarrow \mathbb{P}_f(X)$ a set multifunction which is monotone with respect to the operation of inclusion of sets and which satisfies $\mu(\emptyset) = \{0\}$.

By $|\mu|$, defined on $\mathscr{C}$ and taking values in $[0, \infty]$, we mean the set function defined by $|\mu(A)| = h(\mu(A), \{0\})$, $\forall A \in \mathscr{C}$, where h is the Hausdorff-Pompeiu pseudo-metric (Gavrilu 0x163 [7]).

Definition 1. (i) If $\mu : \mathscr{C} \longrightarrow \mathbb{P}_f(X)$, we consider the variation of μ, $\overline{\mu} : \mathscr{P}(T) \longrightarrow [0, \infty]$, defined by

$$\begin{aligned} \overline{\mu}(A) = \sup \Bigg\{ \sum_{i=1}^{p} \left|\mu(A_i)\right| ; A = \bigcup_{i=1}^{p} A_i, A_i \in \mathscr{C}, \forall i \\ = \overline{1, p}, A_i \cap A_j = \emptyset, i \neq j \Bigg\}, \quad \forall A \in \mathscr{P}(T) \end{aligned} \tag{1}$$

(ii) μ has finite variation if $\overline{\mu}(T) < \infty$.

We firstly recall the following notions (Gavrilut [7, 8], Gavriluţ and Croitoru [5, 6])).

Definition 2. A set function $m : \mathscr{C} \longrightarrow \mathbb{R}_+$, with $m(\emptyset) = 0$, is said to be

(i) null-additive if $m(A \cup B) = m(A), \forall A, B \in \mathscr{C}$, with $m(B) = 0$;

(ii) null-null-additive if $m(A \cup B) = 0, \forall A, B \in \mathscr{C}$, with $m(A) = m(B) = 0$;

(iii) diffused if $m(\{t\}) = 0$, whenever $\{t\} \in \mathscr{C}$;

(iv) monotone (or, fuzzy) if $m(A) \leq m(B), \forall A, B \in \mathscr{C}$, with $A \subseteq B$;

(v) a submeasure (in the sense of Drewnowski [38–40]) if m is monotone and subadditive ($m(A \cup B) \leq m(A) + \nu(B)$, $\forall$ (disjoint) $A, B \in \mathscr{C}$);

(vi) finitely additive if $m(A \cup B) = m(A) + \nu(B)$, $\forall$ disjoint $A, B \in \mathscr{C}$.

Examples 1 (Gavriluţ and Agop [4]). (i) Suppose ($T = \{t_1, t_2, \ldots, t_n\}, m : \mathscr{P}(T) \longrightarrow \mathbb{R}_+$) is a finite measure space, where, $\forall i \in \{1, 2, \ldots, n\}$, t_i represents the particle and m is the mass. The real valued set function associated with the mass is additive in the macroscopic world. On the quantum scale, these statements do not remain valid due to the annihilation and binding energy effects. For instance, if t_1 and t_2 represent an electron and a positron, respectively, then $m(\{t_1\}) = m(\{t_2\}) = 9,11 \times 10^{-31}$ kg, but $m(\{t_1, t_2\}) = m(\{t_1\} \cup \{t_2\}) = 0$.

(ii) By [41], Shannon's entropy is a subadditive real-valued set function.

(iii) In Quantum Mechanics, the wave-particle duality principle states that every fermion (matter particle) and boson (force-carrying particle) are described by a wave-function, i.e., a time-varying function, giving the particle's probability density at each point in space. These wavefunctions often behave like classical waves, having diffraction and interference properties.

Additivity of measures does not remain valid when interference occurs. Precisely, interference allows the union of sets of measure zero to have nonzero measure.

Definition 3 (Gavrilut [7, 8], Gavriluţ and Croitoru [5, 6]). (I) A set multifunction $\mu : \mathscr{C} \longrightarrow \mathbb{P}_f(X)$, with $\mu(\emptyset) = \{0\}$, is said to be

(i) null-additive if $\mu(A \cup B) = \mu(A)$, $\forall A, B \in \mathscr{C}$, with $\mu(B) = \{0\}$;

(ii) null-null-additive if $\mu(A \cup B) = \{0\}, \forall A, B \in \mathscr{C}$, with $\mu(A) = \mu(B) = \{0\}$;

(iii) monotone (with respect to the operation of inclusion of sets): $\mu(A) \subseteq \mu(B), \forall A, B \in \mathscr{C}$, $A \subseteq B$;

(iv) a multisubmeasure if μ is monotone and $\mu(A \cup B) \subseteq \mu(A) \dot{+} \mu(B)$, $\forall$ (disjoint) $A, B \in \mathscr{C}$ (where "$\dot{+}$" represents the Minkowski addition of sets: $\forall M, N \in \mathbb{P}_f(X)$, $M \dot{+} N = \overline{M + N}-$ the symbol "." denotes the closure in the topology induced by the norm of X).

(II) We say that a set $A \in \mathscr{C}$ is

(i) a minimal atom of μ if $\mu(A) \supsetneq \{0\}$ and $\forall B \in \mathscr{C}, B \subseteq A$, one has $\mu(B) = \{0\}$ or $A = B$;

(ii) (Gavrilut [7, 8], Gavriluţ and Croitoru [5, 6]) an atom of μ if $\mu(A) \supsetneq \{0\}$ and $\forall B \in \mathscr{C}, B \subseteq A$, one has $\mu(B) = \{0\}$ or $\mu(A \setminus B) = \{0\}$;

(iii) (Gavrilut [7, 8], Gavriluţ and Croitoru [5, 6]) a pseudo-atom of μ if $\mu(A) \supsetneq \{0\}$ and $\forall B \in \mathscr{C}, B \subseteq A$, one has $\mu(B) = \{0\}$ or $\mu(A) = \mu(B)$.

In the following, we suppose that μ is monotone.

Remark 4. (i) Any minimal atom is also an atom (and a pseudo-atom), so, for $\mathscr{M}\mathscr{A}$, the collection of all minimal atoms of μ, we have

$$\begin{aligned} \mathscr{M}\mathscr{A} = \{&A \in \mathscr{C}; \mu(A) \supsetneq \{0\} \text{ and } \forall B \in \mathscr{C}, B \\ &\subsetneq A, \text{ one has } \mu(B) = \{0\}\} \subseteq\subseteq \mathscr{A} = \{A \in \mathscr{C}; \mu(A) \\ &\supsetneq \{0\} \text{ and } \forall B \in \mathscr{C}, B \subseteq A \text{ one has } \mu(B) \\ &= \{0\} \text{ or } \mu(A \setminus B) = \{0\}\}, \end{aligned} \tag{2}$$

where $\mathscr{A}$ is the collection of all atoms of μ.

(ii) If, moreover, μ is null-additive, then any atom of μ is also a pseudo-atom.

(iii) If A is a minimal atom of μ, then $\forall B \subsetneq A$, one has $\mu(B) \supsetneq \{0\}$.

(iv) If $\mathscr{C} = \mathscr{B}$ and $\mu : \mathscr{B} \longrightarrow \mathbb{P}_f(X)$, then for every minimal atom A of μ and for every $a \in A$, one has $\mu(A \setminus \{a\}) = \{0\}$. If, moreover, μ is null-additive, then $\mu(A) = \mu(\{a\})$. In a physical interpretation, in this case, information is the same in each point.

(v) One can think to the following physical interpretation of an atom/a pseudo-atom: when the collapse occurs, the negligible part corresponds to the corpuscle, while the part which covers the entire set (precisely, the atom) corresponds to the wave. In this way, there is absolute dominance of one over the other results, so an atom can be considered as the "elementary bridge" which includes all the properties of the "matter" (in the form of the corpuscle and of the wave) it originated form. More precisely, in our opinion, we discuss here an Einstein-Podolsky-Rosen (EPR) type bridge (Susskind [17, 42]), in which the corpuscle and the wave are connected and can be interpreted as maximally entangled states of the same "physical objects".

(vi) In fact, atoms (in all their variants) could be considered as singularities of the space-matter metric.

Proposition 5. *If $\mu : \mathscr{C} \longrightarrow \mathscr{P}_f(X)$ is null-null-additive and $A, B \in \mathscr{C}$ are two different minimal atoms of μ, then $A \cap B = \emptyset$.*

Proof. Suppose that there are nondisjoint, different minimal atoms $A, B \in \mathscr{C}$ of μ. Since $A \setminus (A \cap B) = A \setminus B \subsetneq A$ and $A \cap B \subseteq B$, then $[\mu(A \setminus B) = \{0\}$ or $A \setminus B = A]$ and $[\mu(A \cap B) = \{0\}$ or $A \cap B = B]$.

(i) If $\mu(A \setminus B) = \{0\}, \mu(A \cap B) = \{0\}$, since μ is null-null-additive, we get that $\mu(A) = \{0\}$, a contradiction.

(ii) If $A \setminus B = A$, then $A \cap B = \emptyset$, a contradiction.

(iii) If $\mu(A \setminus B) = \{0\}$ and $A \cap B = B$, then $B \subseteq A$, so $\mu(B) = \{0\}$ (or $B = A$, a contradiction), so again since μ is null-null-additive, one has $\mu(A) = \{0\}$, which is a false. ☐

Evidently, if $A \in \mathscr{C}$ is a minimal atom of μ, then another different minimal atom $A_1 \in \mathscr{C}$ of μ can not exist so that $A_1 \subset A$.

Proposition 6. *(i) If T is finite, then $\forall A \in \mathscr{C}$, with $\mu(A) \supsetneq \{0\}$, $\exists$ is a minimal atom of μ, $B \in \mathscr{C}, B \subseteq A$.*

(ii) If, moreover, A is an atom of μ and μ is null-additive, then $\mu(A) = \mu(B)$ and the set B is unique.

Proof. (i) Let be the set $\mathscr{M} = \{M \in \mathscr{C}, M \subseteq A, \mu(M) \supsetneq \{0\}\}$. Obviously, $\mathscr{M} \neq \emptyset$, since $A \in \mathscr{C}$. We observe that any minimal element of $\mathscr{M}$ is a minimal atom of μ. Indeed, let $S \in \mathscr{M}$ be a minimal element of $\mathscr{M}$. Then $D \in \mathscr{M}$ can not exist so that $D \subseteq S$ and $D \neq S(*)$.

Since $S \in \mathscr{M}$, then $S \in \mathscr{C}, S \subseteq A, \mu(S) \supsetneq \{0\}$.

We prove that S is a minimal atom of μ. Indeed, $\forall L \subseteq S, L \in \mathscr{C}$, one has either $\mu(L) = \{0\}$ or $\mu(L) \supsetneq \{0\}$. In the latter case, one has either $L = S$ or $L \neq S$, which is in contradiction with $(*)$.

(ii) If on the contrary there are two different minimal atoms B_1 and B_2 of μ, then $\mu(A \setminus B_1) = \mu(A \setminus B_2) = \{0\}$, whence $\mu(A) = \{0\}$, a contradiction. ☐

Proposition 7 (self-similarity of minimal atoms). *Any subset $B \in \mathscr{C}$ with $\mu(B) \supsetneq \{0\}$ of a minimal atom $A \in \mathscr{C}$ of μ is a minimal atom of μ, too.*

Proof. Let $A \in \mathscr{C}$ be a minimal atom of μ and consider any $B \in \mathscr{C}$, with $\mu(B) \supsetneq \{0\}, B \subseteq A$. We prove that B is a minimal atom of μ. Indeed, for any $E \in \mathscr{C}, E \subseteq B$, then $E \subseteq A$, so either $\mu(E) = \{0\}$ or $E = A$, whence $E = B$. ☐

Remark 8. (i) Suppose that $\mu_1, \mu_2 : \mathscr{C} \longrightarrow \mathbb{P}_f(\mathbb{R})$ are two monotone set multifunctions such that $\mu_1(\emptyset) = \mu_2(\emptyset) = \{0\}$ and $\mu_1(A) \subseteq \mu_2(A)$, $\forall A \in \mathscr{C}$ (for instance,

$\mu_1, \mu_2 : \mathscr{C} \longrightarrow \mathbb{P}_f(\mathbb{R}), \mu_1(A) = [0, m_1(A)], \mu_2(A) = [0, m_2(A)], \forall A \in \mathscr{C}, m_1, m_2 : \mathscr{C} \longrightarrow \mathbb{R}_+$ being monotone, $m_1(A) \leq m_2(A), \forall A \in \mathscr{C}, m_1(\emptyset) = m_2(\emptyset) = 0$). Then any minimal atom of μ_2 is a minimal atom of μ_1.

(ii) If $m : \mathscr{C} \longrightarrow [0, \infty)$ is monotone, $m(\emptyset) = 0$ and $\mu : \mathscr{C} \longrightarrow \mathbb{P}_f(\mathbb{R}), \mu(A) = [0, m(A)], \forall A \in \mathscr{C}$, then $A \in \mathscr{C}$ is an atom/pseudo-atom/minimal atom of μ iff the same is A for m (in the sense of Mesiar et al. [43] and Ouyang et al. [44]).

μ is called the set multifunction induced by the set function m.

(iii) Let be $\mu : \mathscr{C} \longrightarrow \mathbb{P}_f(\mathbb{R}), \mu(A) = [-m_1(A), m_2(A)]$, $\forall A \in \mathscr{C}$, where $m_1, m_2 : \mathscr{C} \longrightarrow \mathbb{R}_+, m_1(\emptyset) = m_2(\emptyset) = 0$. Then $A \in \mathscr{C}$ is a minimal atom of μ iff A is a minimal atom for both m_1 and m_2 (in the sense of Mesiar et al. [43] and Ouyang et al. [44]).

(iv) If $\mu : \mathscr{C} \longrightarrow \mathbb{P}_f(\mathbb{R}), \mu(A) = \{m(A)\}, \forall A \in \mathscr{C}$, where $m : \mathscr{C} \longrightarrow \mathbb{R}_+, m(\emptyset) = 0$, then $A \in \mathscr{C}$ is a minimal atom of μ iff A is a minimal atom for m (in the sense of Mesiar et al. [43] and Ouyang et al. [44]).

In consequence, one can easily construct different examples involving minimal atoms with respect to the set multifunction induced by a set function, starting from the examples given in Mesiar *et al.* [43] and *Ouyang et al.* [44].

Examples 2. (i) Suppose

$$\begin{gathered} T = \{u, v\}, \\ \mu : \mathscr{P}(T) \longrightarrow \mathbb{P}_f(\mathbb{R}), \\ \mu(A) = \begin{cases} [0, 1], & \text{if } A = \{u\} \text{ or } A = T \\ \{0\}, & \text{else} \end{cases}. \end{gathered} \tag{3}$$

Then T is an atom of μ. Also, it is not a minimal atom of μ. $\{u\}$ is an atom and also it is a minimal atom of μ.

(ii) If

$$\begin{gathered} T = \{u, v, w, z\}, \\ \mu : \mathscr{P}(T) \longrightarrow \mathbb{P}_f(\mathbb{R}), \\ \mu(A) = \begin{cases} \{5\}, & \text{if } A = T \\ \{3\}, & \text{if } A = \{u, v, w\}, \{u, v, z\}, \{u, w, z\} \\ \{2\}, & \text{if } A = \{u, v\}, \{u, w\} \\ \{0\}, & \text{else}, \end{cases} \end{gathered} \tag{4}$$

then $\{u, v\}$ and $\{u, w\}$ are minimal atoms of μ.

We observe that our Definition 3-(I)-(i) generalizes to the set valued case of the corresponding notion introduced by Mesiar et al. [43] and Ouyang et al. [44].

(iii) Suppose

$$\begin{gathered} T = \{u, v, w, z\}, \\ \mu : \mathscr{P}(T) \longrightarrow \mathbb{P}_f(\mathbb{R}), \\ \mu(A) = \begin{cases} \{5\}, & \text{if } A = T \\ \{2\}, & \text{if } A \neq T, A \neq \emptyset \\ \{0\}, & \text{if } A = \emptyset. \end{cases} \end{gathered} \tag{5}$$

Then every singleton of μ is a minimal atom.

Remark 9. $\overline{\mu}(A) \geq |\mu(A)|, \ \forall A \in \mathscr{C}$. In consequence, if $A \in \mathscr{C}$ is a minimal atom of $\overline{\mu}$ (in the sense of Mesiar et al. [43] and Ouyang et al. [44]), then A is a minimal atom of μ. Moreover, conversely, if $A \in \mathscr{C}$ is a minimal atom of μ, then it is also an atom of μ, so $\overline{\mu}(A) \geq |\mu(A)|$, whence A is a minimal atom of $\overline{\mu}$.

Remark 10. (i) Any set $A \in \mathscr{C}$ that can be written as $\bigcup_{i=1}^{p} A_i$ (where $\overline{\mu}(A) \geq |\mu(A)| i = \overline{1, p}$, $A_i \in \mathscr{C}$ are different minimal atoms of μ) is partitioned in fact in this way, since by Proposition 5 one has $A_i \cap A_j = \emptyset, i \neq j$.

Since any minimal atom is an atom, then in this case $\overline{\mu}(A_i) = |\mu(A_i)|$, $\overline{\mu}(A) \geq |\mu(A)| i = \overline{1, p}$. In consequence, if, moreover, μ is a multisubmeasure (in the sense of Gavrilut [8]) of finite variation, then $\overline{\mu}$ is finitely additive, so $\overline{\mu}(A) = \sum_{i=1}^{p} |\mu(A_i)|$ (Dinculeanu [45]).

(ii) (Nondecomposability of minimal atoms) Any minimal atom $A \in \mathscr{C}$ can not be partitioned (its only partition is $\{A, \emptyset\}$).

The converse of the last statement also holds.

Proposition 11. *Any nonpartitionable atom $A \in \mathscr{C}$ is a minimal atom.*

Proof. Since A is an atom, then $\mu(A) \supsetneq \{0\}$. On the other hand, because A is nonpartitionable, there can not exist two nonvoid disjoint subsets of A, say $A_1, A_2 \in \mathscr{C}$. Let now be any $E \in \mathscr{C}$, with $E \subseteq A$. One has either $\mu(E) = \{0\}$ (which is fine) or $\mu(E) \supsetneq \{0\}$. In the latter case, the only possibility is $E = A$ (if not, $\{A \setminus E, E\}$ is a partition of A, which is false). □

Corollary 12. *An atom is minimal if and only if it is not partitionable.*

Theorem 13. *If $\mathscr{C} = \mathscr{B}$ and if μ is null-additive and regular (in the sense of Gavrilut [7, 8]), then any minimal atom $A \in \mathscr{B}$ of μ is a singleton.*

Proof. $\exists! a \in A$ so that $\mu(A) = \mu(\{a\})$. Then either $A = \{a\}$, or $\mu(\{a\} = \{0\}$, in which case $\mu(A) = \{0\}$, which is absurd, since A is an atom. □

Theorem 14. *If T is finite, μ is null-additive and $\{A_i\}_{i=\overline{1,p}}$ is the set of all minimal different atoms contained in a set $A \in \mathscr{C}$, with $\mu(A) \supsetneq \{0\}$ (this set exists by Proposition 6-(i)), then $\mu(A) = \mu(\bigcup_{i=1}^{p} A_i)$ (so, the minimal atoms are the only ones which are important from the "measurement" viewpoint).*

Proof. We obviously have $\mu(A \setminus \bigcup_{i=1}^{p} A_i) = \{0\}$. If not, by Proposition 6-(i), another minimal atom of μ exists, so, since μ is null-additive, $\mu(A) = \mu(\bigcup_{i=1}^{p} A_i)$. □

Corollary 15. *If T is finite, μ is a multisubmeasure, $A \in \mathscr{C}$, with $\mu(A) \supsetneq \{0\}$ being arbitrary, and $\{A_i\}_{i=\overline{1,p}}$ is the collection of all minimal different atoms that are contained in A, then $\mu(\bigcup_{i=1}^{p} A_i) = \mu(A) \subseteq \overline{\sum_{i=1}^{p} \mu(A_i)}$. Moreover, $|\mu(A)| \leq \sum_{i=1}^{p} |\mu(A_i)|$.*

Proof. The statement is immediate by Theorem 14 since μ is in particular null-additive and $|\mu|$ is a submeasure. □

3. Towards Quantum Measure Theory by Means of Fractal Mechanics

The main idea in Quantum Measure Theory, or Generalized Quantum Mechanics, is to give a description of the world in terms of histories. A history represents a classical description of the system considered for a given period of time, which can be finite or infinite. If someone tries to describe a system of N particles, then a history will be given by N classical trajectories. If someone deals with a field theory, then a history corresponds to the spatial configuration of the field as a function of time. In either case, Quantum Measure Theory tries to provide a way to describe the world through classical histories by extending the notion of probability theory which is clearly not rich enough to model our universe.

On the other hand, structures, self-structures, and so forth of the Nature can be assimilated to complex systems, taking into account both their functionality and their structure (Mitchell [46] and Nottale [36]). The models used to study the complex systems dynamics are built on the assumption that the physical quantities which describe it (such as density, momentum, and energy) are differentiable (for mathematical models and for applications, see Nottale [36]).

Differential methods fail when facing the physical reality, since the instabilities in the case of dynamics of complex systems, instabilities that can generate both chaos and patterns.

In order to describe such complex systems dynamics, one has to introduce the scale resolution in the expressions of the physical variables that describe these dynamics and in the fundamental equations of "evolution" (density, momentum, and energy equations). Thus, any dynamic variable, dependent, in a classical meaning, on both spatial coordinates and time (Michel and Thomas [47] and Mitchell [46]), becomes, in this new context, dependent also on the scale resolution.

In consequence, instead of working with a dynamic variable, we will deal with different approximations of a strictly nondifferentiable mathematical function. In consequence, any dynamic variable acts as the limit of a functions family. Any function is nondifferentiable for a null resolution scale and differentiable for a nonzero scale resolution.

This approach, well adapted for applications in the field of dynamics of complex systems, where any real determination is conducted at a finite scale resolution, clearly implies the development both of a new geometric structure and of a physical theory (applied to dynamics of complex systems) for which the motion laws, invariant to spatial and temporal coordinates transformations, are integrated with scale laws, invariant at scale transformations.

Such a theory that includes the geometric structure based on the above presented assumptions was developed in the Scale Relativity Theory (Nottale [36]) and more recently in the Scale Relativity Theory with an arbitrary constant fractal dimension (Mercheş and Agop [33]). Both theories define the "fractal physics models" class (Mercheş and Agop [33] and Nottale [36]).

Various theoretical aspects and applications of the Scale Relativity Theory with an arbitrary constant fractal dimension in the field of physics are presented in Mercheş and Agop [33] and Nottale [36]. In this model, we assume that, in the complex systems dynamics, the complexity of interactions is replaced by nondifferentiability. Then, the motions constrained on continuous, differentiable curves in an Euclidean space are replaced with free motions, without any constraints, on continuous, nondifferentiable curves (called fractal curves) in a fractal space. In other words, for time resolution scale that seems to be large when someone compares them with the inverse of the highest Lyapunov exponent (Mandelbrot [37]), the deterministic trajectories can be replaced by a set of potential routes, so that the notion of "definite positions" is substituted by the concept of a set of positions which have a definite probability density (Mandelbrot [37], Mercheş and Agop [33], and Nottale [36]).

In consequence, the motion curves have double identity: the geodesics of the fractal space and streamlines of a fractal fluid, whose entities (the complex system structural units) are replaced with the geodesics and so any external constraints can be seen as a selection of geodesics by means of measuring device.

In such conjecture, Quantum Mechanics becomes a particular case of Fractal Mechanics (for structural units movements of a complex system on Peano curves at Compton scale resolution). In consequence, in our opinion, Quantum Measure Theory could become a particular type of a Fractal Measure Theory.

Let us admit that the entities of any physical system are moving on continuous but nondifferentiable curves (fractal curves). Then its dynamics becomes functional by means of the scale covariance principle: the physics laws which describe the dynamics of the physical systems are invariant with respect to scale transformations (Mazilu and Agop [48] and the Appendix).

Theorem 16. *If the scale covariance principle is functional, then the transition from the dynamics of the classical physics (differential physics) to the dynamics of the fractal physics (nondifferentiable physics) can be implemented by replacing the usual derivative operator d/dt by the scale covariant derivative $\widehat{d}/dt$*

$$\frac{\widehat{d}}{dt} = \partial_t + \widehat{V}^l \partial_l - i\lambda\,(dt)^{(2/D_F)-1}\,\partial_l \partial^l \tag{6}$$

where

$$\partial_t = \frac{\partial}{\partial t},$$

$$\partial_l = \frac{\partial}{\partial X^l},$$

$$\partial_l \partial^l = \frac{\partial}{\partial X_l}\left(\frac{\partial}{\partial X^l}\right) \tag{7}$$

$$\widehat{V}^l = V_D^l - iV_F^l, \quad i = \sqrt{-1}, \ l = 1, 2, 3 \tag{8}$$

In the above relations, X^l are the fractal spatial coordinates, t is the nonfractal temporal coordinate with the role of an affine parameter of the motion curves, $\widehat{V}^l$ is the velocities complex field, V_D^l is the real part of the complex velocity which is independent on the scale resolution dt, and V_F^l is the imaginary part of the complex velocity which is dependent on the scale resolution. λ is the diffusion coefficient associated with the fractal-nonfractal transition. D_F represents the fractal dimension of the motion curve.

For D_F one can choose different definitions, as fractal dimension in the sense of Kolmogorov, fractal dimension in the sense of Hausdorff-Besikovici, and so forth (Mandelbrot [37] and Nottale [36]).

Once chosen the definition of the fractal dimension, it has to be the same in the analysis of the physical systems dynamics.

Proof. The proof of the above statements is given in the Appendix (scale covariant derivative and fractal geodesics) and also in Mercheş and Agop [33]. □

The "fractal operator" (1) plays the role of the scale covariant derivative; namely, it is used to write the fundamental equations of the physical systems dynamics in the same form as in the classical (differentiable) case.

Theorem 17. *If one applies the operator* (1) *to the complex velocity field* (3)*, then the equation of motion (geodesics equation) takes the form:*

$$\frac{\widehat{d}\widehat{V}^i}{dt} = \partial_t \widehat{V}^i + \widehat{V}^l \partial_l \widehat{V}^i - i\lambda (dt)^{(2/D_F)-1} \partial_l \partial^l \widehat{V}^i \equiv 0. \tag{9}$$

In the presence of an external scalar potential U, the equation of motion (geodesics equation) becomes

$$\frac{\widehat{d}\widehat{V}^i}{dt} = \partial_t \widehat{V}^i + \widehat{V}^l \partial_l \widehat{V}^i - i\lambda (dt)^{(2/D_F)-1} \partial_l \partial^l \widehat{V}^i = -\partial^i U \tag{10}$$

Proof. This proof of the above statements is given in the Appendix (fractal geodesics) and also in Mercheş and Agop [33]. □

For irrotational motions of the entities of the physical system, the complex velocity field (3) satisfies the restriction:

$$\varepsilon_{ilk} \partial^l \widehat{V}^k = 0 \tag{11}$$

with ε_{ilk} the Levi-Civita pseudo-tensor.

From here, $\widehat{V}^i$ is given by means of the gradient of the complex scalar function $\ln \Psi$, called the scalar potential of the complex velocities field, as

$$\widehat{V}^i = -2i\lambda (dt)^{(2/D_F)-1} \partial^i \ln \Psi \tag{12}$$

Corollary 18. *A Schrödinger type fractal equation, which is free of any "external constraints",*

$$\lambda^2 (dt)^{(4/D_F)-2} \partial^l \partial_l \Psi + i\lambda (dt)^{(2/D_F)-1} \partial_t \Psi \equiv 0 \tag{13}$$

or a Schrödinger type fractal equation in the presence of an "external constraint" U, respectively,

$$\lambda^2 (dt)^{(4/D_F)-2} \partial^l \partial_l \Psi + i\lambda (dt)^{(2/D_F)-1} \partial_t \Psi - \frac{U}{2}\Psi \equiv 0 \tag{14}$$

can be obtained substituting the relation (7)*, either in the geodesics equation* (4) *or in the motion equation* (5)*.*

Proof. This result can be directly obtained following the procedure from the Appendix (fractal geodesics in the Schrödinger type representation) and also from Mercheş and Agop [33]. □

Remark 19. Assuming now that the entities motions of any physical system take place on Peano curves (i.e., for $D_F = 2$), at Compton scale resolution (i.e., for $\lambda = \hbar/2m_0$, where $\hbar$ is the reduced Planck constant and m_0 the rest mass of the physical system entity), (8) and (9) reduce to the standard Schrödinger equations:

$$\frac{h^2}{2m_0}\Delta\Psi + i\hbar\partial_t\Psi = 0, \tag{15}$$

$$\frac{h^2}{2m_0}\Delta\Psi + i\hbar\partial_t\Psi - U\Psi = 0 \tag{16}$$

respectively.

Practically, we discuss here the fractalization by stochasticization through Markov type stochastic processes. These stochastic processes are compatible with generalized Brownian type motions (Mandelbrot [37] and Nottale [36]).

Remark 20. Since, by means of this mathematical procedure, Quantum Mechanics becomes a particular case of Fractal Mechanics (in the fractal dimension $D_F = 2$ at Compton scale resolution $\lambda = \hbar/2m_0$), all the results of Quantum Mechanics can be extended (generalized) to any scale resolution. Thus, "fundamental concepts" of Quantum Mechanics as quantum entanglement, quantum superposition, quantum information, and so forth have to be substituted by those of "fractal entanglement", "fractal superposition", "fractal information" (Agop et al. [49–51] and Grigorovici et al. [52]), and so forth. Moreover, we shall substitute the notion of an atom/pseudo-atom with that of a fractal atom/fractal pseudo-atom, respectively.

4. Barbilian's Model of the Fractal Differential Geometry

In what follows, we shall use the integral and differential properties of the homographic group.

Theorem 21. *In the case of a physical system dynamics described by means of the stationary Schrödinger type fractal* *equation, the "synchronization" of its entities implies a special symmetry. The explicitation of this symmetry could be achieved through a continuous group with three parameters (the homographic group in Barbilian's form).*

Proof. One can immediately get the conclusion based on the properties of the Schrödinger type fractal equation solutions for the stationary one-dimensional case.

For this reason, the Schrödinger type fractal equation (8) in the one-dimensional case becomes

$$\lambda^2 (dt)^{(4/D_F)-2} \partial_{xx}\Psi(x,t) + i\lambda (dt)^{(2/D_F)-1} \partial_t \Psi = 0 \tag{17}$$

By means of the solution

$$\Psi(x,t) = \theta(x) \exp\left[-\frac{i}{2m_0\lambda (dt)^{(2/D_F)-1}} Et\right] \tag{18}$$

with E the energy of a physical system entity, (12) becomes

$$\partial_{xx}\theta(x) + k_0^2\theta(x) = 0 \tag{19}$$

$$k_0^2 = \frac{E}{2m_0\lambda (dt)^{(4/D_F)-2}} \tag{20}$$

i.e., a stationary Schrödinger type fractal equation.

The most general solution of (14) can be written in the form:

$$\theta(x) = he^{i(k_0x+\varphi)} + \overline{h}e^{-i(k_0x+\varphi)} \tag{21}$$

with h a complex amplitude, $\overline{h}$ its complex conjugate, and φ a phase.

This solution describes physical system entities of the same "characteristic" k_0, in which the entity is identified by means of the parameters $h, \overline{h}$ and $k = e^{i\varphi}$.

Now, a question arises. Which is the relation among the entities of the physical system having the same k_0? The answer to this question could be got if one admits that all we intend here is to find a way to switch from a triplet of numbers—the initial conditions—of an entity, to the same triplet of another entity having the same k_0.

This transition implies a special symmetry which is explicited in the form of a continuous group with three parameters, group that is simple transitive and which could be constructed based on a certain definition of k_0.

Since the ratio between two fundamental solutions of (16) is a solution of Schwartz's nonlinear equation (Mihăileanu [53])

$$\{\tau_0(x), x\} = 2k_0^2 \tag{22}$$

where the curly brackets define Schwartz's derivative of τ_0 with respect to x,

$$\{\tau_0(x), x\} = \partial_x\left(\frac{\partial_{xx}\tau_0}{\partial_x\tau_0}\right) - \frac{1}{2}\left(\frac{\partial_{xx}\tau_0}{\partial_x\tau_0}\right)^2 \tag{23}$$

This equation seems to be a veritable definition of k_0, as a general characteristic of any entity of a physical system which can be swept through a continuous group with three parameters—the homographic group.

One can easily observe that (17) is invariant with respect to the dependent variable change:

$$\tau(x) = \frac{a\tau_0(x) + b}{c\tau_0(x) + d}, \quad a, b, c, d \in \mathbb{R} \tag{24}$$

and this statement can be directly verified.

In this way, $\tau(x)$ characterizes another entity of the same k_0, which allows us to state that, starting from a standard entity, we can sweep the entire physical system of entities having the same k_0, when we are not conditioning (we leave it free) the ratios $a : b : c : d$ in (19).

We can then highlight the correspondence between a homographic transformation and a physical system entity, by associating to every physical system entity, a "personal" $\tau(x)$ by the relation:

$$\tau_1(x) = \frac{h + \overline{h}k\tau_0(x)}{1 + k\tau_0(x)}, \quad k = e^{-2i\varphi} \tag{25}$$

Let us observe that τ_0 and τ_1 can be used freely one in place of another and this leads us to the following transformation group for the initial conditions (Barbilian's group):

$$\begin{aligned} h &\longleftrightarrow \frac{ah + b}{ch + d}, \\ \overline{h} &\longleftrightarrow \frac{a\overline{h} + b}{c\overline{h} + d}, \\ k &\longleftrightarrow \frac{c\overline{h} + d}{ch + d}k \end{aligned} \tag{26}$$

This group is simple transitive: to a given set of values $(a/c, b/c, d/c)$ will correspond a single transformation and only one of the group. ☐

Remark 22. The group (21) works as a group of "synchronization" among the various entities of the physical system, process to which the amplitudes and phases of each of them obviously participate, in the sense that they are correlated, too. More precisely, by means of (21), the phase of k is moved with a quantity depending on the amplitude of the physical system entity at the transition among various physical system entities. But not only that, the amplitude of the physical system entities is also affected homographically.

The usual "synchronization" manifested through the delay of the amplitudes and phases of the physical system entities must represent here only a totally particular case.

Theorem 23. *In the case of group* (21) *we have a differential realization describing the action of* $SL(2\mathbb{R})$.

Proof. The proof of this statement is based on the differential and integral properties of the homographic group. Thus, considering a specific parametrization of the group (21), the infinitesimal generators (Mazilu and Agop [48] and Mercheş and Agop [33])

$$\widehat{B}_1 = \frac{\partial}{\partial h} + \frac{\partial}{\partial \bar{h}},$$
$$\widehat{B}_2 = h\frac{\partial}{\partial h} + \bar{h}\frac{\partial}{\partial \bar{h}}, \quad (27)$$
$$\widehat{B}_3 = h^2\frac{\partial}{\partial h} + \bar{h}^2\frac{\partial}{\partial \bar{h}} + \left(h-\bar{h}\right)k\frac{\partial}{\partial k}$$

satisfy the commutation relations:

$$\left[\widehat{B}_1, \widehat{B}_2\right] = \widehat{B}_1,$$
$$\left[\widehat{B}_2, \widehat{B}_3\right] = \widehat{B}_3, \quad (28)$$
$$\left[\widehat{B}_3, \widehat{B}_1\right] = -2\widehat{B}_2$$

i.e., an action of $SL(2\mathbb{R})$. □

Theorem 24. *The differential realization* (22) *implies a particular differential geometry of fractal type (Barbilian's model of differential geometry of fractal type).*

Proof. Indeed, the structure (of the group (21)) is given by (23) so that the only nonzero structure constants should be

$$C_{12}^1 = C_{23}^3 = -1, \quad C_{31}^2 = -2 \quad (29)$$

In consequence, the invariant quadratic form is given by the "quadratic" tensor of the group (21):

$$C_{\alpha\beta} = C_{\alpha\nu}^{\mu} C_{\beta\mu}^{\nu} \quad (30)$$

(we understand the summation over repeated indices). By (24) and (25), the tensor $C_{\alpha\beta}$ becomes

$$C_{\alpha\beta} = \begin{pmatrix} 0 & 0 & -4 \\ 0 & 2 & 0 \\ -4 & 0 & 0 \end{pmatrix} \quad (31)$$

so the invariant metric of the group (21) (Barbilian metric) is

$$\frac{ds^2}{g^2} = \omega_0^2 - 4\omega_1\omega_2 \quad (32)$$

with g an arbitrary factor of fractal type and $\omega_\alpha, \alpha = 0, 1, 2$ three differential 1-forms (Flanders [54]) that are absolutely invariant through (21). These 1-forms are given by the relations (Barbilian [55]):

$$\omega_0 = -i\left(\frac{dk}{k} - \frac{dh + d\bar{h}}{h - \bar{h}}\right),$$
$$\omega_1 = \frac{dh}{\left(h-\bar{h}\right)k}, \quad (33)$$
$$\omega_2 = \frac{-kd\bar{h}}{\left(h-\bar{h}\right)}$$

so that the metric (27) becomes

$$\frac{ds^2}{g^2} = -\left(\frac{dk}{k} - \frac{dh + d\bar{h}}{h-\bar{h}}\right)^2 + 4\frac{dhd\bar{h}}{\left(h-\bar{h}\right)^2} \quad (34)$$

□

Remark 25. The above results can be rewritten in real terms based on the transformations:

$$\left(h, \bar{h}, k\right) \longrightarrow (u, v, \phi) \quad (35)$$

which can be made explicit through the relations

$$h = u + iv,$$
$$\bar{h} = u - iv, \quad (36)$$
$$k = e^{i\phi}$$

Thus, both the operators (22) and the 1-forms (28) have the expressions:

$$\widehat{M}_1 = \frac{\partial}{\partial u},$$
$$\widehat{M}_2 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \quad (37)$$
$$\widehat{M}_3 = \left(u^2 - v^2\right)\frac{\partial}{\partial u} + 2uv\frac{\partial}{\partial v} + 2v\frac{\partial}{\partial \phi}$$

respectively,

$$\Omega^1 \equiv \omega^0 = d\phi + \frac{du}{v},$$
$$\Omega^2 = \omega^1 = \cos\phi\frac{du}{v} + \sin\phi\frac{dv}{v}, \quad (38)$$
$$\Omega^3 = \omega^2 = -\sin\phi\frac{du}{v} + \cos\phi\frac{dv}{v}$$

while the 2-form (29) reduces to the two-dimensional Lorentz metric

$$-\left(\Omega^1\right)^2 + \left(\Omega^2\right)^2 + \left(\Omega^3\right)^2 = -\left(d\phi + \frac{du}{v}\right)^2 + \frac{du^2 + dv^2}{v^2} \quad (39)$$

Theorem 26. *For the restriction $\omega_0 = 0$, Barbilian's metric is reduced to Lobachewski's metric. Then a parallelism of directions in Levi Civita's sense must be defined.*

Proof. The metric (27) is reduced to the metric of Lobachewski's plane in Poincaré's representation:

$$\frac{ds^2}{g^2} = 4\frac{dhd\bar{h}}{\left(h-\bar{h}\right)^2} \quad (40)$$

for the condition $\omega_0 = 0$, i.e., in real terms (31)

$$d\phi = -\frac{du}{v} \tag{41}$$

By this restriction, the metric (39) in the variables (31) can be reduced to Lobachewski's one in Beltrami's representation:

$$\frac{ds^2}{g^2} = -\frac{du^2 + dv^2}{v^2} \tag{42}$$

The relation (40) defines a parallelism of directions in Levi-Civita's sense. Indeed, since the parallelism angle in Lobachewski's plane is given by the relation (Flanders [54])

$$\begin{aligned} d\phi &= \frac{1}{2}\left\{\frac{\partial}{\partial v}[\ln F(v,u)]\,du - \frac{\partial}{\partial u}[\ln F(v,u)]\,dv\right\} \\ &= -\frac{du}{v} \end{aligned} \tag{43}$$

with $F(u, v) = (1/v^2)$ the conformal factor of the metric (42), we find (41). □

Remark 27. The existence of a parallelism of directions in Levi-Civita's sense on a hyperbolic manifold implies a particular type of synchronization of physical system entities through the phase-amplitude correlation.

Remark 28. The "ensemble" of the initial conditions of the physical system entities corresponding to the same k_0 can be organized as a geometry of the hyperbolic plane.

Remark 29. The existence of the parallelism of directions in Levi-Civita's sense (41) implies either the substitution of the operators (22) with the operators,

$$\begin{aligned} \widehat{B}_1' &= \frac{\partial}{\partial h} + \frac{\partial}{\partial \overline{h}}, \\ \widehat{B}_2' &= h\frac{\partial}{\partial h} + \overline{h}\frac{\partial}{\partial \overline{h}}, \\ \widehat{B}_3' &= h^2\frac{\partial}{\partial h} + \overline{h}^2\frac{\partial}{\partial \overline{h}} \end{aligned} \tag{44}$$

in the case of the representation in complex variables, or the substitution of the operators (32) with the operators,

$$\begin{aligned} \widehat{M}_1' &= \frac{\partial}{\partial u}, \\ \widehat{M}_2' &= u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \\ \widehat{M}_3' &= \left(u^2 - v^2\right)\frac{\partial}{\partial u} + 2uv\frac{\partial}{\partial v} \end{aligned} \tag{45}$$

in the case of the representation in real variables.

5. Probabilities Generated by Means of Harmonic Mappings

In the following, we shall use the integral and differential properties of the homographic group in Barbilian's form (Barbilian [55]).

Theorem 30. *Barbilian's group is measurable.*

Proof. Barbilian's group is simply transitive and, since its structure vector (see the method from Flanders [54])

$$C_\alpha = C^{\nu}_{\nu\alpha} \tag{46}$$

is identically null, as it can be seen from (24), this means that it posseses an invariant function given by relation:

$$F\left(h, \overline{h}, k\right) = -\frac{1}{\left(h - \overline{h}\right)^2 k} \tag{47}$$

which is the inverse of the modulus of the determinant of a linear system (Mercheş and Agop [33]). □

Remark 31. In the space of the variables $(h, \overline{h}, k)$, a priori probabilistic theories, based on the elementary probability

$$dP\left(h, \overline{h}, k\right) = -\frac{dh \wedge d\overline{h} \wedge dk}{\left(h - \overline{h}\right)^2 k} \tag{48}$$

where $\wedge$ denotes the external product of the 1-forms, can be constructed.

Remark 32. This is a physical attractive point of the theory—even though the realization is not a compact one—especially for statistical and stochastic constructions (Mazilu and Agop [48], Mercheş and Agop [33]). In these constructions, the complex numbers h, k are simply related to the fundamental probability characterizing contingencies which, from a Quantum Mechanical point of view, may lay at the basis of the space and space-time structures (Mazilu and Agop [48]). At this juncture, the fractal theories appear as most appropriate. They certainly allow a unitary view of the connection of gravitation, in the Einstenian acceptance, therefore as inertial field, with some other known fields of a classical or quantum nature alike (Mazilu and Agop [48]).

Theorem 33. *The fundamental problem in the dynamic analysis of the fields theory is to get field equations by means of Lobachewski's plane metrics (or related to them) using a variational principle.*

Proof. We suppose that we can describe the field by means of the variables y_i for which we "discovered" the metric (Mazilu and Agop [48])

$$h_{ij}dy^i dy^j \tag{49}$$

in an "ambient space" of the metric

$$\gamma_{\alpha\beta}dx^\alpha dx^\beta. \tag{50}$$

In consequence, the field equations can be obtained from the variational principle (Mazilu and Agop [48])

$$\delta\int L\gamma^{1/2}d^3x = 0 \tag{51}$$

relative to the Lagrange function

$$L = \gamma^{\alpha\beta} h_{ij} \frac{\partial y^i}{\partial x^\alpha} \frac{\partial y^j}{\partial x^\beta} = \gamma^{\alpha\beta} \frac{\partial h/\partial x^\alpha \cdot \partial \overline{h}/\partial x^\beta}{\left(h-\overline{h}\right)^2} = \frac{\nabla h \nabla \overline{h}}{\left(h-\overline{h}\right)^2}. \tag{52}$$

Then the differential equations of the field (Barbilian's field equations) corresponding to the variational principle (51) take the form

$$\begin{aligned} \left(h-\overline{h}\right) \cdot \left(\nabla^2 h\right) &= 2\nabla h \nabla h, \\ \left(h-\overline{h}\right) \cdot \left(\nabla^2 \overline{h}\right) &= 2\nabla \overline{h} \nabla \overline{h} \end{aligned} \tag{53}$$

and have the solution

$$h = -i \frac{\cosh\tau - e^{-i\alpha}\sinh\tau}{\cosh\tau + e^{-i\alpha}\sinh\tau} \tag{54}$$

with

$$\Delta\tau = 0 \tag{55}$$

and α real. □

Remark 34. The field equations (53) given by means of the variational principle (51) relative to the Lagrange function (52) describe through (55) a harmonic map between the ordinary flat space of metric $\gamma_{\alpha\beta}$ and the complex half plane possessing the Poincaré metric, exactly as in the case of classical gravitation of motion. But here the physical interpretation is a little bit different: (53) reveal an intimate interrelationship between inertia and gravitation. In such perspective a theory of space-time free for any contradictions of actual gravitation theory can be conceived by means of harmonic mappings, i.e., a methodology which implies complex potentials (Mazilu and Agop [48]). Only that such functionality induced by complex potentials is reduced to fractal variables (functions depending both on spatial coordinates and on time and scale resolutions).

Theorem 35. *If the square root of the invariant function of Barbilian's group can be assimilated with a wave function, then a stationary Schrödinger type fractal equation with a complex "eigenvalue" can be obtained by means of the harmonic mappings.*

Proof. Taking into account the field equations (53), one can deduce that the "wave" function

$$\Psi = \frac{1}{\nu} e^{i\Phi/2} \tag{56}$$

obtained from (47) satisfies the stationary Schrödinger type fractal equation:

$$\nabla^2\Psi = \left[\left(\frac{\nabla\nu}{\nu}\right)^2 + \frac{\nabla u \nabla \nu}{\nu} - \frac{(\nabla\Phi)^2}{4} + \frac{i}{2}\frac{\nabla\Phi\nabla\nu}{\nu}\right]\Psi \tag{57}$$

with a complex "eigenvalue". □

Remark 36. The complex "eigenvalue" becomes real if the entities of the physical system have either the same (spatial) phase or the same amplitude. In the first case, (57) reduces to

$$\nabla\Psi = \left(\frac{\nabla\nu}{\nu}\right)^2 \Psi \tag{58}$$

its "eigenvalue" being the squared "momentum" responsible for coordinate-momentum uncertainty in the hydrodynamic model of "Fractal Mechanics". In fact, both Barbilian's field equations and the way we define the "wave" function show that a fractal hydrodynamic model of such a "Fractal Mechanics" is more easily approached (see the Appendix, fractal geodesics in the hydrodynamic type representation).

Remark 37. This analysis offers an example for the way how *a priori* generated probabilities, as measure of some parametric groups, can interfere in fundamental physical theories. Thus, *a priori* probabilistic theory in Jaynes' sense must be constructed (Jaynes [56]): "every circumstance left unspecified in the statement of a problem defines an invariance property, which the solution must have if there is to be any definite solution at all. The transformation group which expresses these invariances mathematically imposes definite restrictions on the form of the solution, and in many cases fully determines it".

Remark 38. Since Quantum Mechanics represents a particular case of Fractal Mechanics (motions on Peano curves at Compton scale resolution), the Quantum Measure Theory could be extended to Fractal Measure Theory. Then, the probabilities could be a priori generated as measure of some parameter groups by means of the harmonic mappings.

6. From the Minimal Atom to the Fractal Minimal Atom

The previous results specify the following:

(i) Quantum Mechanics is a particular case of Fractal Mechanics (theory of motion on Peano curves at Compton scale resolution).

(ii) Probabilities in Jaynes' sense are generated through harmonic mappings between the Euclidean space and the hyperbolic one.

These two arguments are sufficient in our opinion since they enable us to give a generalization of the Quantum Measure Theory to the possible Fractal Measure Theory and, implicitly, of certain concepts developed in the framework of such theories. One of these concepts is that of a fractal minimal atom, as a natural generalization for the concept of a minimal atom.

In what follows, we refer to the definition and some properties of the fractal minimal atom.

We suppose that T is an abstract nonvoid set, $\mathscr{C}$ is a lattice of subsets in $\mathscr{P}(T)$, and $\nu : \mathscr{C} \longrightarrow \mathbb{R}_+$ is a set function so that $\nu(\emptyset) = 0$. Evidently, one can immediately generalize the notions of a pseudo-atom/minimal atom, respectively, to this context when $\mathscr{C}$ is only a lattice and not necessarily a ring.

Example 39. (i) If T is a nonempty metric space, then the Hausdorff dimension $\dim_{Haus} : \mathscr{P}(T) \longrightarrow \mathbb{R}$ (Mandelbrot [37]) is a monotone real function. Evidently, $\dim_{Haus}(\emptyset) = 0$.

(ii) For every $d \geq 0$, the Hausdorff measure $H^d : \mathscr{P}(T) \longrightarrow \mathbb{R}$ is an outer measure, so, particularly, it is a submeasure.

Remark 40. (i) The union of two sets M and N having the fractal dimensions D_M and D_N, respectively, has the fractal dimension $D_{M\cup N} = \max\{D_M, D_N\}$.

(ii) The intersection of two sets M and N having the fractal dimensions D_M and D_N, respectively, has the fractal dimension $D_{M\cap N} = D_M + D_N - d$, where d is the embedding Euclidean dimension (Iannaccone and Khokha [57]).

The following definition is then consistent.

Definition 41. A pseudo-atom/minimal atom, respectively, $M \in \mathscr{C}$ of ν having the fractal dimension D_M is said to be a fractal pseudo-atom/fractal minimal atom, respectively.

Proposition 42. *If $M, N \in \mathscr{C}$ are fractal pseudo-atoms of ν and if $\nu(M \cap N) > 0$, then $M \cap N$ is a fractal pseudo-atom of ν and $\nu(M \cap N) = \nu(M) = \nu(N)$.*

7. Conclusions

(i) Minimal atomicity in correspondence with Quantum Measure Theory is discussed. In such context, some physical applications are provided.

(ii) The concept of minimal atomicity is extended in the form of fractal atomicity. Some mathematical properties of fractal minimal atomicity are given. In such approach, an inverse method with respect to the common developments concerning the minimal atomicity concept has been used, observing that Quantum Mechanics identifies as a particular case of Fractal Mechanics when a scale resolution is given. Precisely, we talk about a fractality through Markov type stochastic processes, in which case the standard Schrödinger equation identifies with the geodesics of a fractal space for motions of entities of a physical system on curves of Peano type at Compton scale resolution.

For the one-dimensional stationary case of the fractal Schrödinger type geodesics, a special symmetry induced by the homographic group in the form of Barbilian's group implies the "synchronization" of the entities of a given physical system. The integral and differential properties of the synchronization group, under the restriction of the existence of a parallelism of directions in Levi-Civita's sense, impose correspondences with the "dynamics" of the hyperbolic plane. In such conjecture, the definition of a harmonic mapping between an Euclidian space and a hyperbolic one in the form of a variational principle permits the *a priori* generation of probabilities in Jaynes' sense. An explicitation of such situation specifies the fact that the hydrodynamical variant of a Fractal Mechanics is more easily approached. Moreover, the Quantum Measure Theory can become a particular case of a possible Fractal Measure Theory at a given scale resolution.

Appendix

A. On a Fractal Theory of Motion

Assuming that nondifferentiability is a fundamental property of the motions (Mandelbrot [37]), a correspondence between the interaction processes and fractality of the motion trajectories can be established. Then, consequences of nondifferentiability, scale covariance derivative, fractal geodesics in the Schrödinger type representation, and fractal geodesics in the hydrodynamic representation are obtained.

B. Consequences of Nondifferentiability

Let us assume that the motions of the entities of a given physical system take place on continuous but nondifferentiable curves (fractal curves). In such hypothesis, the following consequences are resulting (Nottale [36]).

(i) Any continuous but nondifferentiable curve is explicitly scale resolution δt dependent; i.e., its length tends to infinity when δt tends to zero.

We mention that a curve is nondifferentiable if it satisfies the Lebesgue theorem (Mandelbrot [37]); i.e., its length becomes infinite when the scale resolution goes to zero. Consequently, in this limit, a curve is as zig-zagged as one can imagine. Thus, it exhibits the property of self-similarity in every one of its points, which can be translated into a property of holography (every part reflects the whole) (Mandelbrot [37], Mitchell [46], and Nottale [36]).

(ii) The physics of phenomena is related to the behavior of a set of functions during the zoom operation of the scale resolution δt. Then, through the substitution principle, δt will be identified with dt; i.e., $\delta t \equiv dt$ and, consequently, it will be considered as an independent variable. We reserve the notation dt for the usual time as in the Hamiltonian physical system dynamics.

(iii) The physical system dynamics is described through fractal variables, i.e., functions depending on both the space coordinates and the scale resolution since the differential time reflection invariance of any dynamical variable is broken. Then, in any point of a physical system fractal curve, two derivatives of the variable field $Q(t, dt)$ can be defined:

$$\begin{aligned} \frac{d_+Q(t,dt)}{dt} &= \lim_{\Delta t \longrightarrow 0_+} \frac{Q(t+\Delta t, \Delta t) - Q(t, \Delta t)}{\Delta t} \\ \frac{d_-Q(t,dt)}{dt} &= \lim_{\Delta t \longrightarrow 0_-} \frac{Q(t, \Delta t) - Q(t-\Delta t, \Delta t)}{\Delta t} \end{aligned} \quad \text{(B.1)}$$

The "+" sign corresponds to the forward physical processes, while the "−" sign corresponds to the backwards ones.

(iv) The differential of the spatial coordinate field $dX^i(t, dt)$ is expressed as the sum of the two differentials, one of them being scale resolution independent (differential part $d_\pm x^i(t)$), and the other one being scale resolution dependent (fractal part $d_\pm \xi^i(t)$), i.e.,

$$d_\pm X^i(t, dt) = d_\pm x^i(t) + d_\pm \xi^i(t, dt). \quad \text{(B.2)}$$

(v) The nondifferentiable part of the spatial coordinate field satisfies the fractal equation:

$$d_{\pm}\xi^i(t,dt) = \lambda_{\pm}^i (dt)^{1/D_F} \tag{B.3}$$

where $\lambda_{\pm}^i$ are constant coefficients through which the fractalization type is specified and D_F defines the fractal dimension of the motion curve (Mandelbrot [37]).

In our opinion, physical processes imply dynamics on motion curves with various fractal dimensions (more precisely, for $D_F = 2$, quantum type physical processes, for $D_F < 2$, correlative type physical processes, and for $D_F > 2$, noncorrelative type physical processes) (Mandelbrot [37] and Nottale [36]).

Because all the physical processes described here can take place simultaneously in the dynamics of a given physical system, it is thus necessary to consider the multifractal behavior of the physical system (Mandelbrot [37]).

(vi) The differential time reflection invariance of any dynamical variable is recovered by combining the derivatives d_+/dt and d_-/dt in the nondifferentiable operator:

$$\frac{\widehat{d}}{dt} = \frac{1}{2}\left(\frac{d_+ + d_-}{dt}\right) - \frac{i}{2}\left(\frac{d_+ - d_-}{dt}\right) \tag{B.4}$$

This is a natural result of the complex prolongation procedure applied to physical system dynamics (Nottale [36]). Applying now the nondifferentiable operator to the spatial coordinate field, it yields the complex velocity field:

$$\widehat{V}^i = \frac{\widehat{d}X^i}{dt} = V_D^i - V_F^i \tag{B.5}$$

with

$$\begin{aligned} V_D^i &= \frac{1}{2}\frac{d_+X^i + d_-X^i}{dt}, \\ V_F^i &= \frac{1}{2}\frac{d_+X^i - d_-X^i}{dt} \end{aligned} \tag{B.6}$$

The real part V_D^i of the complex velocity field is differentiable and scale resolution independent (differentiable velocity field), while the imaginary one V_F^i is nondifferentiable and scale resolution dependent (fractal velocity field).

(vii) In the absence of any external constraint, an infinite number of fractal curves (geodesics) can be found relating any pair of points, and this is true on all scale resolutions of the physical system dynamics. Then, in the fractal space of the physical system, all its entities are substituted with the geodesics themselves so that any external constraint can be interpreted as a selection of geodesics. The infinity of geodesics in the bundle, their nondifferentiability, and the two values of the derivative imply a generalized statistical fluid-like description (in what follows we shall call it a fractal fluid). Then, the average values of the fractal fluid variables must be considered in the previously mentioned sense, so the average of $d_{\pm}X^i$ is

$$\left\langle d_{\pm}X^i \right\rangle \equiv d_{\pm}x^i \tag{B.7}$$

with

$$\left\langle d_{\pm}\xi^i \right\rangle = 0 \tag{B.8}$$

The previous relation (8) implies that the average of the nondifferentiable spatial coordinate field is null.

C. Scale Covariant Derivative

The fractal fluid dynamics can be described through a scale covariant derivative, the explicit form of which is obtained as follows. Let us consider that the nondifferentiable curves are immersed in a 3-dimensional space and that X^i are the spatial coordinate field of a point on the nondifferentiable curve. We also consider a variable field $Q(X^i, t)$ and the following Taylor expansion up to the second order:

$$\begin{aligned} d_{\pm}Q\left(X^i, t\right) &= \partial_t Q dt + \partial_i Q d_{\pm}X^i \\ &+ \frac{1}{2}\partial_l\partial_k Q d_{\pm}X^l d_{\pm}X^k \end{aligned} \tag{C.1}$$

These relations are valid in any point and more for the points X^i on the nondifferentiable curve which we have selected in (9). From here, the main forward and backward values for fractal fluid variables from (9) become

$$\begin{aligned} \langle d_{\pm}Q \rangle &= \langle \partial_t Q dt \rangle + \left\langle \partial_i Q d_{\pm}X^i \right\rangle \\ &+ \frac{1}{2}\left\langle \partial_l\partial_k Q d_{\pm}X^l d_{\pm}X^k \right\rangle \end{aligned} \tag{C.2}$$

We suppose that the average values of all variable field Q and its derivatives coincide with themselves and the differentials $d_{\pm}X^i$ and dt are independent. Therefore, the average of their products coincides with the product of averages. Consequently, (10) becomes

$$d_{\pm}Q = \partial_t Q dt + \partial_i Q \left\langle d_{\pm}X^i \right\rangle + \frac{1}{2}\partial_l\partial_k Q \left\langle d_{\pm}X^l d_{\pm}X^k \right\rangle \tag{C.3}$$

Even the average value of $d_{\pm}\xi^i$ is null, for the higher order of $d_{\pm}\xi^i$ the situation can still be different. Let us focus on the averages $\langle d_{\pm}\xi^l d_{\pm}\xi^k \rangle$. Using (3) we can write

$$\left\langle d_{\pm}\xi^l d_{\pm}\xi^k \right\rangle \pm \lambda_{\pm}^l \lambda_{\pm}^k (dt)^{(2/D_F)-1} dt \tag{C.4}$$

where we accepted that the sign + corresponds to $dt > 0$ and the sign − corresponds to $dt < 0$.

Then, (11) takes the form:

$$\begin{aligned} d_{\pm}Q &= \partial_t Q dt + \partial_i Q \left\langle d_{\pm}X^i \right\rangle + \frac{1}{2}\partial_l\partial_k Q d_{\pm}x^l d_{\pm}x^k \\ &\pm \frac{1}{2}\partial_l\partial_k Q \left[\lambda_{\pm}^l \lambda_{\pm}^k (dt)^{(2/D_F)-1} dt\right] \end{aligned} \tag{C.5}$$

If we divide by dt and neglect the terms that contain differential factors (for details, see the method from Mercheş and Agop [33]), we obtain

$$\frac{d_{\pm}Q}{dt} = \partial_t Q + \nu_{\pm}^i \partial_i Q \pm \frac{1}{2}\lambda_{\pm}^l \lambda_{\pm}^k (dt)^{(2/D_F)-1} \partial_l\partial_k Q \tag{C.6}$$

These relations also allow us to define the local nondifferentiable operators

$$\frac{d_\pm}{dt} = \partial_t + v_\pm^i \partial_i \pm \frac{1}{2} \lambda_\pm^l \lambda_\pm^k (dt)^{(2/D_F)-1} \partial_l \partial_k \tag{C.7}$$

where

$$\begin{aligned} v_+^i &= \frac{d_+ x^i}{dt}, \\ v_-^i &= \frac{d_- x^i}{dt} \end{aligned} \tag{C.8}$$

Under these circumstances, taking into account (4), (5), and (15), let us calculate $\widehat{d}/dt$. It results in

$$\frac{\widehat{d}Q}{dt} = \partial_t Q + \widehat{V}^i \partial_i Q + \frac{1}{4} (dt)^{(2/D_F)-1} D^{lk} \partial_l \partial_k Q \tag{C.9}$$

where

$$\begin{aligned} D^{lk} &= d^{lk} - i\overline{d}^{lk} \\ d^{lk} &= \lambda_+^l \lambda_+^k - \lambda_-^l \lambda_-^k, \\ \overline{d}^{lk} &= \lambda_+^l \lambda_+^k + \lambda_-^l \lambda_-^k \end{aligned} \tag{C.10}$$

The relation (16) also allows us to define the scale covariant derivative in the form (Mercheş and Agop [33]):

$$\frac{\widehat{d}}{dt} = \partial_t + \widehat{V}^i \partial_i + \frac{1}{4} (dt)^{(2/D_F)-1} D^{lk} \partial_l \partial_k \tag{C.11}$$

D. Fractal Geodesics

Let us now consider the functionality of the scale covariance principle (the physics laws are invariant with respect to scale transformations, Nottale [36]). Then the passage from the classical (differentiable) physics to the fractal (nondifferentiable) physics can be implemented by replacing the standard time derivative d/dt with the "nondifferentiable operator" $\widehat{d}/dt$. Thus, this operator plays the role of the scale covariant derivative; namely, it is used to write the fundamental equations of fractal fluid dynamics in the same form as in the classic (differentiable) case. Under these conditions, applying the operator (18) to the complex velocity field (5), in the absence of any external constraint, the geodesics take the following form:

$$\frac{\widehat{d}\widehat{V}^i}{dt} = \partial_t \widehat{V}^i + \widehat{V}^l \partial_l \widehat{V}^i + \frac{1}{4} (dt)^{(2/D_F)-1} D^{lk} \partial_l \partial_k \widehat{V}^i = 0 \tag{D.1}$$

This means that the local acceleration $\partial_t \widehat{V}^i$, the local convection $\widehat{V}^l \partial_l \widehat{V}^i$, and the local dissipation $D^{lk} \partial_l \partial_k \widehat{V}^i$ make their balance in any point of the nondifferentiable curve. Moreover, the presence of the complex coefficient of viscosity-type $(1/4)(dt)^{(2/D_F)-1} D^{lk}$ in the fractal fluid dynamics specifies that it is a rheological medium. So, it has memory, as a datum, by its own structure.

If the fractalization is achieved by Markov type stochastic processes, which involve generalized Brownian type movements (Tabor [42]) of the fractal fluid entities, then (Nottale [36])

$$\lambda_+^i \lambda_+^l = \lambda_-^i \lambda_-^l = 2\lambda \delta^{il} \tag{D.2}$$

where δ^{il} is Kronecker's pseudo-tensor.

In such conjecture, the scale covariant derivative becomes

$$\frac{\widehat{d}}{dt} = \partial_t + \widehat{V}^l \partial_l - i\lambda (dt)^{(2/D_F)-1} \partial^l \partial_l \tag{D.3}$$

so that the geodesics equation of the fractal fluid takes the simple form

$$\frac{\widehat{d}\widehat{V}^i}{dt} = \partial_t \widehat{V}^i + \widehat{V}^l \partial_l \widehat{V}^i - i\lambda (dt)^{(2/D_F)-1} \partial^l \partial_l \widehat{V}^i = 0 \tag{D.4}$$

or more, by separating the motions on differential and fractal scale resolutions,

$$\begin{aligned} \frac{\widehat{d}V_D^i}{dt} &= \partial_t V_D^i + V_D^l \partial_l V_D^i \\ &\quad - \left[V_F^l + \lambda (dt)^{(2/D_F)-1} \partial^l\right] \partial_l V_F^i = 0 \\ \frac{\widehat{d}V_F^i}{dt} &= \partial_t V_F^i + V_D^l \partial_l V_F^i \\ &\quad + \left[V_F^l + \lambda (dt)^{(2/D_F)-1} \partial^l\right] \partial_l V_D^i = 0 \end{aligned} \tag{D.5}$$

In the presence of an external scalar potential U, the geodesics equation takes the form

$$\frac{\widehat{d}\widehat{V}^i}{dt} = \partial_t \widehat{V}^i + \widehat{V}^l \partial_l \widehat{V}^i - i\lambda (dt)^{(2/D_F)-1} \partial^l \partial_l \widehat{V}^i = -\partial^i U \tag{D.6}$$

These results are even more general than those obtained by Nottale [36] since his theory "operates" only with dynamics in fractal dimension 2.

E. Fractal Geodesics in the Schrödinger Type Representation

For irrotational motions of the fractal fluid, the complex velocity field $\widehat{V}^l$(5) takes the form:

$$\widehat{V}^l = -2i\lambda (dt)^{(2/D_F)-1} \partial^l \ln \Psi \tag{E.1}$$

Substituting (23) in (21) and using the method from Mercheş and Agop [33], up to an arbitrary phase factor which may be set at zero by a suitable choice of the phase of Ψ, we obtain the fractal type Schrödinger equation (without constraints):

$$\lambda^2 (dt)^{(4/D_F)-2} \partial_p \partial^p \Psi + i\lambda (dt)^{(2/D_F)-1} \partial_t \Psi = 0 \tag{E.2}$$

The standard Schrödinger equation

$$\frac{\hbar^2}{2m_0}\partial_p\partial^p\Psi + i\hbar\partial_t\Psi = 0 \tag{E.3}$$

can be obtained from (24) for nonrelativistic motions on Peano curves, $D_F = 2$ (Nottale [36]), at Compton scale $\lambda = \hbar/2m_0$ (Nottale [36]).

In the case of nondifferentiable dynamics with constraints, for instance, under the action of a scalar potential U, following the same procedure as before, one obtains the fractal type Schrödinger equation (with constraints):

$$\lambda^2 (dt)^{(4/D_F)-2}\,\partial^l\partial_l\Psi + i\lambda (dt)^{(2/D_F)-1}\,\partial_t\Psi - \frac{U}{2}\Psi = 0 \tag{E.4}$$

For nonrelativistic motions on Peano curves, $D_F = 2$, at Compton scale $\lambda = \hbar/2m_0$, (30) takes the standard form:

$$\frac{\hbar^2}{2m_0}\partial_l\partial^l\Psi + i\hbar\partial_t\Psi - U\Psi = 0 \tag{E.5}$$

F. Fractal Geodesics in the Hydrodynamic Type Representation

If $\psi = \sqrt{\rho}\exp(iS)$, with $\sqrt{\rho}$ the amplitude and S the phase of ψ, the complex velocity field (23) takes the explicit form:

$$\begin{aligned}\widehat{V}^i &= -2i\lambda (dt)^{2/D_F-1}\,\partial^i \ln\Psi\\ V_D^i &= 2\lambda (dt)^{2/D_F-1}\,\partial^i S\\ V_F^i &= \lambda (dt)^{2/D_F-1}\,\partial^i \ln\rho\end{aligned} \tag{F.1}$$

Substituting (28) in (21) and separating the real and the imaginary parts, up to an arbitrary phase factor which may be set at zero by a suitable choice of the phase of ψ, we obtain (see the method from Mercheş and Agop [33])

$$\partial_t V_D^i + \left(V_D^l\partial_l\right)V_D^i = -\partial^i(Q) \tag{F.2}$$

$$\partial_t\rho + \partial^i\left(\rho V_D^i\right) = 0 \tag{F.3}$$

with Q the specific nondifferentiable (fractal) potential:

$$\begin{aligned}Q &= -2\lambda^2 (dt)^{4/D_F-2}\,\frac{\partial^l\partial_l\sqrt{\rho}}{\sqrt{\rho}}\\ &= -\frac{V_F^l V_{Fl}}{2} - \lambda (dt)^{2/D_F-1}\,\partial_i V_F^i\end{aligned} \tag{F.4}$$

We note that the presence of an external scalar potential U modifies (29) in the form:

$$\partial_t V_D^i + \left(V_D^l\partial_l\right)V_D^i = -\partial^i(Q+U) \tag{F.5}$$

Equation (29) represents the specific momentum conservation law, while (30) represents the states density conservation law. Equations (29)-(31) define the fractal hydrodynamics model and imply the following:

(i) Any entity of the physical system is in a permanent interaction with a fractal medium through the specific nondifferentiable potential.

(ii) The physical system can be identified with a fractal fluid, whose dynamics is described by the fractal hydrodynamic model.

(iii) The fractal velocity field V_F^i does not represent actual motion but contributes to the transfer of the specific momentum and to the energy focus. This may be clearly seen from the absence of V_F^i from the states density conservation law and from its role in the variation principle (Nottale [36]).

(iv) Any interpretation of the specific fractal potential should take cognizance of the "self" nature of the specific momentum transfer. While the fractal energy is stored in the form of the mass motion and fractal potential energy, some other part is available elsewhere and only the total is conserved. It is the conservation of the fractal energy and the fractal momentum that ensure fractal reversibility and the existence of fractal eigenstates but denies a generalized Brownian motion fractal force of interaction with an external medium (Tabor [42]).

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

All authors declare that they have no conflicts of interest.

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